Sine-Gordon solitons: Particles obeying relativistic dynamics

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We define a dynamical variable associated with the position of a particle representing the sine-Gordon soliton. Simple relativistic dynamic equations for this particle under the influence of forces are derived, including the case of dissipation. The fluctuation-dissipation theorem for this particle is obtained as well as some results on steady-state velocities and slowing of the thermalization time for ultrarelativistic solitons.

There has recently been a substantial amount of interest in the motion of solitonlike excitations in nonlinear systems under the action of various perturbations.[1–5] In specific situations, such as the sine-Gordon systems, these effects may be treated either numerically,[6–8] by perturbation theory or similar approximations[3–5,9] for small disturbances, or by energy-balance and steady-state considerations.[9–11] In many types of situations, it has been found that the solitons behave like particles driven by external forces (including dissipative and Langevin-type[12] ones).[2,13–15] Theoretical derivations of such behavior have tended to be rather circumspect,[16] and consequently their validity has at times been questioned. In particular, apparent non-Newtonian effects have been noted in the acceleration of solitons,[6] as well as a seeming disagreement between their "inertial" and "rest" masses.[2,17] When the velocities approach the speed of light for the model, the solitons in the sine-Gordon chain also exhibit relativistic-type effects[1–6] with some modifications in the dissipative case.[11]

In this paper we confirm the expectation[3] that solitons behave like particles. We directly address this question by making a heuristically obvious definition of the coordinate Q associated with the motion of the soliton in space. The conjugate momentum P is found, and the relativistic-type kinematics is set up. The way in which external forces in the underlying sine-Gordon equation are translated to those acting upon the particle is clarified. The basic Newton's second law $\dot{P}=F$ is obtained under general conditions (including dissipation, noise, uniform, and local perturbations) and, as a result, many of the properties which have previously been derived approximately and sometimes rather laboriously for soliton dynamics may be simply and exactly derived. The apparent non-Newtonian behavior[6] is explained as well as the "mass paradox."[2,6] We believe that we presented here a direct, simple, and powerful method for treating soliton dynamics. As typical simple applications we derive the slowing down of the "thermalization" time for ultrarelativistic solitons, the Langevin equation for relativistic solitons, and the terminal steady-state velocity as a function of an external uniform force in the presence of dissipation. We also discuss the scattering of the soliton by a localized force.

We consider the general sine-Gordon equation for a classical field θ,

$$\theta_{tt}-\theta_{xx}+\sin\theta=\mathscr{F}(x,t)-G\theta_t \ . \tag{1}$$

In appropriate dimensionless units, G is a viscosity and $\mathscr{F}$ is an external driving force, which may include Langevin[12]-type noise terms appropriate, say, to a temperature T.[13,15] We start with the kinematics of a free soliton ($G=0$, $\mathscr{F}=0$). The appropriate

Lagrangian density $\mathscr{L}$ is given by[18]

$$\mathscr{L}=\tfrac{1}{2}\theta_t^2-\tfrac{1}{2}\theta_x^2-(1-\cos\theta) . \quad (2)$$

For a soliton traveling with a velocity v such that $\tan(\theta/4)=e^{-\gamma(x-vt)}$ the energy is given by[1]

$$E=\int_{-\infty}^{+\infty}[\tfrac{1}{2}\theta_t^2+\tfrac{1}{2}\theta_x^2+(1-\cos\theta)]dx=m_0\gamma , \quad (3)$$

where $m_0=8$ in these units and $\gamma\equiv(1-v^2)^{-1/2}$.

One would like to define a particlelike coordinate[16] associated with the soliton, which for a free soliton should have the same mass and energy and moving with the same velocity v, as the soliton. Consider the form of θ_x; it is positive, symmetrically peaked around the center of the soliton, and has a width comparable with that associated with the latter. Furthermore, its integral over all x is a conserved quantity for a single soliton. We are thus led to view $\theta_x/2\pi$ as the spatial distribution for the soliton. We therefore define the coordinate Q by

$$Q\equiv\frac{1}{2\pi}\int_{-\infty}^{+\infty}x\theta_x dx \quad (4)$$

with a velocity given then by

$$\dot{Q}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}x\theta_{xt}dx . \quad (5)$$

The canonical momentum conjugate to Q (using $\Pi\equiv\partial\mathscr{L}/\partial\theta_t=\theta_t$ which is the momentum density conjugate to θ) is

$$P\equiv\int_{-\infty}^{+\infty}\Pi\theta_x dx . \quad (6)$$

It is straightforward to check that P and Q satisfy the relation $[Q,P]_{\mathrm{PB}}=1$, (where PB means Poisson bracket), as long as $\int_{-\infty}^{+\infty}\theta_x dx=2\pi$.

For the free soliton we obtain

$$Q=Q_0+vt , \quad \dot{Q}=v , \quad P=m_0\gamma v \quad (7)$$

using Eqs. (4)–(6) and the relation $\theta_t=-v\theta_x$. Obviously, the energy of this particle is equal to $m=m_0\gamma$ as in Eq. (3).

Next we consider the effects of forces on the particle. From Eq. (6)

$$\dot{P}=\int_{-\infty}^{+\infty}\theta_{tt}\theta_x dx , \quad (8)$$

where the term $\int_{-\infty}^{+\infty}\theta_{xt}\theta_t dx=\tfrac{1}{2}\theta_t^2\,|_{-\infty}^{+\infty}$ vanishes under the assumption that at $x\to\pm\infty$, θ_t vanishes. This assumption is sometimes violated like in situations where the force depends on time, but Eq. (8) is still valid as long as θ_t^2 behaves similarly at $x=\pm\infty$.

Using Eq. (1) without dissipation ($G=0$) we find, for a force $\mathscr{F}(x,t)$ satisfying the above assumption,

$$\dot{P}=\int_{-\infty}^{+\infty}\mathscr{F}(x,t)\theta_x dx\equiv F , \quad (9)$$

where again the surface terms vanish. In the particular case of a force uniform in space $\mathscr{F}(t)$, $\dot{P}=2\pi\mathscr{F}(t)$. Thus the force driving the particle is 2π times the force operating on the field θ. In cases where $P=mv$ (see below) and for $v\ll 1$ this yields $\dot{v}=2\pi\mathscr{F}/m_0$. This result has been viewed as a paradox[2,6]—the inertial mass being different from the rest mass. However, it is simply due to the relation between $\mathscr{F}$ and the force F driving the particle.[17]

The relation $P=mv$ was obtained for free solitons and may, sometimes and with appropriate modification, be generalized to situations where transients are not important. If the force F is switched on at $t=0$, there is a transient time, inversely proportional to G, in which the soliton form is distorted and therefore $P\neq mv$. The non-Newtonian effects of Ref. 6 are just this transient. While for $v\ll 1$, $\dot{v}\neq F/m_0$ (in fact,[6] $v\propto t^2$), Eq. (9) which is the correct formulation of Newton's law is valid.

We now add the effect of a finite dissipation G, which changes Eq. (9) to

$$\dot{P}=-GP+F . \quad (10)$$

Thus for $F=0$, P decays with a time constant G^{-1}. Using the relations $P=mv$ (valid for slow changes and for velocities that are not too high, see below), $E=m$, we find $\dot{v}=-Gv/\gamma^2$ and $\dot{E}=-Gv^2E$. The velocity decay of ultrarelativistic solitons is extremely slow. These results can also be obtained by a direct calculation of dE/dt, using Eq. (3). They had been derived by perturbation theory in Refs. 3–5 and are clearly seen in the numerical simulations of Ref. 13. (In fact, this is the reason why the fast solitons have not thermalized in these calculations, a fact which is of importance *vis-à-vis* the long-time dynamics.[13–15])

For an external force which depends only on time one can directly show that

$$\frac{dE}{dt}=F(t)\dot{Q} \quad \text{or} \quad \frac{dE}{dP}=\dot{Q} . \quad (11)$$

In view of Eqs. (9)–(11), one might be tempted to conclude that the soliton behaves exactly like a point particle. This is of course incorrect, because the relationship between E and p (or between p and v) has not been shown to remain $E^2=p^2+m_0^2$, as it would be for a free soliton. In fact, as the soliton is accelerated other degrees of freedom may get excited so that only part of the energy and momentum imparted to the system according to Eqs. (9)–(11) resides in the bare soliton. These matters have been investigated using perturbation theory,[2,3] and it remains to be seen whether our approach can also be used to discuss them.

For a "relativistic" soliton with a small but finite dissipation G, it has been found that the Lorentz-type contraction of its spatial width by a factor γ is valid only as long as $\gamma \ll 1/G$.[11,19,20] For $\gamma \gtrsim 1/G$ the width saturates at a value proportional to G. Thus the last relation in Eq. (7) between P and v is modified $P = m_0 \gamma_{\text{eff}} v$, where $\gamma_{\text{eff}} \to \gamma$ for $\gamma \lesssim G^{-1}$ and $\gamma_{\text{eff}} \propto G$ for $\gamma \gtrsim G^{-1}$. $\gamma_{\text{eff}} = \gamma$ for the case of no dissipation. The case of a large G has been treated in Refs. 19 and 20. We emphasize that Eq. (10) is unaffected by this.

In dissipative systems one should also consider the effect of Langevin-type noise terms, where $\mathscr{F}(x,t)$ has an uncorrelated Gaussian white-noise part $r(x,t)$ satisfying $\langle r(x',t') r(x,t) \rangle = 2GT \times \delta(x-x')\delta(t-t')$ (T is the temperature measured in the appropriate units). The Langevin force operating on the particle is shown in the limit $kT \ll m_0 c^2$ to satisfy

$$\langle R(t) R(t+\tau) \rangle = 2GTm_0 \gamma \delta(\tau) , \tag{12}$$

which is the appropriate fluctuation-dissipation theorem[12] for the particle considered here. In deriving (12) a coarse-grained θ_x has to be assumed. At long times, the particle will diffuse with a diffusion coefficient given in this limit by $kT/(m_0 G)$.

Thus if interaction with a heat bath is assumed for the above sine-Gordon problem, an effective heat bath at the same temperature is obtained for the particle, which will reach thermal equilibrium and perform diffusive motion after a thermalization time given by the discussion following Eq. (10). If interactions among the solitons and antisolitons can be neglected, this justifies the assumption that this gas will get into the appropriate equilibrium for the Boltzmann gas. Of course, to get the complete picture one has also to include the nucleation and annihilation of kink-antikink pairs,[15] which is not contained in our treatment.

In the presence of a uniform force $\mathscr{F}$ in Eq. (1) and a finite G, the soliton will reach a steady-state terminal velocity v_0, given in terms of $\mathscr{F}$ by $\mathscr{F} = m_0 \gamma_{\text{eff}} G v_0 / 2\pi$, in agreement, whenever $\gamma_{\text{eff}} = \gamma$, with the energy-balance equation result.[9–11]

For an external force localized in space and constant in time, $\mathscr{F}(x)$, the soliton experiences a localized force given by (7). In the case of a spatial range of $\mathscr{F}$ much larger than the width of the soliton in the presence of the force this yields $F(Q) = 2\pi \mathscr{F}(x=Q)$. This is now a straightforward classical one-dimensional scattering problem for the particle.

To summarize, we have established the kinematics and dynamics of the classical relativistic particle associated with the sine-Gordon soliton under the influence of external forces. This should facilitate the treatment of many further interesting problems[3] such as perturbations in the parameters of the equation, periodic forces, and particle interactions.

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