PHYSICAL REVIEW A VOLUME 27, NUMBER 6 JUNE 1983

Quantum nondemolition measurements on coupled harmonic oscillators

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(Received 3 May 1982)

Quantum nondemolition measurements on an harmonic-oscillator detector are discussed for two detector-meter coupling schemes: parametric amplification and parametric frequency conversion. A time-dependent solution for the density operator of the coupled detector-meter system, including damping, is obtained. A sequence of measurements via meter state reduction is analyzed.

I. INTRODUCTION

The current experimental efforts to detect gravitational radiation have led to a new consideration of how quantum fluctuations in the detection and associated amplification process limit the accuracy with which a classical force may be monitored. This is due to the fact that the displacements produced by gravitational radiation reaching terrestrial detectors are so small as to displace a harmonic oscillator by less than the quantum-mechanical uncertainties in the ground state of the oscillator. The detector must, then, be treated quantum mechanically. Since quantum fluctuations are intrinsic to any detection apparatus it is obviously important to consider ways to circumvent the limits such fluctuations place on the accuracy with which a weak classical force may be monitored. This problem was first recognized by Braginsky and Vorontsov,[1] and several people have suggested schemes to overcome it.[2–5] These measurement schemes have been given the general title of quantum nondemoliton (QND) measurements.

In the detection of gravitational radiation, one is faced with the problem of making a sequence of measurements on a single quantum system, the detector (e.g., a bar-type harmonic oscillator), the results of which must be completely predictable in the absence of the gravitational wave. For most dynamic variables this is not possible since intrinsic quantum-mechanical uncertainties in the results of a measurement lead to fluctuations in the results of successive measurements. However, there do exist variables, the so called QND variables for which such a predictable sequence of measurements is possible, providing the first measurement is sufficiently precise. The QND measurement scheme thus seeks to find appropriate QND variables and devise ways of coupling them to an amplifier-readout stage so that they may be measured. In the usual analysis of such a QND measurement we treat the first stage of the readout system as another quantum system coupled to the detector. This second stage is referred to as the meter.

In this paper we consider both the detector and meter to be harmonic oscillators and consider two models of detector-meter coupling, which may be achieved, for example, in the coupling of two electromagnetic field modes via a nonlinear crystal.

The first, a parametric amplifier coupling,[6,7] was suggested by Hillery and Scully.[8] The second model considers a parametric frequency conversion coupling.[9]

In the first part of this paper we consider the two models from a completely deterministic point of view. The Heisenberg equations of motion are solved and the QND observables are identified. It is shown that the parametric amplifier interaction is not back-action evading, however, the frequency converter interaction is back-action evading in a more general sense.

The failure of the parametric amplifier as a back-action evading interaction illustrates how quantum noise, even in the absence of damping, can limit the predictability of a QND measurement.

The accuracy of a QND measurement is further degraded when damping via spontaneous emission is taken into account. We model damping of the two oscillator modes by coupling the modes to a heat bath. Since we are only interested in the effects of quantum noise we consider the heat bath to be at absolute zero.

The free evolution of the system with damping, is then treated using a master equation and associated Fokker-Planck equation. The generalized P distribution of Drummond and Gardiner[10,11] is used to describe the nonclassical states which arise in this problem.

We then proceed to the second part of a QND measurement—readout of the meter variable and associated state reduction. This allows an analysis of a sequence of measurements, and shows that despite the failure of back-action evasion for the parametric amplifier a QND measurement is still possible for sufficiently large coupling strength.

II. QUANTUM NONDEMOLITION MEASUREMENTS

The theory of QND measurements has been described in detail by Caves *et al.*[5] Here we shall just summarize some of the results necessary for an understanding of the following analysis.

Consider an observable $\mathscr{A}$, with the corresponding Hermitian Schrödinger picture operator being $\hat{A}(t)$ (where there may be an explicit time dependence). If in any sequence of measurements of $\mathscr{A}$, the results of each measurement can be predicted precisely from the results of a proceeding measurement, then $\mathscr{A}$ is a QND observable. This requires that[5]

$$\hat{A}(t)=\mathscr{f}(\hat{A}(t_0),t,t_0) \;, \tag{2.1}$$

where $\mathscr{f}$ is an arbitrary function of $\hat{A}(t_0)$, t, and the initial time t_0. In the interaction picture, Eq. (2.1) implies

$$[A^I(t'),A^I(t)]=0 \;. \tag{2.2}$$

Equation (2.2) means that if the system begins in an eigenstate of $\mathscr{A}$ it remains in this eigenstate for all times. Observables which are conserved in the absence of interactions with the external world clearly satisfy Eq. (2.2).

We can also define a generalized QND variable as follows:

$$\hat{A}(t)=\mathscr{f}(\hat{A}(t_0),\hat{B}_i,t,t_0) \;, \tag{2.3}$$

where the Hermitian operators $\hat{B}_i$ commute with one another and with $\hat{A}(t_0)$. If a system begins in an eigenstate of a generalized QND observable it remains in this eigenstate, only if the initial state was also an eigenstate of the $\hat{B}_i$'s.

Equation (2.2) allows us to find the detector QND variables. However, we need to know whether the coupling to the meter places any additional constraints on $\hat{A}(t)$ in order that it remain a QND variable. It is interesting that the meter need introduce no further fluctuations into the QND variable we are trying to measure, if the following criteria holds. The evolution of a QND observable $\hat{A}(t)$ is completely unaffected by the interaction with the measuring apparatus provided that $\hat{A}(t)$ is the only observable of the detector that appears in the detector-meter interaction Hamiltonian. If this condition holds we say that we have evaded the back-action of the meter.

Back-action evasion means that if the detector is placed in a near eigenstate of the QND observable it remains in a near eigenstate under free evolution of the coupling detector-meter system. It should be noted that even though a QND observable maintains its QND property [Eq. (2.2)] in the presence of the interaction Hamiltonian, the measurement scheme will not be back-action evading unless the criteria, specified above, holds.

III. MODELS FOR DETECTOR-METER COUPLING

We take a model for the detector-meter system where the detector and meter are taken to be harmonic oscillators. The Hamiltonian for this model may be written as

$$H=\hbar\omega_a a^\dagger a+\hbar\omega_b b^\dagger b+H_I+a\Gamma_a^\dagger+a^\dagger\Gamma_a +b\Gamma_b^\dagger+b^\dagger\Gamma_b \;, \tag{3.1}$$

where a refers to the detector mode and b refers to the meter mode. Γ_a and Γ_b are heat bath operators for the detector and meter modes, respectively. ω_a and ω_b are the oscillator frequencies of the two modes. We assume that the heat baths are at absolute zero.

The interaction Hamiltonian H_I takes the following forms for the two models we are considering.

1. Parametric amplifier

$$H_I^{(1)}=-\hbar\kappa\{a^\dagger b^\dagger\exp[-i(\omega_a+\omega_b)t]+\text{c.c.}\} \;. \tag{3.2}$$

2. Parametric frequency converter

$$H_I^{(2)}=\hbar\kappa\{a^\dagger b\exp[-i(\omega_a-\omega_b)t]+\text{c.c.}\} \;. \tag{3.3}$$

These interaction Hamiltonians may be achieved, for example, in the coupling of two electromagnetic field modes in a nonlinear crystal. The interaction represented in Eq. (3.2) was first suggested as a QND scheme by Hillery and Scully.[8]

Before proceeding, we first define some important oscillator observables. In the Schrödinger picture we define the Hermitian operators, $\hat{X}_i(t)$ and $\hat{Y}_i(t)$ $(i=1,2)$ by the following expressions:

$$\hat{X}_1(t)=\left[\frac{\hbar}{2\omega_a}\right]^{1/2}(a\,e^{i\omega_a t}+a^\dagger e^{-i\omega_a t})\ , \tag{3.4}$$

$$\hat{X}_2(t)=\frac{1}{i}\left[\frac{\hbar}{2\omega_a}\right]^{1/2}(a\,e^{i\omega_a t}-a^\dagger e^{-i\omega_a t})\ , \tag{3.5}$$

$$\hat{Y}_1(t)=\left[\frac{\hbar}{2\omega_b}\right]^{1/2}(b\,e^{i\omega_b t}+b^\dagger e^{-i\omega_b t})\ , \tag{3.6}$$

$$\hat{Y}_2(t)=\frac{1}{i}\left[\frac{\hbar}{2\omega_b}\right]^{1/2}(b\,e^{i\omega_b t}-b^\dagger e^{-i\omega_b t})\ , \tag{3.7}$$

and

$$[\hat{X}_1,\hat{X}_2]=i\hbar/\omega_a\ .$$

Thus, $\hat{X}_1,\hat{X}_2$ and $\hat{Y}_1,\hat{Y}_2$ are operators corresponding to the real and imaginary components of the detector (meter) mode complex amplitude. Let us first consider the parametric amplifier interaction, $H_I^{(1)}$.

We may write the interaction Hamiltonian in the interaction picture as

$$H_I^{(1)}=-\kappa\sqrt{\omega_a\omega_b}(\hat{X}_1\hat{Y}_1-\hat{X}_2\hat{Y}_2)\ , \tag{3.8}$$

which is explicitly time independent.

The Heisenberg equations of motion in the interaction picture are

$$\frac{d\hat{X}_1(t)}{dt}=\kappa\left[\frac{\omega_b}{\omega_a}\right]^{1/2}\hat{Y}_2(t)\ , \tag{3.9a}$$

$$\frac{d\hat{X}_2(t)}{dt}=\kappa\left[\frac{\omega_b}{\omega_a}\right]^{1/2}\hat{Y}_1(t)\ , \tag{3.9b}$$

$$\frac{d\hat{Y}_1(t)}{dt}=\kappa\left[\frac{\omega_a}{\omega_b}\right]^{1/2}\hat{X}_2(t)\ , \tag{3.9c}$$

$$\frac{d\hat{Y}_2(t)}{dt}=\kappa\left[\frac{\omega_a}{\omega_b}\right]^{1/2}\hat{X}_1(t)\ . \tag{3.9d}$$

These equations have the following solutions:

$$\hat{X}_1(t)=\hat{X}_1(0)\cosh\kappa t+\left[\frac{\omega_b}{\omega_a}\right]^{1/2}\hat{Y}_2(0)\sinh\kappa t\ , \tag{3.10a}$$

$$\hat{X}_2(t)=\hat{X}_2(0)\cosh\kappa t+\left[\frac{\omega_b}{\omega_a}\right]^{1/2}\hat{Y}_1(0)\sinh\kappa t\ , \tag{3.10b}$$

$$\hat{Y}_1(t)=\hat{Y}_1(0)\cosh\kappa t+\left[\frac{\omega_a}{\omega_b}\right]^{1/2}\hat{X}_2(0)\sinh\kappa t\ , \tag{3.10c}$$

$$\hat{Y}_2(t)=\hat{Y}_2(0)\cosh\kappa t+\left[\frac{\omega_a}{\omega_b}\right]^{1/2}\hat{X}_1(0)\sinh\kappa t\ . \tag{3.10d}$$

It is now clear that $\hat{X}_1(t)$ is a QND variable of this system, i.e.,

$$[\hat{X}_1(t),\hat{X}_1(t')]=0\ .$$

From Eqs. (3.10a) and (3.10c) we have

$$\hat{X}_1(t)=\left[\frac{\omega_b}{\omega_a}\right]^{1/2}\left[\hat{Y}_2(t)\coth\kappa t-\frac{\hat{Y}_2(0)}{\sinh\kappa t}\right]\ . \tag{3.11}$$

Using this equation, we can infer values for $\hat{X}_1(t)$ by making measurements of $\hat{Y}_2(t)$. A scheme for the measurement of such quadrature phase amplitudes based on homodyne detection has been proposed by Yuen and Shapiro.[12]

As was stated earlier, this interaction model is not back-action evading. Clearly, the interaction Hamiltonian does not satisfy the fundamental back-action evading criteria. However, it is more instructive to demonstrate the failure of back-action evasion more directly.

We first define the variance in $\hat{X}_1(t)$ as follows:

$$\Delta\hat{X}_1(t)^2=\langle\hat{X}_1^2(t)\rangle-\langle\hat{X}_1(t)\rangle^2\ . \tag{3.12}$$

This determines the possible spread of measured results of $\hat{X}_1(t)$ on an ensemble of identically prepared systems. Using Eq. (3.10a), we find

$$\Delta X_1^2(t)=\Delta X_1^2(0)\cosh^2\kappa t+\frac{\omega_b}{\omega_a}\Delta Y_2^2(0)\sinh^2\kappa t\ . \tag{3.13}$$

We then see that even if $\Delta Y_2^2(0)$ is zero, $\Delta X_1^2(t)$ will grow with time, unless $\Delta X_1^2(0)=0$, i.e., the detector is initially in an $\hat{X}_1(0)$ eigenstate. If the detector was initially in a near eigenstate of $\hat{X}_1(0)$ it will not remain so.

We now consider the situation for the parametric frequency converter. The Hamiltonian in the interaction picture may be written as

$$H_I^{(2)}=\kappa\sqrt{\omega_a\omega_b}(\hat{X}_1\hat{Y}_1+\hat{X}_2\hat{Y}_2)\ . \tag{3.14}$$

The equations of motion are

$$\frac{d\hat{X}_1(t)}{dt}=\kappa\left[\frac{\omega_b}{\omega_a}\right]^{1/2}\hat{Y}_2(t)\ , \tag{3.15a}$$

$$\frac{d\hat{X}_2(t)}{dt}=-\kappa\left[\frac{\omega_b}{\omega_a}\right]^{1/2}\hat{Y}_1(t)\ , \tag{3.15b}$$

$$\frac{d\hat{Y}_1(t)}{dt}=\kappa\left[\frac{\omega_a}{\omega_b}\right]^{1/2}\hat{X}_2(t)\ , \tag{3.15c}$$

$$\frac{dY_2(t)}{dt}=-\kappa\left[\frac{\omega_a}{\omega_b}\right]^{1/2}\hat{X}_1(t)\ . \tag{3.15d}$$

The solutions are

$$\hat{X}_1(t)=\hat{X}_1(0)\cos\kappa t+\left[\frac{\omega_b}{\omega_a}\right]^{1/2}\hat{Y}_2(0)\sin\kappa t\ , \tag{3.16a}$$

$$\hat{X}_2(t)=\hat{X}_2(0)\cos\kappa t-\left[\frac{\omega_b}{\omega_a}\right]^{1/2}\hat{Y}_1(0)\sin\kappa t\ , \tag{3.16b}$$

$$\hat{Y}_1(t)=\hat{Y}_1(0)\cos\kappa t+\left[\frac{\omega_a}{\omega_b}\right]^{1/2}\hat{X}_2(0)\sin\kappa t\ , \tag{3.16c}$$

$$\hat{Y}_2(t)=\hat{Y}_2(0)\cos\kappa t-\left[\frac{\omega_a}{\omega_b}\right]^{1/2}\hat{X}_1(0)\sin\kappa t\ . \tag{3.16d}$$

Once again we identify $\hat{X}_1(t)$ as the QND observable for the detector. From Eq. (3.16a) and (3.16d) we find

$$\hat{X}_1(t)=\left[\frac{\omega_b}{\omega_a}\right]^{1/2}\left[-\hat{Y}_2(t)\cot\kappa t+\frac{\hat{Y}_2(0)}{\sin\kappa t}\right]\ , \tag{3.17}$$

which allows values for $\hat{X}_1(t)$ to be inferred from measurements of $\hat{Y}_2(t)$.

It would appear at first inspection that this interaction is also not back-action evading. It does not satisfy the fundamental criterion, that the QND observable be the only detector observable in the interaction Hamiltonian. However, this time the situation is not so bad. From Eq. (3.16a) we have

$$\Delta X_1^2(t)=\Delta X_1^2(0)\cos^2\kappa t+\frac{\omega_b}{\omega_a}\Delta Y_2^2(0)\sin^2\kappa t\ . \tag{3.18}$$

If $\Delta Y_2^2(0)=0$ then $\Delta X_1^2(t)\leq\Delta X_1^2(0)$ for all time, that is, if the detector is placed in a near eigenstate of $\hat{X}_1(t)$ it will remain in a near eigenstate of $\hat{X}_1(t)$ and at certain times it will be exactly in an eigenstate of $\hat{X}_1(t)$. This is all that is required for a predictable sequence of measurements. As will be shown in Sec. V, a measurement of $\hat{Y}_2(0)$ places the detector in a very near eigenstate of $\hat{X}_1(0)$. Since the free evolution does not take it out of this eigenstate, subsequent measurements of $\hat{Y}_2(t)$ at a later time t, only serve to put the detector more closely in an eigenstate of $\hat{X}_1(t)$. Thus, a sequence of measurements will always yield the same result for $\hat{X}_1(t)$.

We conclude that the QND observable is not shielded from the back action of the meter in the case of the parametric amplifier coupling, however, in the case of the parametric frequency converter coupling it is effectively shielded.

IV. INCLUSION OF DAMPING

We have just shown how the fundamental dynamics of a system can prevent us from making a predictable sequence of measurements of a QND observable. We now wish to consider how damping due to the coupling of the system to external reservoirs effects the predictability of a sequence of measurements. We shall assume that thermal fluctuations may be eliminated and hence consider only spontaneous emission into a zero-temperature heat bath.

Through the use of standard techniques[13] the following master equation may be derived from the full Hamiltonian, Eq. (3.1) (in the interaction picture):

$$\frac{\partial\rho}{\partial t}=\frac{1}{i\hbar}[H_I,\rho]+\frac{\gamma_1}{2}(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a)+\frac{\gamma_2}{2}(2b\rho b^\dagger-b^\dagger b\rho-\rho b^\dagger b)\ , \tag{4.1}$$

where ρ is the density operator for the coupled system and where γ_1 and γ_2 are the damping constants for the detector and meter, respectively.

It should be noted however that in deriving Eq. (4.1) we have assumed that each system is damped independently regardless of the strength of the coupling between the two modes. It has been shown by Walls[14] that this is only true if κ is not too large. Since a large κ is desirable for fast QND measurements (as shown in Sec. V), further investigation of this point is warranted.

Using the complex P representation of Drummond and Gardiner[10,11] we expand ρ in terms of coherent states,

$$\rho=\int_{C_i} d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 P(\vec{z},t) \times \frac{|\alpha_1,\alpha_2\rangle\langle\beta_1^*,\beta_2^*|}{\langle\beta_1^*,\beta_2^*|\alpha_1,\alpha_2\rangle}, \quad (4.2)$$

where $\vec{z}^{\,T}=(\alpha_1,\beta_1,\alpha_2,\beta_2)$ and we have the following correspondences:

$$a\leftrightarrow\alpha_1 ,$$

$$a^\dagger\leftrightarrow\beta_1 ,$$

$$b\leftrightarrow\alpha_2 ,$$

$$b^\dagger\leftrightarrow\beta_2 .$$

There are actually four independent contour integrals ($i=1,4$) involved in Eq. (4.2) in the complex space of each variable. We are free to choose these contours to obtain a normalizable P function, providing partial integration is defined. Substitution of Eq. (4.2) into (4.1) yields the following Fokker-Planck equation for the complex P representation:

$$\frac{\partial P}{\partial t}(\vec{z},t)=(\nabla_{\vec{z}}^T A\vec{z}+\tfrac{1}{2}\nabla_{\vec{z}}^T D\nabla_{\vec{z}})P(\vec{z},t) , \quad (4.3)$$

where

$$\nabla_{\vec{z}}^T=\left[\frac{\partial}{\partial\alpha_1},\frac{\partial}{\partial\beta_1},\frac{\partial}{\partial\alpha_2},\frac{\partial}{\partial\beta_2}\right]$$

and, for the two models, we obtain the following.

Parametric amplifier:

$$A=\begin{bmatrix} \gamma_1/2 & 0 & 0 & -i\kappa \\ 0 & \gamma_1/2 & i\kappa & 0 \\ 0 & -i\kappa & \gamma_2/2 & 0 \\ i\kappa & 0 & 0 & \gamma_2/2 \end{bmatrix}, \quad (4.4)$$

$$D=\begin{bmatrix} 0 & 0 & i\kappa & 0 \\ 0 & 0 & 0 & -i\kappa \\ i\kappa & 0 & 0 & 0 \\ 0 & -i\kappa & 0 & 0 \end{bmatrix}. \quad (4.5)$$

Parametric frequency converter:

$$A=\begin{bmatrix} \gamma_1/2 & 0 & i\kappa & 0 \\ 0 & \gamma_1/2 & 0 & -i\kappa \\ i\kappa & 0 & \gamma_2/2 & 0 \\ 0 & -i\kappa & 0 & \gamma_2/2 \end{bmatrix}, \quad (4.6)$$

$$D=0 . \quad (4.7)$$

The general solution to Eq. (4.3) is

$$P(\vec{z},t)=\exp[-\tfrac{1}{2}(\vec{z}-\langle\vec{z}\rangle)^T\sigma^{-1}(\vec{z})(\vec{z}-\langle\vec{z}\rangle)] , \quad (4.8)$$

a multivariate Gaussian, where $\langle\vec{z}\rangle$ is

$$\langle\vec{z}(t)\rangle^T=[\langle a(t)\rangle,\langle a^\dagger(t)\rangle,\langle b(t)\rangle,\langle b^\dagger(t)\rangle] \quad (4.9)$$

and $\sigma(\vec{z})$ is the covariance matrix.

The initial states of the detector and meter are taken to be general squeezed states,[15]

detector $|\nu_1,r_1\rangle$,

meter $|\nu_2,r_2\rangle$.

These states are such that

$$\langle a(0)\rangle=\nu_1,\ \langle b(0)\rangle=\nu_2 ,$$

$$\Delta X_1^2(0)=\frac{\hbar}{2\omega_a}e^{-2r_1},\ \Delta X_2^{(0)}=\frac{\hbar}{2\omega_a}e^{2r_1} , \quad (4.10)$$

$$\Delta Y_1^2(0)=\frac{\hbar}{2\omega_b}e^{-2r_2},\ \Delta Y_2^2(0)=\frac{\hbar}{2\omega_b}e^{2r_2} .$$

r_1 and r_2 are referred to as squeeze parameters. With these initial conditions we now discuss the particular form Eq. (4.8) takes for the two interaction models.

Parametric amplifier interaction. For the interaction model represented by Eq. (3.2) (parametric amplifier coupling) we find

$$\langle\vec{z}(t)\rangle=\begin{bmatrix} \tfrac{1}{2}(q^2e^{-\Delta t/4}+p^2e^{\Delta t/4})\nu_1+\frac{4i\kappa}{\Delta}\sinh\left[\frac{\Delta t}{4}\right]\nu_2^* \\ \tfrac{1}{2}(q^2e^{-\Delta t/4}+p^2e^{\Delta t/4})\nu_1^*-\frac{4i\kappa}{\Delta}\sinh\left[\frac{\Delta t}{4}\right]\nu_2 \\ \tfrac{1}{2}(p^2e^{-\Delta t/4}+q^2e^{\Delta t/4})\nu_2+\frac{4i\kappa}{\Delta}\sinh\left[\frac{\Delta t}{4}\right]\nu_1^* \\ \tfrac{1}{2}(p^2e^{-\Delta t/4}+q^2e^{\Delta t/4})\nu_2^*-\frac{4i\kappa}{\Delta}\sinh\left[\frac{\Delta t}{4}\right]\nu_1 \end{bmatrix}\exp[-(\gamma_1+\gamma_2)t/4] , \quad (4.11)$$

where

$$\Delta=[(\gamma_2-\gamma_1)^2+16\kappa^2]^{1/2}\,,$$

$$p=\left[1+\frac{\gamma_2-\gamma_1}{\Delta}\right]^{1/2},$$

$$q=\left[1-\frac{\gamma_2-\gamma_1}{\Delta}\right]^{1/2}.$$

The full expression for the covariance matrix is somewhat involved and is given in the Appendix. From the covariance matrix we obtain

$$\Delta X_1^2(t)=\frac{\hbar}{2\omega_a}\left[1+[(e^{-2r_1}-1)\alpha^2+(e^{2r_2}-1)\beta^2]\exp[-(\gamma_1+\gamma_2)t/2]+\frac{16\kappa^2}{\Delta^2}\left[\frac{x}{p^2}-\frac{2y}{pq}+\frac{z}{q^2}\right]\right], \quad (4.12)$$

$$\Delta Y_2^2(t)=\frac{\hbar}{2\omega_b}\left[1+[(e^{2r_2}-1)\alpha^2+(e^{-2r_1}-1)\beta^2]\exp[-(\gamma_1+\gamma_2)t/2]+\frac{16\kappa^2}{\Delta^2}\left[\frac{x}{p^2}-\frac{2y}{pq}+\frac{z}{q^2}\right]\right], \quad (4.13)$$

where

$$\alpha=\frac{8\kappa^2}{\Delta^2}\left[\frac{e^{\Delta t/4}}{p^2}+\frac{e^{-\Delta t/4}}{q^2}\right],$$

$$\beta=\frac{4\kappa}{\Delta}\sinh\left[\frac{\Delta t}{4}\right],$$

$$x=\frac{-8\kappa^2}{\Delta(\gamma_1+\gamma_2+\Delta)}\times\{1-\exp[-(\gamma_1+\gamma_2+\Delta)t/2]\}\,,$$

$$y=\frac{2\kappa(\gamma_2-\gamma_1)}{(\gamma_2+\gamma_1)}\{1-\exp[-(\gamma_1+\gamma_2)t/2]\}\,,$$

$$z=\frac{8\kappa^2}{\Delta(\gamma_1+\gamma_2-\Delta)}\times\{1-\exp[-(\gamma_1+\gamma_2-\Delta)t/2]\}\,.$$

We now discuss some simplifying cases.

Consider the situation where $\kappa\gg\gamma_1,\gamma_2$. If the detector is initially in an eigenstate of $\hat{X}_1$ (i.e., $r_1\to\infty$) and the meter is in a coherent state ($r_2=0$) we have for large t

$$\Delta X_1^2(t)=\frac{\hbar\omega}{2\omega_a}(\sinh^2\kappa t+\tfrac{1}{2})\,, \quad (4.14)$$

$$\Delta Y_2^2(t)=\frac{\hbar}{2\omega_b}(\cosh^2\kappa t-\tfrac{1}{2})\,. \quad (4.15)$$

We can again see the failure of back-action evasion for this system. The presence of quantum fluctuations in $\hat{Y}_2(0)$ feed back into the QND variable $\hat{X}_1(t)$ and increase its variance, even if spontaneous damping is entirely negligible.

We now assume the system starts in a simultaneous eigenstate of $\hat{X}_1(0)$ and $\hat{Y}_2(0)$ ($r_1\to\infty$, $r_2\to-\infty$). Then,

$$\Delta X_1^2(t)=\Delta Y_2^2(t)=0 \quad (4.16)$$

and the system remains in this simultaneous eigenstate for all time. This is due to the fact that $\hat{X}_1(t)$ is actually a generalized QND variable for the total detector-meter system.

We now consider the large damping case: $\gamma_1=\gamma_2=\gamma$ and $\gamma\gg\kappa$. If the system again starts in a simultaneous eigenstate of $\hat{X}_1(0)$ and $\hat{Y}_2(0)$, we find

$$\Delta X_1^2(t)=\frac{\hbar}{2\omega_a}(1-e^{-\gamma t})\,, \quad (4.17)$$

$$\Delta Y_2^2(t)=\frac{\hbar}{2\omega_b}(1-e^{-\gamma t})\,. \quad (4.18)$$

Not surprisingly, the system simply relaxes to a coherent state as if there were no coupling. For a QND measurement we would require the measurement time to be much shorter than the time over which the system is damped.

Parametric frequency converter interaction. For the interaction equation (3.3) (parametric frequency converter coupling) the means are given by

$$\langle \vec{z}(t) \rangle = \begin{bmatrix} \left[\cos\left[\frac{\Lambda t}{4}\right] - \frac{(\gamma_1-\gamma_2)}{\Lambda}\sin(\Lambda t)\right] v_1 - \frac{4i\kappa}{\Lambda}\sin\left[\frac{\Lambda t}{4}\right] v_2 \\ \left[\cos\left[\frac{\Lambda t}{4}\right] - \frac{(\gamma_1-\gamma_2)}{\Lambda}\sin(\Lambda t)\right] v_1^* + \frac{4i\kappa}{\Lambda}\sin\left[\frac{\Lambda t}{4}\right] v_2^* \\ \frac{-4i\kappa}{\Lambda}\sin\left[\frac{\Lambda t}{4}\right] v_1 + \left[\cos\left[\frac{\Lambda t}{4}\right] + \frac{(\gamma_1-\gamma_2)}{\Lambda}\sin\left[\frac{\Lambda t}{4}\right]\right] v_2 \\ \frac{4i\kappa}{\Lambda}\sin\left[\frac{\Lambda t}{4}\right] v_1^* + \left[\cos\left[\frac{\Lambda t}{4}\right] + \frac{(\gamma_1-\gamma_2)}{\Lambda}\sin\left[\frac{\Lambda t}{4}\right]\right] v_2^* \end{bmatrix} \exp[-(\gamma_1+\gamma_2)t/4] , \tag{4.19}$$

where

$$\Lambda = [16\kappa^2 - (\gamma_1-\gamma_2)^2]^{1/2} .$$

Once again, the full expression for the covariance matrix is given in the Appendix. Here we just quote the result for $\Delta X_1^2(t)$ and $\Delta Y_2^2(t)$:

$$\Delta X_1^2(t) = \frac{\hbar}{2\omega_a}\left[1 + \left\{\left[\cos\left[\frac{\Lambda t}{4}\right] - \frac{(\gamma_1-\gamma_2)}{\Lambda}\sin\left[\frac{\Lambda t}{4}\right]\right]^2 (e^{-2r_1}-1) + \left[\frac{4\kappa}{\Lambda}\right]^2 \sin^2\left[\frac{\Lambda t}{4}\right](e^{2r_2}-1)\right\}\exp[-(\gamma_1+\gamma_2)t/2]\right] , \tag{4.20}$$

$$\Delta Y_2^2(t) = \frac{\hbar}{2\omega_b}\left[1 + \left\{\left[\cos\left[\frac{\Lambda t}{4}\right] + \frac{(\gamma_1-\gamma_2)}{\Lambda}\sin\left[\frac{\Lambda t}{4}\right]\right]^2 (e^{2r_2}-1) + \left[\frac{4\kappa}{\Lambda}\right]^2 \sin^2\left[\frac{\Lambda t}{4}\right](e^{-2r_1}-1)\right\}\exp[-(\gamma_1+\gamma_2)t/2]\right] . \tag{4.21}$$

Considerable simplification follows for $\gamma_1=\gamma_2=\gamma$ and we now discuss this case. When the meter is initially in a $\hat{Y}_2(0)$ eigenstate $r_2 \rightarrow -\infty$ we find

$$\Delta X_1^2(t) = \frac{\hbar}{2\omega_a}[1 + (e^{-2r_1}\cos^2\kappa t - 1)e^{-\gamma t}] . \tag{4.22}$$

This is simply a particular case of Eq. (3.18) with damping included. We can see that providing $\gamma \ll \kappa$, the system is back-action evading, over a certain number of periods, however, the presence of damping has destroyed back-action evasion for all time. We would require our measurement time τ to be $\ll 1/\gamma$. Once again we note that for large damping, the system simply relaxes to a coherent state with zero amplitude.

If the initial state of the detector is an $\hat{X}_1(0)$ eigenstate and the initial state of the meter is a $\hat{Y}_2(0)$ eigenstate we find

$$\Delta X_1^2(t) = \frac{\hbar}{2\omega_a}(1 - e^{-\gamma t}) , \tag{4.23}$$

$$\Delta Y_2^2(t) = \frac{\hbar}{2\omega_b}(1 - e^{-\gamma t}) . \tag{4.24}$$

In the absence of damping, the system remains in a simultaneous eigenstate of $\hat{X}_1(t)$, $\hat{Y}_2(t)$. This is again a consequence of the fact that $\hat{X}_1(t)$ and $\hat{Y}_2(t)$ are generalized QND variables for the coupled system. Damping will destroy this property.

V. METER STATE REDUCTION

A complete analysis of a QND measurement involves a determination of the free evolution of the system and the nonunitary effect of meter state reduction upon readout of a meter variable. In Sec. IV we discussed free evolution. In this section we determine the effect of meter state reduction. It will be shown that no matter how small the measurement time is, the coupling can be made sufficiently large, so that the detector is left in an eigenstate of the detector QND variable. This is the usual limit of an arbitrarily quick and accurate quantum measurement.[5]

At some point the free evolution of the combined system must be suspended, and a readout of the observable $\hat{Y}_2(t)$ made. Let this readout value be $y_2(t)$. The value of $\hat{X}_1(t)$, $x_1(t)$, is then inferred from Eqs. (3.11) and (3.17). Clearly, this inference is only exact if the system is in a simultaneous eigenstate of $\hat{X}_1(t)$ and $\hat{Y}_2(t)$. If this is not the case, there will be a distribution of values $x_1(t)$ made over a series of measurements. For a QND measurement of $\hat{X}_1(t)$ we require that after a couple of measurements the inferred value of $\hat{X}_1(t)$ settles down to a particular (time-dependent) value. This would be possible if the variance in $\hat{X}_1(t)$ decreased as a result of each measurement and did not increase as a result of free evolution. It will be shown here that for both interaction models, $\Delta X_1^2(t)$ does decrease as a result of measurement; however, as was shown in Sec. III, only the frequency converter interaction satisfies the second condition. It should be noted that the time of free evolution is in fact the time for a measurement of the QND variable $\hat{X}_1(t)$.

Upon readout of the meter value $y_2(t)$, the meter collapses into an eigenstate of $\hat{Y}_2(t)$ with eigenvalue $y_2(t)$. This causes a nonunitary change in the state of the detector, which then becomes the initial state for the next measurement. (We could however reprepare the state of the meter before the next measurement.) Using a projection operator technique, Caves *et al.*[5] have considered the effect of meter state reduction on system with the meter modeled as a free mass. We adopt a similar techinque here.

In the Schrödinger picture, the total system is represented at time t by the density operator $\rho^s(t)$. Upon reduction of the meter state, the density operator for the total system after readout is

$$\rho^s(t)_R = N[\langle y_2(t),t\,|\rho^s(t)|\,y_2(t),t\rangle] \times |y_2(t),t\rangle\langle y_2(t),t\,| \;, \tag{5.1}$$

where $|y_2(t),t\rangle$ is the eigenstate of $\hat{Y}_2(t)$ with eigenvalue $y_2(t)$, and N is a normalization constant. The density operator for the detector after readout is then

$$\rho_D^s(t)_R = N\langle y_2(t),t\,|\rho^s(t)|\,y_2(t),t\rangle \;. \tag{5.2}$$

If we introduce an interaction picture defined by

$$\rho^s(t) = \exp\left[-\frac{i}{\hbar}(H_m+H_D)t\right]\rho^I(t) \times \exp\left[\frac{i}{\hbar}(H_m+H_D)\right] , \tag{5.3}$$

where $H_m(H_D)$ is the free Hamiltonian for the meter (detector). The state of the detector after readout in this picture is

$$\rho_D^I(t)_R = N\langle y_2(t),0\,|\rho^I(t)|\,y_2(t),0\rangle \;, \tag{5.4}$$

where we have used the property[1]

$$|y_2(t),t\rangle = \exp\left[\frac{-i}{h}H_m(t)\right]|y_2(t),0\rangle \;. \tag{5.5}$$

$|y_2(t),0\rangle$ is an eigenstate of $\hat{Y}_2(0)$. If we now expand $\rho^I(t)$ in terms of the complex P representation and use Eq. (5.4), the state of the detector after readout is given by the following representation:

$$P_R(\alpha_1,\beta_1,t) = \oint_{C_2}\oint_{C_2^\dagger} d\alpha_2 d\beta_2 P(\vec{z},t)\frac{\langle y_2(t),0\,|\,\alpha_2\rangle\langle\beta_2^*\,|\,y_2(t),0\rangle}{\langle\beta_2^*\,|\,\alpha_2\rangle} \;, \tag{5.6}$$

where $P(\vec{z},t)$ is the full complex P representation for the coupled detector-meter system at time t, and C_2, and $C_2^\dagger$ are the same contours as would be used in calculating moments from $P(\vec{z},t)$.

To evaluate this integral we note that $\hat{Y}_2(0)=\hat{P}/\omega$ and thus $|y_2(t),0\rangle$ is in fact a momentum eigenstate. The momentum representation for coherent states may then be used to show

$$\langle y_2(t),0\,|\,\alpha_2\rangle = \left[\frac{1}{\hbar\omega_b\pi}\right]^{1/2}\exp\left[-\frac{\omega_b}{2\hbar}y_2^2(t) - i\left[\frac{2\omega_b}{\hbar}\right]^{1/2}\alpha_2 y_2(t) - \tfrac{1}{2}|\alpha_2|^2 + \tfrac{1}{2}\alpha_2^2\right] . \tag{5.7}$$

If we use Eq. (4.8), Eq. (5.6) may be written as

$$P_R(\alpha,\beta,t) = \oint\oint d\alpha_2 d\beta_2 \exp\{-\tfrac{1}{2}[(\vec{z}-\langle\vec{z}\rangle)^T\sigma^{-1}(\vec{z}-\langle\vec{z}\rangle)] + (\vec{z}-\vec{x})^T B(\vec{z}-\vec{x})\} \;, \tag{5.8}$$

where

$$\vec{x}^T=\left[0,0,i\left[\frac{\omega_b}{2\hbar}\right]^{1/2}y_2(t),-i\left[\frac{\omega_b}{2\hbar}\right]^{1/2}y_2(t)\right],$$

$$B=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&1\\0&0&1&-1\end{bmatrix}.$$

The contours used to evaluate the integrals are either the real or imaginary axes in α_2, β_2 complex space, depending on the sign of the quadratic term in the argument of the exponential in Eq. (5.8). Upon integration, the P representation becomes a two-dimensional Gaussian. It may be written in the following form:

$$P_R(\alpha,\beta_1,t)=\exp[-\tfrac{1}{2}(\vec{z}_1-\langle\vec{z}_1\rangle_R)^T\sigma_R^{-1}(\vec{z}_1-\langle\vec{z}_1\rangle_R)]\ , \tag{5.9}$$

where $\vec{z}_1^T=(\alpha_1,\beta_1)$.

The general form of the mean $\langle\vec{z}_1\rangle_R$ and covariance $\sigma_R(\vec{z}_1)$, after measurements including damping, for the two interaction models is given in the appendix. It will be sufficient here to give the results for $\Delta X_1^2(t)_R$ and $\langle X_1(t)\rangle_R$ for the two interaction models, in the absence of damping.

Parametric amplifier interaction. We assume both detector and meter were initially in coherent states with $\langle Y_2(0)\rangle=0$. After a readout of $\hat{Y}_2(t)$ with results $y_2(t)$, the state of the detector is such that

$$\Delta X_1^2(t)_R=\frac{\hbar}{2\omega_a}\frac{1}{\cosh 2\kappa t}\ , \tag{5.10}$$

$$\Delta X_2^2(t)_R=\frac{\hbar}{2\omega_a}\cosh 2\kappa t\ , \tag{5.11}$$

$$\langle X_1(t)\rangle_R=\frac{\langle X_1(t)\rangle}{\cosh 2\kappa t}+\frac{2}{\coth^2\kappa t+1}x_1(t)\ , \tag{5.12}$$

where $x_1(t)$ is the inferred value for $\hat{X}_1(t)$.

The limit for arbitrarily fast and accurate measurement is $\kappa t\to\infty$. In this limit we find

$$\begin{aligned}&\Delta X_1^2(t)_R\to 0\ ,\\&\Delta X_2^2(t)_R\to\infty\ ,\\&\langle X_1(t)\rangle_R\to x_1(t)\ .\end{aligned}\tag{5.13}$$

Thus, after such a perfect measurement of $\hat{X}_1(t)$, the detector is in an eigenstate of $\hat{X}_1(t)$ with eigenvalue equal to the result obtained from the measurement.

At the end of such a perfectly accurate measurement the coupled detector-meter system is in an $\hat{X}_1(t),\hat{Y}_2(t)$ eigenstate and it will remain in this eigenstate in the absence of damping. However, if the measurement is not perfectly accurate, the detector will only be in a near eigenstate of $\hat{X}_1(t)$, and because of the failure of back-action evasion it will move out of this near eigenstate during the free evolution stage of the next measurement.

If we make a second measurement, the same time t after the first measurement we find

$$\Delta\hat{X}_1^2(2t)_R=\frac{\hbar}{2\omega_a}\frac{1}{(2\cosh^4\kappa t-1)}\ .$$

Despite the increase in fluctuations of $\hat{X}_1$ under free evolution, the second readout reduces the fluctuations to a greater extent than the first readout. (We wish to thank Dr. M. Hillery for drawing our attention to this point.)

At a further time t after the second readout the variance in $\hat{X}_1$ has grown to

$$(\hbar/2\omega_a)\cosh^2\kappa t[(2\cosh^4 t-1)^{-1}]$$

which may be made arbitrarily small for sufficiently large κ. Thus all measurements subsequent to the second may determine $\hat{X}_1$ to an arbitrary degree of certainty and yield a determinate sequence of results.

Frequency converter interaction. In this interaction model we find, for initial squeezed states,

$$\Delta X_1^2(t)_R=\frac{\hbar}{2\omega_a}\left[1+\frac{2}{\left[d+\dfrac{f^2}{1-g}\right]-1}\right]\ , \tag{5.14}$$

$$\Delta X_2^2(t)_R=\frac{\hbar}{2\omega_a}\left[1-\frac{2}{\left[d+\dfrac{f^2}{1-g}\right]+1}\right]\ , \tag{5.15}$$

$$\langle X_1(t)\rangle_R = \langle X_1(t)\rangle + \frac{2if}{(g-1)\left[d+\frac{f^2}{1-g}-1\right]} \times[\langle Y_2(t)\rangle - y_2(t)]\left[\frac{\omega_b}{\omega_a}\right]^{1/2}, \tag{5.16}$$

where

$$d=\sin^2(\kappa t)\coth r_2 - \cos^2(\kappa t)\coth r_1 ,$$

$$g=\sin^2(\kappa t)\coth r_1 - \cos^2(\kappa t)\coth r_2 ,$$

$$f = i\sin(\kappa t)\cos(\kappa t)(\coth r_1+\coth r_2) .$$

We now consider various initial conditions.

(i) Detector and meter both in coherent states,

$$r_1=r_2=0 .$$

In the parametric frequency convertor initial coherent states remain coherent.[9] This means that if the initial P function for the system was

$$P(\vec{z},0)=\delta(\alpha_1-\nu_1)\delta(\beta_1-\nu_1^*)\delta(\alpha_2-\nu_2) \times\delta(\beta_2-\nu_2^*) , \tag{5.17}$$

then at time t it remains factorized,

$$P(\vec{z},t)=\delta(\alpha_1-\langle\alpha_1(t)\rangle)\delta(\beta_1-\langle\beta_1(t)\rangle) \times\delta(\alpha_2-\langle\alpha_2(t)\rangle)\delta(\beta_2-\langle\beta_2(t)\rangle) , \tag{5.18}$$

where, upon projection over meter variables, we obtain the state of the detector after measurement as

$$P(\alpha_1,\beta_1,t)_R = N\delta(\alpha_1-\langle\alpha_1(t)\rangle)\delta(\beta_1-\langle\beta_1(t)\rangle) . \tag{5.19}$$

The state of the detector is unchanged upon readout. This may also be seen from Eq. (5.14) and (5.16). In the limit $r_1=r_2=0$ we find, after a readout of time t_1,

$$\Delta X_1^2(t_1)_R=\Delta X_2^2(t_1)_R=\frac{\hbar}{2\omega_a} , \tag{5.20}$$

$$\langle X_1(t_1)\rangle_R=\langle X_1(t_1)\rangle . \tag{5.21}$$

The detector is now in a coherent state with the meter in a $\hat{Y}_2(t)$ eigenstate. We now let the system evolve freely for a time τ and make another measurement at time $t_2=t_1+\tau$. This leads to the second limit.

(ii) Detector in a coherent state ($r_1\to 0$) and meter in a $\hat{Y}_2(t)$ eigenstate ($r_2\to-\infty$). This limit depends on how close we are to a zero of $\sin(\kappa\tau)$. We find that, provided

$$\sin^2(\kappa\tau) \gg \tanh r_1$$

(with $r_1\to 0$), then

$$\Delta X_1^2(t_2)_R\to 0 , \quad \Delta X_2^2(t_2)_R\to\infty , \quad \langle\hat{X}_1(t_2)\rangle_R\to x_1(t_2) . \tag{5.22}$$

In the case $\sin^2(\kappa\tau)=0$ we find the state of the detector is unchanged. This is not surprising since, at every half period, (i.e., the zeros of $\sin\kappa\tau$), the variances return to their initial values. At the zeros of $\sin^2\kappa\tau$, the meter is already fully squeezed and a measurement has no effect. We conclude that it is necessary to make measurement times such that $\sin(\kappa\tau)\neq 0$.

The limit for arbitrarily fast and accurate measurements for the parametric frequency converter is, however, somewhat different to the limit for the parametric amplifier. In the case of the parametric frequency converter, the detector is left in an eigenstate of the variable $\hat{X}_1(t)$ with eigenvalue equal to the measured result when the values of $\kappa\tau$ are sufficiently far from the zeros of $\sin(\kappa\tau)$. If we wish to make such a perfectly accurate measurement as this for an arbitrarily small measurement time τ, we need to make κ sufficiently large so that $\kappa\tau$ always remains sufficiently far from the zeros of $\sin\kappa\tau$. This can be achieved if we vary the coupling strength so that $\kappa\tau=\text{const}$. In the case of the parametric amplifier, however, this was not sufficient. There we required $\kappa\tau$ to become very large for a perfect quantum measurement.

These conclusions are summarized in Fig. 1, where we have plotted $\Delta X_1^2(t)_R$ against measurement time for various initial states. In Fig. 1(a) the detector is initially in a coherent state and the meter is put into a near eigenstate of $\hat{Y}_2(0)$. This represents a near ideal measurement where zero fluctuations in $\hat{X}_1(t)$ may be achieved. In Fig. 1(b) we consider a nonideal measurement on the meter $[\Delta Y_2^2(0)=0.02]$. Consequently the uncertainty in $\hat{X}_1(t)$ cannot be reduced below the initial meter variance.

Once the system is in an eigenstate of $\hat{X}_1(t)$ and $\hat{Y}_2(t)$ it will remain in this eigenstate for all time. We can thus consider these first two measurements as state preparation steps. We can then perform further measurements of $\hat{X}_1(t)$ with all inferred values $x_1(t)$ predictable. This is what is required of a QND measurement.

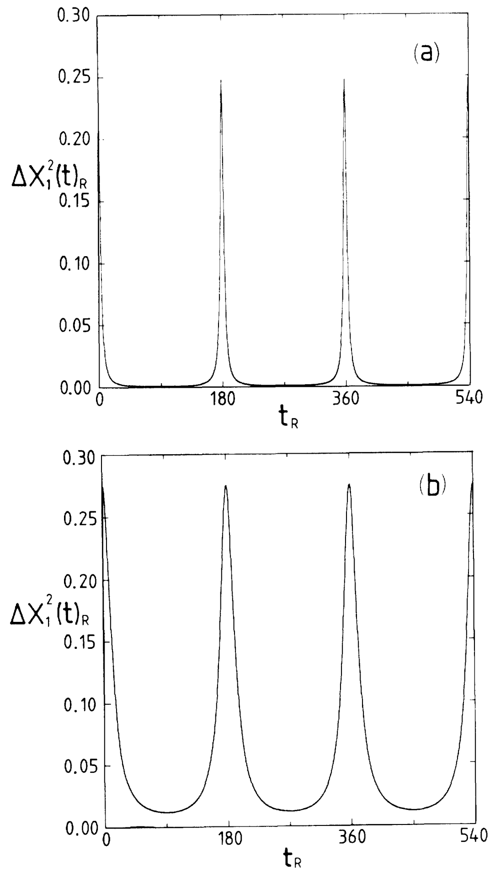

FIG. 1. Variance of $\hat{X}_1(t)$ after readout vs time of readout for two sets of initial conditions: (a) $\Delta\hat{X}_1^2(0)=0.25$, $\Delta Y_2^2(0)=0.0$ and (b) $\Delta X_1^2(0)=0.27$, $\Delta Y_2^2(0)=0.02$. For both curves $\kappa=1.0$.

Caves[5] has also introduced the concept of a stroboscopic QND variable. This is a variable for which the QND condition (2.1) is only satisfied at carefully selected times. This means that the variance of the QND variable is identically zero at certain times during the free evolution of the coupled detector-meter system.

It may appear at first sight that $\hat{X}_1(t)$ is a stroboscopic QND variable for the frequency converter. While it is true that for the meter initially in a $\hat{Y}_2(0)$ eigenstate, the variance of $\hat{X}_1(t)$ does become zero every period, $\hat{X}_1(t)$ is not a true stroboscopic QND variable. This is due to the fact that if a measurement of $\hat{Y}_2(t)$ is made at a time when $\Delta X_1^2(t)=0$, under future evolution both $\Delta Y_2^2(t)$ and $\Delta\hat{X}_1^2(t)$ will remain identically zero for all time.

VI. CONCLUSION

We have analyzed the problem of making QND measurements on a harmonic-oscillator detector, including damping. Two detector-readout stage coupling schemes have been considered: parametric amplification and parametric frequency conversion. In a QND measurement one is required to make a sequence of measurements of an observable on a single system such that after a sufficient number of measurements, the results of subsequent measurements are entirely predictable.

There are two ways in which this predictability for a QND observable may be limited or destroyed. The first arises from the reversible free evolution of the coupled system and corresponds to the failure of the detector to evade the back action of the meter. The second is due to fluctuations which arise from an irreversible coupling to heat bath. It has been shown in this paper that the parametric frequency converter is immune to the first kind of noise whereas the parametric amplifier is not. However, neither interaction is immune from the second source of noise. We need to choose our coupling strength sufficiently large so that, no matter how small the measurement time is, the effect of irreversible damping is negligible.

A sequence of measurements has been discussed by taking into account the effect of meter state reduction upon readout. This demonstrated that both the frequency converter and parametric amplifier interactions could be used to make a predictable sequence of measurements of an harmonic-oscillator quadrature phase amplitude, despite the fact that neither is truly back-action evading.

ACKNOWLEDGMENTS

The research reported here has been supported in part by the United States Army through its European Research Office. G. Milburn was supported by a New Zealand Universities Scholarship. One of us (D.F.W.) wishes to thank Dr. C. M. Caves for helpful comments.

APPENDIX

Here we quote the expressions for the covariance matrix for the two coupling models, at time t, after evolving from the initial states given in Eqs. (4.10).

Parametric amplifier interaction. The covariance matrix may be written as the sum of two terms:

$$\sigma(\vec{z})=\sigma_1(\vec{z})+\sigma_2(\vec{z}) , \tag{A1}$$

where

$$\sigma_1(\vec{z})=\begin{bmatrix} 0 & a & -ib & 0 \\ a & 0 & 0 & ib \\ -ib & 0 & 0 & a \\ 0 & ib & a & 0 \end{bmatrix} . \tag{A2}$$

and

$$a=\frac{8\kappa^2}{\Delta^2}\left[\frac{x}{p^2}-\frac{2y}{pq}+\frac{z}{q^2}\right] , \tag{A3}$$

$$b=\frac{2\kappa}{\Delta}\left[x-z\frac{(\gamma_2-\gamma_1)y}{2\kappa}\right] , \tag{A4}$$

$$\sigma_2(\vec{z})=\begin{bmatrix} s_1\alpha^2-s_2\beta^2 & c_1\alpha^2+c_2\beta^2 & i\alpha\beta(c_1+c_2) & -i\alpha\beta(s_1-s_2) \\ c_1\alpha^2+c_2\beta^2 & s_1\alpha^2-s_2\beta^2 & i\alpha\beta(s_1-s_2) & -i\alpha\beta(c_1+c_2) \\ i\alpha\beta(c_1+c_2) & i\alpha\beta(s_1-s_2) & -s_1\beta^2+s_2\alpha^2 & c_1\beta^2+c_2\alpha^2 \\ -i\alpha\beta(s_1-s_2) & -i\alpha\beta(c_1+c_2) & c_1\beta^2+c_2\alpha^2 & -s_1\beta^2+s_2\alpha^2 \end{bmatrix} \exp\left[\frac{-(\gamma_1+\gamma_2)}{2}\right] , \tag{A5}$$

where

$$s_1=-\sinh 2r_1 ,$$

$$c_1=\cosh 2r_1-1 ,$$

$$s_2=-\sinh 2r_2 ,$$

$$c_2=\cosh 2r_2-1 ,$$

and α and β are given in Eqs. (4.12) and (4.13).

Parametric frequency converter. Since the diffusion matrix in the Fokker-Planck equation is zero, the first term $\sigma_1(\vec{z})$ is zero in this case and only $\sigma_2(\vec{z})$, arising from the time evolution of the initial covariance matrix, occurs. Then

$$\sigma(\vec{z})=e^{-At}\langle\vec{z}(0),\vec{z}^T(0)\rangle e^{-A^Tt}$$

$$=\begin{bmatrix} a_1 & c_1 & a_2 & c_2 \\ c_1 & b_1 & -c_2 & b_2 \\ a_2 & -c_2 & a_1 & -c_1 \\ c_2 & b_2 & -c_1 & b_1 \end{bmatrix} \exp\left[-\frac{(\gamma_1+\gamma_2)t}{2}\right] , \tag{A6}$$

where

$$a_1=-\eta_1c_-^2+\eta_2s_k^2 ,$$

$$b_1=\eta_1s_k^2-\eta_2c_+^2 ,$$

$$c_1=is_k(\eta_1c_-+\eta_2c_+) ,$$

$$a_2=\xi_1c_-^2+\xi_2s_k^2 ,$$

$$b_2=\xi_1s_k^2+\xi_2c_+^2 ,$$

$$c_2=is_k(\xi_1c_--\xi_2c_+) ,$$

and

$$\eta_1=\tfrac{1}{2}\sinh 2r_i ,$$

$$\xi_i=\sinh^2 r_i ,$$

$$c_\pm=\cos\left[\frac{\Lambda t}{4}\right]\pm\frac{(\gamma_1-\gamma_2)}{\Lambda}\sin\left[\frac{\Lambda t}{4}\right] ,$$

$$s_k=\frac{4\kappa}{\Lambda}\sin\left[\frac{\Lambda t}{4}\right] .$$

Next we quote the expressions for the covariance matrix and means for the state of the detector after readout of the meter variable $y_2(t)$.

Parametric amplifier interaction:

$$\sigma(\vec{z}_1)_R=(C^{-1}+D^{-1})^{-1} , \quad (A7)$$

where

$$C^{-1}=\frac{1}{a^2-b^2}\begin{bmatrix}0 & a\\ a & 0\end{bmatrix} ,$$

$$D^{-1}=\frac{b^2}{a[2(a^2-b^2)+a]}\begin{bmatrix}1 & -\left[1+\frac{a}{a^2-b^2}\right]\\ -\left[1+\frac{a}{a^2-b^2}\right] & 1\end{bmatrix} ,$$

and a and b are given in Eqs. (A3) and (A4). The mean after readout is

$$\langle\vec{z}_1\rangle_R=\langle\vec{z}_1(t)\rangle+(C^{-1}+D^{-1})^{-1}\vec{\omega} , \quad (A8)$$

where

$$\vec{\omega}=\frac{a^2-b^2}{b}\left[\frac{2\omega_b}{\hbar}\right]^{1/2}[y_2(t)-\langle Y_2(t)\rangle]\begin{bmatrix}1\\1\end{bmatrix} .$$

Parametric frequency converter. We take the case $\gamma_1=\gamma_2=0$, then

$$\sigma(\vec{z}_1)_R=\frac{1}{v^2-1}\begin{bmatrix}v & 1\\ 1 & v\end{bmatrix} , \quad (A9)$$

where

$$v=d+\frac{f^2}{1-g} .$$

The mean is given by

$$\langle\vec{z}_1(t)\rangle_R=\langle\vec{z}_1(t)\rangle-\frac{if}{(g-1)(v-1)}\left[\frac{2\omega_b}{\hbar}\right]^{1/2}\times[y_2(t)-\langle Y_2(t)\rangle]\begin{bmatrix}1\\1\end{bmatrix} . \quad (A10)$$

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