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Wigner-Kirkwood expansion of N-body Green's function: The case with magnetic field

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A systematic method is given to compute the semiclassical Wigner-Kirkwood expansion of the off-diagonal thermal kernel of an N-body quantum system.

I. INTRODUCTION

We consider the N-body quantum system with Hamiltonian ($\mu=1,2,\ldots,dN$; d is the dimension of space $M=dN$)

$$\hat{H}=\sum_{\mu=1}^{M}\frac{1}{2m}[\hat{p}_\mu-A_\mu(\hat{\vec{q}})]^2+V(\hat{\vec{q}})\ ,$$

where $\hat{p}_\mu=-i\hbar\partial/\partial q^\mu$ in the coordinate representation and $\vec{q}=(q^1,\ldots,q^M)$. Our purpose is to give a systematic method for computing the semiclassical Wigner-Kirkwood expansion of the off-diagonal thermal kernel

$$P(\vec{X}\mid\vec{X}_0;\beta)\equiv\langle\vec{X}\mid\exp(-\beta\hat{H})\mid\vec{X}_0\rangle$$

$[\beta=(kT)^{-1}]$. Such an expansion has recently been obtained in Ref. 1 for the case in which the Hamiltonian $\hat{H}$ does not contain linear terms in $\hat{p}_\mu$ [i.e., $A_\mu(\hat{q})=0$]. Our method uses a functional integral representation of the heat kernel and is a generalization of a method that we have presented in Ref. 2.

In Sec. III we treat the exactly solvable model of N harmonic oscillators in a constant magnetic field.

II. WIGNER-KIRKWOOD EXPANSION

A. Functional integral representation of the heat kernel

The heat kernel $P(\vec{q}\mid\vec{X}_0;\beta)\equiv\langle\vec{q}\mid U(\beta)\mid\vec{X}_0\rangle$, $U(\beta)\equiv\exp(-\beta\hat{H})$, satisfies the differential equation ($\eta=\hbar^2$)

$$\frac{\partial}{\partial\beta}P(\vec{q}\mid\vec{X}_0;\beta)=\mathscr{L}\left[\frac{\partial}{\partial\vec{q}},\vec{q};\eta\right]P(\vec{q}\mid\vec{X}_0;\beta) \tag{1}$$

with ($\partial_\mu\equiv\partial/\partial q^\mu$ and a sum over repeated indices is understood)

$$\mathscr{L}\equiv\frac{\eta}{2m}\partial_\mu\partial_\mu-\frac{i\sqrt{\eta}}{2m}(\partial_\mu A_\mu(\vec{q})+A_\mu(\vec{q})\partial_\mu)-\tilde{V}(\vec{q})\ , \tag{2}$$

where $\tilde{V}\equiv(1/2m)A_\mu A_\mu+V$. Comparing with Eq. (1) of Ref. 2 we can see that the method we used there to obtain a WKB-type (Wentzel-Kramers-Brillouin) expansion needs a slight generalization here due to the dependence on $\sqrt{\eta}$ of the term linear in the derivatives in (2). If this term is absent the results of Ref. 2 can be used directly and lead to the expansion obtained in Ref. 1.

Following Ref. 2 we call $\hat{Q}^\mu$ and $\hat{P}_\mu$ the operators of multiplication by q^μ and $-i\eta\partial_\mu$, respectively; the commutator is $[\hat{Q}^\mu,\hat{P}_\nu]=i\eta\delta_{\mu\nu}$. Then $U(\beta)$ satisfies

$$i\eta\frac{\partial U(\beta)}{\partial\beta}=\hat{H}'(\hat{\vec{Q}},\hat{\vec{P}})U(\beta)\ ,\quad U(0)=1 \tag{3}$$

with

$$\hat{H}'=i\eta\mathscr{L}(\vec{q}\to\hat{\vec{Q}},-i\eta\partial_\mu\to\hat{P}_\mu)$$

given by

$$\hat{H}'=-\frac{i}{2m}\hat{P}_\mu^2+\frac{i\sqrt{\eta}}{2m}(\hat{P}_\mu A_\mu(\hat{\vec{Q}})+A_\mu(\hat{\vec{Q}})\hat{P}_\mu)-i\eta\tilde{V}(\hat{\vec{Q}})\ , \tag{4}$$

and consequently one can write for $P(\vec{X}\mid\vec{X}_0;\beta)$ the functional integral representation[2,3]

$$P(\vec{X}\mid\vec{X}_0;\beta)=\int_{\gamma_1(1/2)}\mathscr{D}Q\mathscr{D}_\eta P\exp\left[\frac{i}{\eta}\int_0^\beta dt[P_\mu(t)\dot{Q}^\mu(t)-H'(\vec{Q}(t),\vec{P}(t))]\right]\delta(\vec{Q}(\beta)-\vec{X})\delta(\vec{Q}(0)-\vec{X}_0)\ , \tag{5}$$

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where $H'(\vec{Q},\vec{P})$ is obtained from $\hat{H}'$ by the replacement $\hat{Q}\to\vec{Q}(t)$, $\hat{P}\to\vec{P}(t)$, i.e.,

$$H' = -\frac{i}{2m}P_\mu^2 + \frac{i\sqrt{\eta}}{m}P_\mu A_\mu(\vec{Q}) - i\eta\tilde{V}(\vec{Q}) .$$

In (5) the symbol $\gamma_1(\frac{1}{2})$ stands for midpoint discretization[3,4] and defines the functional integral in (5) as $\lim I_n$, $n\to\infty$, with

$$I_n = \int \prod_{i=1}^{n} d\vec{Q}_i \prod_{j=1}^{n+1} \frac{d\vec{P}_j}{(2\pi\eta)^M} \exp\left[\frac{i\epsilon}{\eta}\sum_{j=1}^{n+1}\left[P_{j\mu}\frac{\Delta Q_j^\mu}{\epsilon} + \frac{i}{2m}P_{j\mu}^2 - \frac{i\sqrt{\eta}}{m}P_{j\mu}A_\mu(\bar{\vec{Q}}_j) + i\eta\tilde{V}(\vec{Q}_j)\right]\right] , \tag{6}$$

where $(n+1)\epsilon=\beta$, $\vec{Q}_0=\vec{X}_0$, $\vec{Q}_{N+1}=\vec{X}$, and $\bar{\vec{Q}}_j=\frac{1}{2}(\vec{Q}_{j-1}+\vec{Q}_j)$. At this point we can see according to the discussion in Ref. 2 that the associated "classical" problem for the semiclassical WKB-type expansion (in $\hbar=\sqrt{\eta}$ here) is determined by the Hamiltonian $H_1=H'(\eta=0)\equiv-(i/2m)P_\mu^2$. But before doing the corresponding displacement in (5) and (6) it is more convenient here due to the simplicity of H' to perform the Gaussian $d\vec{P}_j$ integration and to work with the configuration-space path integral. Doing this in (6) one has

$$I_n = \left[\frac{m}{2\pi\eta\epsilon}\right]^{M/2} \int \prod_{i=1}^{n} \left[\frac{m}{2\pi\eta\epsilon}\right]^{M/2} d\vec{Q}_i \exp\left\{\epsilon\sum_{j=1}^{n+1}\left[-\frac{m}{2\eta}\left[\frac{\Delta Q_{j\mu}}{\epsilon}\right]^2 + \frac{i}{\sqrt{\eta}}\frac{\Delta Q_j^\mu}{\epsilon}A_\mu(\bar{\vec{Q}}_j) - V(\vec{Q}_j)\right]\right\} . \tag{7}$$

We now do in (7) the displacement $\vec{Q}(t)=\vec{x}(t)+\sqrt{\eta}\,\vec{q}(t)$ which in the discrete is $\vec{Q}_j=\vec{x}_j+\sqrt{\eta}\,\vec{q}_j$, $\vec{x}_j=\vec{x}(t_j)$, $t_j=j\epsilon$, $j=0,1,\ldots,(n+1)$, and $\vec{x}(t)$ is the solution of the classical problem determined by $H_1=H'(\eta=0)$ [here the free-particle problem for the boundary conditions $\vec{x}(0)=\vec{X}_0, \vec{x}(\beta)=\vec{X}$]. Consequently,

$$\vec{x}(t)=\vec{X}_0+t\frac{\Delta\vec{X}}{\beta} , \quad \Delta\vec{X}\equiv\vec{X}-\vec{X}_0 , \tag{8}$$

and we see then that $\vec{q}_0=\vec{q}_{n+1}=0$. One obtains $[\Delta\vec{x}_j\equiv\vec{x}(t_j)-\vec{x}(t_{j-1}), \Delta\vec{q}_j\equiv\vec{q}_j-\vec{q}_{j-1}]$

$$\begin{aligned} I_n = \left[\frac{m}{2\pi\eta\epsilon}\right]^{M/2} \int \prod_{i=1}^{n} \left[\frac{m}{2\pi\epsilon}\right]^{M/2} d\vec{q}_i \exp\Bigg\{\epsilon\sum_{j=1}^{n+1}\Bigg[& -\frac{m}{2\eta}\left[\frac{\Delta x_j^\mu}{\epsilon}+\sqrt{\eta}\frac{\Delta q_j^\mu}{\epsilon}\right]^2 \\ & + \frac{i}{\sqrt{\eta}}\left[\frac{\Delta x_j^\mu}{\epsilon}+\sqrt{\eta}\frac{\Delta q_j^\mu}{\epsilon}\right]A_\mu(\vec{x}_j+\sqrt{\eta}\,\bar{\vec{q}}_j) \\ & - V(\vec{x}_j+\sqrt{\eta}\,\vec{q}_j)\Bigg]\Bigg\} \end{aligned} \tag{9}$$

with $\bar{\vec{q}}_j=\frac{1}{2}(\vec{q}_{j-1}+\vec{q}_j)$. We note that the argument of A_μ in (9) should be $\bar{\vec{x}}_j+\sqrt{\eta}\bar{\vec{q}}_j$, $\bar{\vec{x}}_j=\frac{1}{2}(\vec{x}_{j-1}+\vec{x}_j)$, instead of $\vec{x}_j+\sqrt{\eta}\,\bar{\vec{q}}_j$ as we wrote; however, the difference is $O(\epsilon\sqrt{\epsilon})$ since $\Delta q_j=O(\sqrt{\epsilon})$ and consequently plays no role.[3,4] Finally, we have for $P(\vec{X}\mid\vec{X}_0;\beta)$ the path-integral representation

$$\begin{aligned} P(\vec{X}\mid\vec{X}_0;\beta) = \int_{\gamma_1(1/2)} \mathscr{D}\vec{q}\,\exp\Bigg[-\frac{1}{\eta}\int_0^\beta dt\,\Bigg[& \frac{m}{2}(\dot{x}^\mu+\sqrt{\eta}\,\dot{q}^\mu)^2 - i\sqrt{\eta}(\dot{x}^\mu+\sqrt{\eta}\,\dot{q}^\mu) \\ & \times A_\mu(\vec{x}(t)+\sqrt{\eta}\,\vec{q}(t)) + \eta V(\vec{x}+\sqrt{\eta}\,\vec{q})\Bigg]\Bigg]\delta(\vec{q}(0))\delta(\vec{q}(\beta)) . \end{aligned} \tag{10}$$

In (10) $\gamma_1(1/2)$ indicates midpoint discretization as specified in (9); this is important in the term $(\dot{x}^\mu+\sqrt{\eta}\,\dot{q}^\mu)A_\mu(\vec{x}+\sqrt{\eta}\,\vec{q}(t))$ since it tells us that we can do partial integrations using the normal rules of calculus.[3-5]

B. Semiclassical expansion

The next step is to develop the integrand in the argument of the exponential in (10) in powers of $\hbar=\sqrt{\eta}$. In the first term

$$\int dt\, \dot{x}^{\mu}\dot{q}^{\mu} = -\int dt\, q^{\mu}\ddot{x}^{\mu}=0$$

since $\ddot{x}^{\mu}=0$, the development of V is

$$-\sum_{n\geq 0}\frac{\hbar^n}{n!}c_{\mu_1,\ldots,\mu_n}(t)q^{\mu_1}(t)\cdots q^{\mu_n}(t)\ ,\quad c_{\mu_1,\ldots,\mu_n}(t)=-\partial_{\mu_1}\cdots\partial_{\mu_n}V(\vec{x}(t))\ , \tag{11}$$

and that of the A_μ term is (see Appendix A)

$$\begin{aligned}\int_0^{\beta} dt\, \dot{Q}^{\mu}A_{\mu}(\vec{Q})\,|_{\vec{Q}=\vec{x}+\hbar\vec{q}} = \int_0^{\beta} dt[\,\dot{x}^{\mu}A_{\mu}(\vec{x}(t))-\hbar\dot{x}^{\mu}F_{\mu\nu}(\vec{x}(t))q^{\nu}(t)]\\ -\sum_{n\geq 2}\frac{\hbar^n}{n!}\int_0^{\beta} dt[(n-1)\dot{q}^{\mu}q^{\rho_1}\cdots q^{\rho_{n-2}}(\partial_{\rho_1}\cdots\partial_{\rho_{n-2}}F_{\mu\nu})q^{\nu}\\ +\dot{x}^{\mu}q^{\rho_1}\cdots q^{\rho_{n-1}}(\partial_{\rho_1}\cdots\partial_{\rho_{n-1}}F_{\mu\nu})q^{\nu}]\ .\end{aligned} \tag{12}$$

In (12), $F_{\mu\nu}\equiv\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ is always evaluated at $\vec{x}(t)$ and $\dot{x}^{\mu}(t)=\Delta X^{\mu}/\beta$. Using these developments we obtain from (10)

$$P(\vec{X}\mid\vec{X}_0;\beta)=\exp\left[\int_0^{\beta} dt\left[-\frac{m}{2\eta}(\dot{x}^{\mu})^2+\frac{i}{\hbar}\dot{x}^{\mu}A_{\mu}(\vec{x}(t))-V(\vec{x}(t))\right]\right]I\ , \tag{13}$$

$$I=\int_{\gamma_1(1/2)}\mathscr{D}q\,\exp\left[\int_0^{\beta} dt\left[-\frac{m}{2}(\dot{q}^{\mu})^2-iq^{\nu}F_{\mu\nu}\dot{x}^{\mu}+\bar{K}(\vec{q},t,\hbar)\right]\right]\delta(\vec{q}(0))\delta(\vec{q}(\beta))\ , \tag{14}$$

where $\bar{K}=\sum_{n\geq 1}\hbar^n\bar{K}_n$ with

$$\begin{aligned}\bar{K}_n=\frac{1}{n!}c_{\mu_1,\ldots,\mu_n}(t)q^{\mu_1}\cdots q^{\mu_n}\\ -\frac{i}{(n+1)!}(n\dot{q}^{\mu}q^{\rho_1}\cdots q^{\rho_{n-1}}\partial_{\rho_1}\cdots\partial_{\rho_{n-1}}F_{\mu\nu}q^{\nu}+\dot{x}^{\mu}q^{\rho_1}\cdots q^{\rho_n}\partial_{\rho_1}\cdots\partial_{\rho_n}F_{\mu\nu}q^{\nu})\ .\end{aligned} \tag{15}$$

For practical purposes it is convenient to eliminate in (14) the term $b_{\nu}(t)q^{\nu}$,

$$b_{\nu}(t)=-iF_{\mu\nu}(x(t))\frac{\Delta X^{\mu}}{\beta}\ ,$$

doing a translation $\vec{q}(t)\to\vec{y}(t)+\vec{q}(t)$, and determining $\vec{y}(t)$ by

$$\ddot{y}^{\mu}(t)=-\frac{1}{m}b_{\mu}(t)\ ,\quad y^{\mu}(0)=y^{\mu}(\beta)=0\ . \tag{16}$$

The solution of (16) is

$$y^{\mu}(t)=\int_0^{\beta} dt'\Delta(t,t')b_{\mu}(t') \tag{17}$$

with

$$\begin{aligned}\Delta(t,t')=-\frac{1}{m\beta}[\Theta(t-t')t'(t-\beta)\\ +\Theta(t'-t)t(t'-\beta)]\ .\end{aligned} \tag{18}$$

Owing to the boundary conditions of (16) one still has $q(0)=q(\beta)=0$ and one obtains from (14)

$$I=\exp\left[\int_0^{\beta} dt\left[-\frac{m}{2}(\dot{y}^{\mu})^2+b_{\mu}y^{\mu}\right]\right]\int_{\gamma_1(1/2)}\mathscr{D}q\,\exp\left[\int_0^{\beta} dt\left[-\frac{m}{2}(\dot{q}^{\mu})^2+\bar{K}(\vec{y}+\vec{q},t,\hbar)\right]\right]\delta(\vec{q}(0))\delta(\vec{q}(\beta))\ . \tag{19}$$

Using (16) one has

$$\int_0^{\beta} dt\left[-\frac{m}{2}(\dot{y}^{\mu})^2+b_{\mu}y^{\mu}\right]=\tfrac{1}{2}\int_0^{\beta} dt\, b_{\mu}(t)y^{\mu}(t)\ . \tag{20}$$

We call $\tilde{I}^c$ the functional integral in (19). In order to set up the $\hbar$ expansion we compute it as usual in the form

$$\tilde{I}^c = \exp\left[\int_0^\beta dt\, \bar{K}(\vec{y}+\vec{q},t,\hbar)\right]\Big|_{\vec{q}=(1/i)(\delta/\delta\vec{J})} Z_0[\vec{J}\,]\,|_{\vec{J}=0}\ , \tag{21}$$

where

$$Z_0[\vec{J}\,] = \int \mathscr{D}\,\vec{q}\,\exp\left[\int_0^\beta dt\left[-\frac{m}{2}(\dot{q}^{\mu})^2 + iJ_\mu(t)q^\mu\right]\right]\delta(\vec{q}(0))\delta(\vec{q}(\beta))\ . \tag{22}$$

The value of this Gaussian path integral is well known; in fact, since

$$\mathscr{D}\,\vec{q} = \eta^{-M/2}\left[\frac{m}{2\pi\epsilon}\right]^{M/2}\prod_{i=1}^{n}\left[\frac{m}{2\pi\epsilon}\right]^{M/2} d\,\vec{q}_i$$

it is the same one (apart from the factor $\eta^{-1/2}$) as in Ref. 2, formula (27), for $H_1 = -(i/2m)p_\mu^2$, as it can be seen reintroducing the $\mathscr{D}\,\vec{p}$ integration. Its value is

$$Z_0[\vec{J}\,] = \left[\frac{m}{2\pi\eta\beta}\right]^{M/2}\exp\left[-\tfrac{1}{2}\int_0^\beta dt\,dt'J_\mu(t)\Delta(t,t')J_\mu(t')\right]\ , \tag{23}$$

where $\Delta(t,t')=\Delta(t',t)$ was defined in (18). We define now $K(q)=\bar{K}(y+q)-\bar{K}(y)$, which is of the form $K=\sum_{n\geq 1}\hbar^n K_n$ with $K_n(q)=\bar{K}_n(y+q)-\bar{K}_n(y)$, $\bar{K}_n$ given by (15). Then one obtains from (21)

$$\tilde{I}^c = \exp[\,\bar{W}\,]Z_0[0]I^c\ ,$$

$$\bar{W} = \sum_{n\geq 1}\hbar^n\,\bar{W}_n = \int_0^\beta dt\sum_{n\geq 1}\hbar^n\bar{K}_n(\vec{y}\,)\ , \tag{24}$$

$$I^c = \exp\left[\int_0^\beta dt\,K(\vec{q}(t),\dot{\vec{q}}(t))\right]\Big|_{\vec{q}=(1/i)(\delta/\delta\vec{J}\,)}$$

$$\times\tilde{Z}_0[\vec{J}\,]\,|_{\vec{J}=0}\ ,\quad \tilde{Z}_0[J]=Z_0[J]/Z_0[0]\ . \tag{25}$$

At this point we should remark that the original midpoint discretization in (10) implies that the integral in the exponential in (21) is to be interpreted as[3,4]

$$\int_0^\beta dt\,K(q(t),\dot{q}(t)) = \lim_{\epsilon\to 0+}\int_0^\beta dt\,K[\tfrac{1}{2}(q(t+\epsilon)+q(t-\epsilon))\,,\dot{q}(t)]\ , \tag{26}$$

and that this is necessary for the action of

$$\exp\left[\int dt\,K\right]$$

on $\tilde{Z}_0[J]$ to be well defined as we shall see. Using the identity

$$F\left[\frac{1}{a}\frac{\partial}{\partial\vec{J}}\right]Z(\vec{J}\,)\,|_{\vec{J}=0} = Z\left[\frac{1}{a}\frac{\partial}{\partial\vec{q}}\right]F(\vec{q}\,)\,|_{\vec{q}=0}$$

we write I^c as

$$I^c = \tilde{Z}_0\left[\frac{1}{i}\frac{\partial}{\partial\vec{q}}\right]\exp\left[\int_0^\beta dt\,K(\vec{q},\dot{\vec{q}}\,)\right]\Bigg|_{\vec{q}=0}\ . \tag{27}$$

Since K is a power series in $\hbar$, (25) shows that

$$I^c = 1 + \sum_{n\geq 1}\hbar^n I_n^c\ .$$

Each term in the development of

$$\exp\left[\int dt\,K\right]$$

will be of the form

$$\int\prod_i dt_i\prod_j dt'_j\,q^{\mu_1}(t_1)\cdots q^{\mu_l}(t_l)\dot{q}^{\rho_1}(t'_1)\cdots\dot{q}^{\rho_k}(t'_k)f_{\mu_1,\ldots,\mu_l,\rho_1,\ldots\rho_k}(t_i,t'_j) \tag{28}$$

and the action of $\tilde{Z}_0$ on (28) tells us that we must do there all two point contractions

$$\{q^\mu(t)q^\nu\}(t') = \delta_{\mu\nu}\Delta(t,t') = \underset{t\qquad t'}{\text{———}}\ , \tag{29a}$$

$$\{\dot{q}^\mu(t)q^\nu\}(t') = \delta_{\mu\nu}S(t,t') = \underset{t\qquad t'}{\sim\!\text{——}}\ , \tag{29b}$$

$$\{\dot{q}^\mu(t)\dot{q}^\nu\}(t') = \delta_{\mu\nu}D(t,t') = \underset{t\qquad t'}{\sim\sim\sim} \tag{29c}$$

with

$$S(t,t') = \frac{\partial}{\partial t}\Delta(t,t')$$

$$= -\frac{1}{m\beta}[\Theta(t-t')t' + \Theta(t'-t)(t'-\beta)]\ , \tag{30a}$$

$$D(t,t')=\frac{\partial^2}{\partial t\,\partial t'}\Delta(t,t')$$
$$=\frac{1}{m}\left[\delta(t-t')-\frac{1}{\beta}\right]. \qquad (30b)$$

We have drawn in (29) a graphical representation of each contraction in order to represent in the usual way by Feynman graphs each term in the expansion of (27). We note that due to the form of K (linear in $\dot q$) the contraction (denoted by curlie braces) $\{\dot q^\mu(t)\dot q^\nu\}(t')$ will always occur at $t\neq t'$ and consequently there is no problem with the $\delta(t-t')$ in (30b). However, one will have in the expansion $\{q^\mu(t)q^\nu\}(t')$ and $\{\dot q^\mu(t)q^\nu\}(t')$ at $t=t'$, the former is defined since $\Delta(t,t')$ has no discontinuity at $t=t'$ and takes the value $-(1/m\beta)t(t-\beta)$; but this is not the case for the latter since $S(t+\epsilon,t)\neq S(t,t+\epsilon)$, $\epsilon\to+0$. But (26) tells us that

$$\{\dot q^\mu(t)q^\nu\}(t)=\tfrac{1}{2}[\{\dot q^\mu(t)q^\nu\}(t+\epsilon)+\{\dot q^\mu(t)q^\nu\}(t-\epsilon)]$$

which is

$$\{\dot q^\mu(t)q^\nu\}(t)=\delta_{\mu\nu}\frac{1}{2m\beta}(\beta-2t)\,. \qquad (31)$$

Finally, one then has

$$P(\vec X\mid\vec X_0;\beta)=P_{\rm WKB}(\vec X\mid\vec X_0;\beta)\exp(\overline{W})I^c\,,$$
$$P_{\rm WKB}(\vec X\mid\vec X_0;\beta)=\left[\frac{m}{2\pi\eta\beta}\right]^{M/2}\exp\left[-\frac{m}{2\eta\beta}\Delta X^\mu\Delta X^\mu+\int_0^\beta dt\left[\frac{i}{\hbar\beta}\Delta X^\mu A_\mu(\vec x(t))-V(\vec x(t))\right]\right.$$
$$\left.-\frac{\Delta X^\rho\Delta X^\sigma}{2\beta^2}\int_0^\beta dt\,dt'F_{\mu\rho}(\vec x(t))F_{\mu\sigma}(\vec x(t'))\Delta(t,t')\right]. \qquad (32)$$

All the singular dependence on $\hbar$ is in (32) since I^c is a power series in $\hbar$ containing all the corrections to the semiclassical WKB approximation. Formula (27) shows that I^c is of the form $I^c=\exp(W)$, $W=\sum_{n>1}\hbar^n W_n$, with W given by (27) but keeping only the connected graphs. The first correction term W_1 is

$$W_1=\tilde Z_0\left[\frac{1}{i}\frac{\delta}{\delta\vec q}\right]\int_0^\beta dt\,K_1(\vec q,\dot{\vec q})\,|_{\vec q=0}$$
$$=-\frac{i\Delta X^\rho}{2m\beta^2}\int_0^\beta dt\,t(t-\beta)\partial_\mu F_{\mu\rho}(\vec x(t))\,. \qquad (33)$$

For W_2 one obtains, when $\Delta\vec X=0$, the result

$$W_2(\Delta\vec X=0)$$
$$=\frac{\beta^2}{24m}[\beta\partial_\mu V(X_0)\partial_\mu V(X_0)-2\partial_\mu\partial_\mu V(X_0)]$$
$$-\frac{1}{48}\left[\frac{\beta}{m}\right]^2F_{\mu\nu}(X_0)F_{\mu\nu}(X_0)\,. \qquad (34)$$

We remark that, apart from the phase factor $\Delta X^\mu A_\mu$ in (32) giving the well-known transformation property of the kernel under a gauge transformation, the rest of (32) and the corrections are gauge invariant as they should be since they are all expressed in terms of $F_{\mu\nu}$.

C. High-temperature expansion

We notice that W_n is always a polynomial in β. In order to see this, it is enough to scale $t\to\beta t$ in the development of (27), then one can see that $y^\mu\sim O(\beta)$, $\dot y^\mu\sim O(\beta^0)$, $\Delta\sim O(\beta)$, $S\sim O(\beta^0)$, $D\sim O(\beta^{-1})$, and consequently one can count $q\sim O(\beta^{1/2})$, $\dot q\sim O(\beta^{-1/2})$; moreover, $\vec x(t)\to\vec X_0+t\,\Delta\vec X$, and each integration over dt gives a factor β. As a consequence of all this one can easily check (using $\dot x^\mu=\Delta X^\mu/\beta$) that each W_n is a finite polynomial in β with power β^l, $l\geq n$ (the same for $\overline{W}_n$). Thus we have obtained at the same time the high-temperature expansion.

One should notice that in (32) the last term in the argument of the exponential is $O(\beta)$ and should then be included in the corrections, while the rest of (32) gives the singular dependence in β. In Appendix B we explain briefly how to obtain the high-temperature expansion as a direct application of the methods we have developed in Ref. 2.

D. Classical partition function

The classical partition function can be obtained putting $\Delta\vec X=0$ in $P_{\rm WKB}$ and integrating over $d\vec X_0$ dividing by $N!$ to take into account the indistinguishability of particles,[6] i.e.,

$$Z_{\rm cl}=(N!)^{-1}\int d\vec X_0 P_{\rm WKB}(\Delta\vec X=0)$$

with

$$P_{\rm WKB}(\Delta\vec X=0)=\left[\frac{m}{2\pi\eta\beta}\right]^{M/2}\exp[-\beta V(\vec X_0)]\,.$$

One easily checks then doing the $d\vec p$ integration that

$$Z_{\rm cl}=\frac{1}{N!(2\pi\hbar)^M}\int d\vec{q}\,d\vec{p}\exp[-\beta H_{\rm cl}(\vec{q},\vec{p})]\ , \tag{35}$$

where

$$H_{\rm cl}=\frac{1}{2m}(p_\mu-A_\mu)^2+V(\vec{q})\ .$$

III. HARMONIC OSCILLATOR IN CONSTANT MAGNETIC FIELD

Consider the system of N three-dimensional harmonic oscillators in a constant magnetic field. It can be solved by making some simple considerations. Because there is no interaction between the particles the N-body Green's function is the product of N one-body Green's functions and it is enough to consider one of them. The one-particle Hamiltonian is

$$\hat{H}=\sum_{\mu=1}^{3}\frac{1}{2m}[\hat{p}_\mu+A_\mu(\hat{\vec{p}})]^2+V(\hat{\vec{q}}) \tag{36}$$

with, in the case of a magnetic field in the third direction,

$$A_\mu(\hat{q})=-\sigma q_2\delta_{\mu 1}\ , \tag{37a}$$

$$V(\hat{q})=\tfrac{1}{2}\lambda(\hat{q}_1^2+\hat{q}_2^2+\hat{q}_3^2)\ . \tag{37b}$$

Applying the general method of Sec. II one obtains a functional integral representation of the form (5) with $H'=H_a'+H_b'$

$$H_a'=-i\left[\frac{1}{2m}(P_1^2+P_2^2)+\frac{\sqrt{\eta}\sigma}{m}P_1Q_2+\eta\left[\frac{1}{2m}\sigma^2Q_2^2+\tfrac{1}{2}\lambda(Q_1^2+Q_2^2)\right]\right]\ , \tag{38a}$$

$$H_b'=-i\left[\frac{1}{2m}P_3^2+\eta\tfrac{1}{2}\lambda Q_3^2\right]\ . \tag{38b}$$

Because the third degree of freedom is decoupled from the first two the one-particle propagator is itself a product of two factors P_a and P_b, belonging to Hamiltonian operators $\hat{H}'_a$ and $\hat{H}'_b$, respectively. $\hat{H}'_a$ is quadratic and has no ordering problems. Therefore we can immediately write the associated propagator in the form[2,7]

$$P_a=\left[\frac{1}{2\pi i\eta}\right]\left[\det-\frac{\partial^2\mathscr{A}_{\rm cl}}{\partial Q_0^\alpha\partial Q^\beta}\right]^{1/2}\exp\left[\frac{i}{\eta}\mathscr{A}_{\rm cl}\right] \tag{39}$$

with $\mathscr{A}_{\rm cl}$ the classical action for the two-dimensional system with Hamiltonian H_a' and Lagrangian [here $\vec{Q}=(Q_1,Q_2)$]

$$\mathscr{L}'_a(\dot{\vec{Q}},\vec{Q})=i(\tfrac{1}{2}m\dot{\vec{Q}}^2+i\sqrt{\eta}\sigma Q_2\dot{Q}_1+\tfrac{1}{2}\eta\lambda\vec{Q}^2)\ . \tag{40}$$

The extremal path for the boundary conditions $\vec{Q}(\beta)=\vec{Q},\vec{Q}(0)=\vec{Q}_0$ is given by

$$\vec{Q}(t)=U(t)\frac{\sinh[\omega(\beta-t)]}{\sinh(\omega\beta)}\vec{Q}_0+U(t-\beta)\frac{\sinh(\omega t)}{\sinh(\omega\beta)}\vec{Q} \tag{41}$$

with

$$U(t)=\exp\left[\frac{1}{2}\frac{\sqrt{\eta}\sigma}{m}\begin{bmatrix}0&-i\\ i&0\end{bmatrix}t\right]\ , \tag{42}$$

$$\omega^2=\frac{1}{4}\left[\frac{\sqrt{\eta}\sigma}{m}\right]^2+\frac{\eta\lambda}{m}\ . \tag{43}$$

Computing the action and the Van Vleck determinant, one then obtains

$$P_a=\left[\frac{m\omega}{2\pi\eta\sinh(\omega\beta)}\right]\exp\left[-\frac{m\omega}{2\eta\sinh(\omega\beta)}[\cosh(\omega\beta)(\vec{Q}^2+\vec{Q}_0^2)-2(\vec{Q},U(\beta)\vec{Q}_0)]-\frac{i\sigma}{2\sqrt{\eta}}(Q_2Q_1-Q_{02}Q_{01})\right]\ . \tag{44}$$

The propagator associated to H_b' is simply

$$P_b=\left[\frac{m\omega_0}{2\pi\eta\sinh(\omega_0\beta)}\right]^{1/2}\exp\left[-\frac{m\omega_0}{2\eta\sinh(\omega_0\beta)}[\cosh(\omega_0\beta)(Q_3^2+Q_{03}^2)-2Q_3Q_{03}]\right] \tag{45}$$

with $\omega_0^2=\eta\lambda/m$.

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APPENDIX A

We put $\alpha^\mu(u,t)=x^\mu(t)+uq^\mu(t)$ and

$$M(u)=\int_0^\beta dt\,\dot\alpha^\mu(u,t)A_\mu(\alpha(u,t))\equiv\left\langle\frac{\partial\alpha}{\partial t},A(\alpha)\right\rangle . \tag{A1}$$

Then the quantity we want to develop in powers of $\hbar$ is

$$M(\hbar)=\sum_{n\geq 0}\frac{\hbar^n}{n!}\frac{\partial^n M}{\partial u^n}\bigg|_{u=0} .$$

One has

$$\frac{\partial M}{\partial u}=\left\langle\frac{\partial}{\partial u}\frac{\partial\alpha}{\partial t},A\right\rangle+\left\langle\frac{\partial\alpha}{\partial t},\frac{\partial\alpha^\sigma}{\partial u}\partial_\sigma A\right\rangle \tag{A2}$$

but the first term gives

$$\left\langle\frac{\partial}{\partial t}\frac{\partial\alpha}{\partial u},A\right\rangle=-\left\langle\frac{\partial\alpha}{\partial u},\frac{\partial\alpha^\sigma}{\partial t}\partial_\sigma A\right\rangle$$

by partial integration and then

$$\frac{\partial M}{\partial u}=-\left\langle\frac{\partial\alpha}{\partial t},F\left[\frac{\partial\alpha}{\partial u}\right]\right\rangle , \tag{A3}$$

where

$$F\left[\frac{\partial\alpha}{\partial u}\right]\equiv F_{\mu\nu}(\vec\alpha(u,t))\frac{\partial\alpha^\nu}{\partial u} .$$

Owing to the form of (A3) we see then that all the successive derivatives will be expressed in terms of $F_{\mu\nu}$. Proceeding in the same way one obtains

$$\frac{\partial^n M}{\partial u^n}=-(n-1)\left\langle\frac{\partial}{\partial t}\frac{\partial\alpha}{\partial u},F^{(n-2)}\left[\frac{\partial\alpha}{\partial u}\right]\right\rangle -\left\langle\frac{\partial\alpha}{\partial t},F^{(n-1)}\left[\frac{\partial\alpha}{\partial u}\right]\right\rangle , \tag{A4}$$

where

$$F^{(n)}\left[\frac{\partial\alpha}{\partial u}\right]\equiv\frac{\partial\alpha^{\sigma_1}}{\partial u}\cdots\frac{\partial\alpha^{\sigma_n}}{\partial u}(\partial_{\sigma_1}\cdots\partial_{\sigma_n}F_{\mu\nu})\frac{\partial\alpha^\nu}{\partial u}$$

since $\vec\alpha(0,t)=\vec{x}(t)$, $\partial\alpha^\mu/\partial u=q^\mu(t)$, one then obtains (12).

APPENDIX B

The β expansion can be obtained as follows. Consider

$$\bar P(\vec q\mid\vec X_0;\beta,t)\equiv\langle\vec q\mid\exp(-t\beta\hat H)\mid\vec X_0\rangle , \tag{B1}$$

then $\bar P(t=1)=P(\vec q\mid\vec X_0;\beta)$. One has

$$\frac{\partial}{\partial t}\bar P(\vec q\mid\vec X_0;\beta,t)=\mathscr{L}\left[\frac{\partial}{\partial\vec q},\vec q;\beta\right]\bar P \tag{B2}$$

with

$$\mathscr{L}=\frac{\beta\hbar^2}{2m}\partial_\mu\partial_\nu-\frac{i\hbar}{2m}\beta(\partial_\mu A_\mu+A_\mu\partial_\mu)-\beta V(q) . \tag{B3}$$

Equation (B2) is exactly of the form of Eq. (1) of Ref. 2 with β playing the role of η; and we can then use without changes the results of Ref. 2.

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