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Quantization of the damped harmonic oscillator

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By means of a canonical transformation that can be implemented as a unitary change of representation, we show how to quantize the damped harmonic oscillator.

I. INTRODUCTION

There has been a revival of the problem of quantizing nonconservative systems. See Refs. 1 and 2 and especially Ref. 3 which is an in depth review of nonpotential systems. In this work we discuss the quantization of the mechanical system described by

$$\ddot{x}+\lambda\dot{x}+\omega^2 x=0\ . \tag{1.1}$$

In Refs. 1 and 2 and in some of the references quoted there a variety of methods of dealing with this problem are given.

Here we treat (1.1) as a Hamiltonian system and show that by means of a canonical transformation, it can be reduced to a standard harmonic oscillator with the appropriate frequency.

We then proceed to the quantum case by implementing the canonical transformations between the obvious observables, solving for the function determining the change of representation, and obtaining as a result that the quantization of the system with Hamiltonian

$$H=\frac{e^{-\lambda t}}{2}p^2+\frac{\omega^2x^2}{2}e^{\lambda t} \tag{1.2}$$

corresponding to (1.1) can be done the standard way, i.e., $p\rightarrow p_{\mathrm{op}}=-i(\partial/\partial x)$, $x\rightarrow x_{\mathrm{op}}$ (equals multiplication by x).

II. SOLUTION BY MEANS OF CANONICAL TRANSFORMATIONS

Consider the Hamilton equations

$$\dot{x}=e^{-\lambda t}p,\ \ \dot{p}=-\omega^2 e^{\lambda t}x\ , \tag{2.1}$$

associated with (1.2). In order to solve them, we perform the canonical transformation[4] given by the generating function

$$F_2(x,P)=xPe^{\lambda t/2}-\frac{\lambda}{4}x^2e^{\lambda t} \tag{2.2}$$

which yields the change of variables

$$\begin{aligned} Q&=e^{\lambda t/2}x,\\ P&=e^{-\lambda t/2}p+\frac{\lambda}{2}e^{\lambda t/2}x, \end{aligned} \tag{2.3}$$

and a new Hamiltonian given by

$$H^1=H+\frac{\partial F_2}{\partial t}=\tfrac{1}{2}p^2+\frac{\omega^2(\lambda)}{2}Q^2\ , \tag{2.4}$$

where $\omega^2(\lambda)=\omega^2-\lambda^2/4$. We should remark that there is no generating function of the type $F_1(x,Q)$ yielding (2.3) and (2.4). Solving (2.1) is equivalent to solving the Hamilton pair associated with (2.4), and with the use of (2.3) obtaining the solution to (2.1).

III. THE QUANTUM CASE: $\hbar=1$

We shall follow Refs. 4 and 5 and say that corresponding to (2.4) is the transformation

$$\begin{aligned} \hat{Q}&=e^{\lambda t/2}\hat{x},\\ \hat{P}&=e^{-\lambda t/2}\hat{p}+\frac{\lambda}{2}e^{\lambda t/2}\hat{x}\ , \end{aligned} \tag{3.1}$$

where the caret denotes operators.

We shall find a transformation function of the type $\langle x\,|P)$, where $\hat{P}\,|P)=P\,|P)$ and $\hat{x}\,|x\rangle=x\,|x\rangle$. It is easy to see that the solution to

$$\begin{aligned} P\langle x\,|P)&=\langle x\,|\hat{P}\,|P)\\ &=e^{-\lambda t/2}\left[-i\frac{\partial}{\partial x}\langle x\,|P)\right]+\frac{\lambda}{2}e^{\lambda t/2}x\langle x\,|P) \end{aligned}$$

is given by

$$\langle x\,|P)=\frac{e^{\lambda t/4}}{(2\pi)^{3/2}}\exp\left[i\left[xPe^{\lambda t/2}-\frac{\lambda}{4}x^2e^{\lambda t})\right]\right]\ , \tag{3.2}$$

where the role of the normalization factor $e^{\lambda t/4}/(2\pi)^{3/2}$ is twofold. First, it ensures that

$$\int (P'\,|\,x\rangle dx\langle x\,|\,P)=\delta(P'-P) \tag{3.3}$$

holds, and second, that (3.6) is satisfied, which we shall now make more explicit.

Let $\psi(P,t)$ be such that

$$i\frac{\partial\psi}{\partial t}(P,t)=\tfrac{1}{2}[P^2-\omega^2(\lambda)]\psi(P,t)\ , \tag{3.4}$$

i.e., $\psi(P,t)$ denotes a possible curve of states for

$$\hat{H}=\tfrac{1}{2}[\hat{P}^2+\omega^2(\lambda)\hat{Q}^2]$$

expressed in the momentum representation. Let

$$\psi(x,t)=\int \langle x\,|\,P)\psi(P,t)dP \tag{3.5}$$

then $\psi(x,t)$ satisfies

$$\frac{i\partial\psi(x,t)}{\partial t}=-\frac{e^{-\lambda t}}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}+\frac{\omega^2}{2}e^{\lambda t}x^2\psi(x,t) \tag{3.6}$$

which justifies the application of the standard quantization rule to the Hamiltonian (1.2).

To verify (3.6) apply $i\partial/\partial t$ to both sides of

$$e^{-\lambda t/4}\exp(i\lambda x^2e^{\lambda t}/4)\psi(x,t) =\frac{1}{(2\pi)^{3/2}}\int \exp(ixPe^{\lambda t/2})\psi(P,t)dP$$

(with $\mathscr{L}$ the left-hand side and $\mathscr{R}$ the right-hand side) and be careful when integrating by parts. One obtains

$$\frac{i\partial}{\partial t}\mathscr{L}=e^{-\lambda t/4}\exp(i\lambda x^2e^{\lambda t}/4)\left[-\frac{i\lambda}{4}\psi(x,t)-\frac{\lambda^2x^2e^{\lambda t}}{4}\psi(x,t)+\frac{i\partial\psi}{\partial t}(x,t)\right],$$

$$\frac{i\partial}{\partial t}\mathscr{R}=e^{-\lambda t/4}\exp(i\lambda x^2e^{\lambda t}/4)\left[-\frac{e^{-\lambda t}}{2}\frac{\partial^2\psi}{\partial x^2}+\frac{\omega^2e^{\lambda t}}{2}x^2\psi(x,t)-\frac{i\lambda\psi}{4}(x,t)-\frac{\lambda^2x^2}{4}e^{\lambda t}\psi(x,t)\right],$$

and by equating the last two, (3.6) is obtained.

It follows from (3.2) that

$$\langle x\,|\,Q)=\int \langle x\,|\,P)(P\,|\,Q)dQ =e^{\lambda t/4}\exp(-i\lambda x^2e^{\lambda t}/4)\delta(Q-xe^{\lambda t/2}). \tag{3.7}$$

This last identity and (3.2) can be used to verify that

$$\int\ |\,x\rangle dx\langle x\,|=\int\ |\,Q)dQ(Q\,|=\int\ |\,P)dP(P\,| =\int\ |\,P\rangle dp\langle p\,|$$

and, therefore, if any of them is a spectral resolution of the identity, so are the others. Also, a simple integration shows that

$$\int \langle x^1\,|\,P)dP(P\,|\,x\rangle=e^{\lambda t/2}\delta(xe^{\lambda t/2}-x^1e^{\lambda t/2}) =\delta(x-x^1)=\langle x\,|\,x^1\rangle$$

which together with (3.3) amounts to the unitarity of the transformation determined by (2.2). This implies that $|\,\psi(x,t)\,|^2$ is a genuine conserved probability density.

In the Sec. IV, we shall see that the "dissipative" behavior of the quantum case is rather analogous to that of the classical case.

To finish this section note that

$$[\hat{P},\hat{Q}]=[\hat{p},\hat{x}]$$

which implies that the commutation rules remain the same in both coordinates. To analyze what happens with the uncertainty principle we proceed as follows. Note that for H as in (3.6) or (1.2), the quantum analog of (2.1) is

$$\frac{d\hat{x}}{dt}=-i[\hat{x},H]=e^{-\lambda t}\hat{p}.$$

If we let $\hat{x}_{\text{phys}}=\hat{x}$, $\hat{p}_{\text{phys}}=(m=1)(d\hat{x}/dt)=e^{-\lambda t}\hat{p}$, it follows that

$$[\hat{x}_{\text{phys}},\hat{p}_{\text{phys}}]=[\hat{x},\hat{p}]e^{-\lambda t}=ie^{-\lambda t}$$

from which the standard computations would yield

$$\Delta X_{\text{phys}}\Delta P_{\text{phys}}=e^{-\lambda t/2}$$

which implies that the uncertainty decreases exponentially with time. Those interested in why one should consider $\hat{p}_{\text{phys}}=md\hat{x}/dt$ and not $\hat{p}$ should consult Ref. 3.

IV. COMPARISON OF CLASSICAL AND QUANTUM CASES

We shall first examine the dependence on λ of the motion of the system described by

$$H^1=\frac{P^2}{2}+\frac{\omega^2(\lambda)}{2}Q^2$$

in the quantum and classical cases.

(a) $\omega^2(\lambda)=\omega^2-\lambda^2/4>0$. In this case the classical orbits are closed in phase space and the standard discrete spectrum and normalized bound states appear in the quantum case.

(b) $\omega^2(\lambda)=0$. Now both cases describe a free particle. The closed orbits and discrete spectrum change into open orbits and continuous spectrum plus unnormalized eigenstates, respectively.

(c) $\omega^2(\lambda)<0$. In this case the potential is that of a repulsive force, and an unbounded motion appears in both the classical and quantum cases.

To compare further the classical and quantum cases, note that if x_0,p_0 denote the initial data, the solution to (1.1) is

$$x(t)=e^{-\lambda t/2}\left\{x_0\cos[\omega(\lambda)t] + \frac{P_0+\lambda x_0/2}{\omega(\lambda)}\sin[\omega(\lambda)t]\right\}, \tag{4.1}$$

where the quantity in brackets is $Q(t)$ for $Q_0=x_0$, $P_0=p_0+\lambda/2x_0$.

When $t\to\infty$, the particle ends up at $x=0$. An analog of this can be obtained as follows. From $\hat{Q}=e^{\lambda t/2}\hat{x}$ it follows that

$$(Q^1\,|\,f(\hat{x})\,|\,Q)=f(Qe^{-\lambda t/2})\delta(Q^1-Q)$$

from which it follows that

$$\int f(x)\,|\,\psi(x,t)\,|^2dx=\int f(e^{-\lambda t/2}Q)\,|\,\psi(Q,t)\,|^2dQ \tag{4.2}$$

which in the particular case $f(x)=x$ reads as

$$\int x\,|\,\psi(x,t)\,|^2dx=e^{-\lambda t/2}\int Q\,|\,\psi(Q,t)\,|^2dQ.$$

When $\omega(\lambda)>0$, stationary states are possible, and a finer analog to (4.1) is obtained. Namely, assume that $|\,\psi(Q,t)\,|^2dQ=|\,\psi(Q)\,|^2dQ$ and $\int|\,\psi(Q)\,|^2dQ=1$, then taking limits as $t\to\infty$ in (4.2) we obtain

$$\lim_{t\to\infty}\int f(x)\,|\,\psi(x,t)\,|^2dx=f(0)=f(0)\int|\,\psi(Q)\,|^2dQ\,,$$

i.e., the particle distribution concentrates at $x=0$ as $t\to\infty$.

We can also compute the expected value of the mechanical energy

$$E=\frac{\dot{x}^2}{2}+\frac{\omega^2}{2}x^2$$

in a stationary state of the Hamiltonian $\frac{1}{2}[P^2+\omega^2(\lambda)Q^2]$. A small computation shows that

$$\langle\psi_n\,|\,E\,|\,\psi_n\rangle=e^{-\lambda t}(n+\tfrac{1}{2})\frac{\omega^2}{\omega(\lambda)}\,,$$

where ψ_n is the eigenstate of $\frac{1}{2}[P^2+\omega^2(\lambda)Q^2]$ with energy $(n+\frac{1}{2})\omega(\lambda)$. Observe that as $\lambda\to0$

$$\langle\psi_n\,|\,E\,|\,\psi_n\rangle\to(n+\tfrac{1}{2})\omega$$

as it should. Also, $\langle\psi_n\,|\,E\,|\,\psi_n\rangle\to0$ as $t\to\infty$, i.e., mechanical energy is dissipated.

V. COMMENTS

We have shown that the standard quantization rule can be applied to the system described by the Hamiltonian (2.1) if it can be applied to the Hamiltonian (3.5), the two descriptions being related by a unitary transformation.

If we take classical dissipativity to mean the fact that, from any starting initial state, the particle ends up at $x=0,p=0$, this is reflected in the quantum case in the fact that in states stationary in the (Q,P) representation, the particle ends up concentrated at $x=0$ as described in the Sec. IV.

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