Effect of laser fluctuations on squeezed states in a degenerate parametric amplifier

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The effect of phase and amplitude fluctuations of the pump mode on the quantum-statistical properties of the signal mode is considered. It is shown that these fluctuations in the laser field tend to decrease the squeezing of the signal field.

I. INTRODUCTION

One scheme to detect gravitational waves is based on Michelson interferometry.[1] The sensitivity of this device is limited by quantum fluctuations. It has been proposed recently that a technique which uses the so-called squeezed states of the radiation field could be employed to reduce the photon-counting fluctuation in the interferometer, thereby increasing the sensitivity of the device.

A number of nonlinear optical systems have been considered recently that generate squeezed states. These include the degenerate parametric amplifier,[2,3] degenerate four-wave mixing,[4] resonance fluorescence,[5] free-electron lasers,[6] optical bistability,[7] Jaynes-Cummings model,[8] and the multiphoton absorption process.[9] In a squeezed state, the fluctuation of one variable is reduced below its symmetrical quantum limit at the expense of the conjugate one so that the uncertainty relation is not violated. Specifically, let $a^\dagger$ and a be the creation and the destruction operators of a single-mode electromagnetic field with $[a,a^\dagger]=1$. Then the Hermitian amplitude operators a_1 and a_2 which are defined as $a=a_1+ia_2$ satisfy the commutation relation $[a_1,a_2]=i/2$. The corresponding uncertainty relation is

$$\Delta a_1\,\Delta a_2 \geq \tfrac{1}{4}\ . \tag{1}$$

A state of the field is squeezed if one of the amplitudes a_i $(i=1,2)$ satisfies

$$(\Delta a_i)^2 < \tfrac{1}{4}\ . \tag{2}$$

We call a squeezed state a "squeezed coherent state" if it is a minimum uncertainty state, i.e., in addition to (2), we obtain

$$\Delta a_1\,\Delta a_2 = \tfrac{1}{4}\ . \tag{3}$$

A parametric amplifier is a particularly important example of the systems that are predicted to exhibit squeezed states. Unlike many other systems, squeezed "coherent" states are generated in this nonlinear optical device. This, however, assumes a perfectly coherent monochromatic pump with a stabilized intensity. It is well known that this is an ideal situation and, in practice, the laser pump has a finite bandwidth which arises due to the phase fluctuations of the field. Moreover, the amplitude fluctuations of the field are also present in many situations of interest.

Some properties of the signal mode with time-dependent pump amplitude and phase were studied by Raiford.[10] This, however, did not include the stochastic character of the pump field.

In this paper we discuss the effect of the phase and the amplitude fluctuations of the laser pump in a parametric amplified on the "squeezing" property of the signal field. In Sec. II we describe the Hamiltonian in the parametric approximation and discuss the amplitude and the phase fluctuations of the laser field. In Sec. III we solve the Heisenberg equation for the signal mode operators exactly in the presence of the amplitude fluctuations in the pump mode. In Sec. IV, we study the effect of the phase fluctuations of the pump field on the squeezing in the signal mode.

II. LASER FLUCTUATIONS IN A DEGENERATE PARAMETRIC AMPLIFIER

In a degenerate parametric amplifier, a pumping field of frequency 2ω interacts with a nonlinear medium and gives rise to a field of frequency ω. This process is described, at exact resonance and with rotating-wave approximation, by the (interaction-picture) Hamiltonian

$$H=\frac{i\lambda\beta}{2}[a^2e^{i\phi}-(a^\dagger)^2e^{-i\phi}]\ , \tag{4}$$

where a $(a^\dagger)$ are the annihilation (creation) operators for the signal field, λ is an appropriate coupling constant, and β and ϕ are the real amplitude and the phase of the pump field. We have made the parametric approximation in which the pump field is treated classically and the pump depletion is neglected. This approximation is justified in the limits

$$\lambda t \to 0 \ ,$$
$$\beta \to \infty \ , \tag{5}$$
$$\lambda \beta t = Et = C \ ,$$

where $E = \lambda\beta$ and C is a constant. The Hamiltonian (4) can be rewritten as

$$H = \frac{iE}{2}[a^2 e^{i\phi} - (a^\dagger)^2 e^{-i\phi}] \ . \tag{6}$$

We then obtain the following Heisenberg equations of motion for the signal mode:

$$\dot{a} = -Ee^{-i\phi}a^\dagger \ , \tag{7a}$$

$$\dot{a}^\dagger = -Ee^{i\phi}a \ . \tag{7b}$$

From laser theory it is well known that, for a laser operating far above threshold, we can write

$$\langle b \rangle \equiv Ee^{-i\phi} = [E_0 + \delta E(t)]e^{-i\phi(t)} \ , \tag{8}$$

where $\delta E(t)$ and $\phi(t)$ are the random variables representing the amplitude and the phase fluctuations of the pump field, respectively. The laser Fokker-Planck equation leads to Gaussian amplitude fluctuations around the coherent part E_0 and a phase diffusion of ϕ.[11]

The Gaussian amplitude fluctuations are described in good approximation by the Ornstein-Uhlenbeck stochastic process[12]:

$$\langle \delta E(t) \rangle = 0 \ ,$$
$$\langle \delta E(t) \delta E(t') \rangle = I_A \Gamma e^{-\Gamma |t - t'|} \ , \tag{9}$$

where I_A measures the variance of amplitude fluctuations, and the bandwidth Γ is the laser linewidth due to amplitude fluctuations. Due to the proper normalization of the correlation function (9), I_A is also the Rabi frequency of the laser amplitude noise and in the limit of a flat spectrum, i.e., $\Gamma \to \infty$, we have $\langle \delta E(t) \delta E(t') \rangle = 2I_A \delta(t - t')$ which is characteristic of a Gaussian noise with diffusion I_A.

The random phase of the laser field performs a Brownian motion described by the Wiener-Levy stochastic process:

$$\langle \phi(t) \rangle = 0 \ ,$$
$$\langle \phi(t) \phi(t') \rangle = D(t + t' - |t - t'|) \ . \tag{10a}$$

We assume that the initial phase is zero, i.e., $\phi(0) = 0$. The derivative of this diffusion process "without friction" is a white noise

$$\langle \dot{\phi}(t) \dot{\phi}(t') \rangle = 2D\delta(t - t') \tag{10b}$$

with diffusion D. Such a model of phase fluctuations leads to a Lorentzian power spectrum of the laser light with phase-induced bandwidth D, i.e.,

$$\langle e^{i\phi(t) - i\phi(t')} \rangle = e^{-D|t - t'|} \ . \tag{11}$$

Far above threshold, amplitude fluctuations and phase fluctuations are independent and they can be treated as separate independent stochastic processes.

If we assume that the initial state of the pumped mode at $t = 0$ is vacuum, we obtain the following simplified formulas for the variances of the Hermitian amplitudes a_1 and a_2:

$$(\Delta a_1)^2 = \tfrac{1}{4}\langle a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a \rangle \ , \tag{12a}$$

$$(\Delta a_2)^2 = -\tfrac{1}{4}\langle a^2 + (a^\dagger)^2 - aa^\dagger - a^\dagger a \rangle \ , \tag{12b}$$

where the expectation value angle bracket denotes both the quantum average with the initial vacuum state and a stochastic average over the random variables of the driving field (ϕ or δE). From these formulas it is clear that we need to calculate only two expectation values, $\langle a^2 \rangle$ and $\langle aa^\dagger + a^\dagger a \rangle$, in order to obtain the variance of the Hermitian amplitudes a_1 and a_2 of the pumped mode.

III. AMPLITUDE FLUCTUATIONS

Let us first discuss only amplitude fluctuations. We are going to use the Heisenberg equations of motion (7) with $\phi = 0$ and $E = E_0 + \delta E(t)$.

For the discussion of squeezing we need to calculate only $\langle a^2 \rangle$ and $\langle a^\dagger a + aa^\dagger \rangle$ averaged over the random fluctuations of the Ornstein-Uhlenbeck amplitude $\delta E(t)$. From the Heisenberg equations of motion we obtain a closed form of the needed equations which we shall write in the following compact matrix form:

$$\dot{\underline{\psi}} = \underline{M}_0 \underline{\psi} + i\,\delta E(t) \underline{M}\,\underline{\psi} \ , \tag{13}$$

where the operator-valued vector $\underline{\psi}$ is given by the definition

$$\underline{\psi}(t) = \begin{bmatrix} a^2(t) \\ [a^\dagger(t)]^2 \\ a(t)a^\dagger(t) + a^\dagger(t)a(t) \end{bmatrix} . \tag{14a}$$

The matrices $\underline{M}_0$ and $\underline{M}$ are as follows:

$$\underline{M}_0 = \begin{bmatrix} 0 & 0 & -E_0 \\ 0 & 0 & -E_0 \\ -2E_0 & -2E_0 & 0 \end{bmatrix} ,$$
$$\underline{M} = -i \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} . \tag{14b}$$

We note that $[\underline{M}_0,\underline{M}]=0$. This important property allows us to solve and to average the differential operator equation (13) with a multiplicative non-white noise $\delta E(t)$ exactly. Because $\underline{M}_0$ and $\underline{M}$ commute, no time ordering is necessary in the evolution operator for Eq. (13). We need only to use the following expectation value for the Gaussian stochastic process $\delta E(t)$:

$$\left\langle \exp\left[i \int_0^t d\tau \underline{M}\, \delta E(\tau) \right] \right\rangle = \exp\left[-\tfrac{1}{2}\underline{M}^2 \int_0^t d\tau_1 \int_0^t d\tau_2 \langle \delta E(\tau_1)\delta E(\tau_2)\rangle \right] . \tag{15}$$

From this formula we obtain the following exact equation satisfied by $\langle \underline{\psi}(t)\rangle$:

$$\langle \dot{\underline{\psi}}\rangle = [\underline{M}_0 - \underline{M}^2 f(t)]\langle \underline{\psi}\rangle , \tag{16}$$

where

$$f(t) = I_A t + \frac{I_A}{\Gamma}(e^{-\Gamma t}-1) , \tag{17}$$

and with the initial value at $t=0$

$$\langle \underline{\psi}(0)\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{18}$$

The differential stochastic equation (18) with a multiplicative noise (9) gives a nontrivial example of only few exactly soluble models with a Ornstein-Uhlenbeck stochastic amplitude fluctuations. We note that our solution holds for an arbitrary correlation function $\langle \delta E(t)\delta E(t')\rangle$ with the only modification coming in the final form of the f function (17) obtained by a double integration of the correlation function of the stochastic process $\delta E(t)$. We are now left with a simple 3×3 linear differential equation with time-dependent coefficients. This matrix equation can be diagonalized in a straightforward way leading to the following solutions:

$$\langle a^2(t)\rangle = -\tfrac{1}{2} e^{4f(t)} \sinh(2E_0 t) , \tag{19}$$

$$\langle a^\dagger(t)a(t) + a(t)a^\dagger(t)\rangle = e^{4f(t)}\cosh(2E_0 t) . \tag{20}$$

Inserting these solutions in Eqs. (12a) and (12b) we obtain the following expressions for the variance of the Hermitian operators a_1 and a_2 with Gaussian amplitude fluctuations:

$$(\Delta a_1)^2 = \tfrac{1}{4} \exp\left[4I_A t + \frac{4I_A}{\Gamma}(e^{-\Gamma t}-1) - 2E_0 t \right] , \tag{21a}$$

$$(\Delta a_2)^2 = \tfrac{1}{4} \exp\left[4I_A t + \frac{4I_A}{\Gamma}(e^{-\Gamma t}-1) + 2E_0 t \right] . \tag{21b}$$

In Fig. 1 we have plotted $(\Delta a_1)^2$ vs $E_0 t$ for different values of I_A/E_0 and Γ/E_0. It is seen that the effect of amplitude fluctuation is to decrease the squeezing.

IV. PHASE FLUCTUATIONS

In this section we discuss the influence of only phase fluctuations on variance of Hermitian amplitudes a_1 and a_2. We shall use the Heisenberg equations of motion (12) with $\delta E=0$ and with the Gaussian random phase $\phi(t)$ given by Eq. (10a).

We first determine the expectation value of the operator as $aa^\dagger + a^\dagger a$. From the Heisenberg equations of motion we obtain the following stochastic multiplicative equation:

$$\dot{\underline{\psi}} = [\underline{M}_0 + i\dot{\phi}(t)\underline{M}]\underline{\psi} , \tag{22}$$

with

$$\underline{\psi} = \begin{bmatrix} a^\dagger a + aa^\dagger \\ a^2 e^{i\phi} \\ (a^\dagger)^2 e^{-i\phi} \end{bmatrix} , \tag{23a}$$

and

$$\underline{M}_0 = \begin{bmatrix} 0 & -2E_0 & -2E_0 \\ -E_0 & 0 & 0 \\ -E_0 & 0 & 0 \end{bmatrix} , \quad \underline{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} . \tag{23b}$$

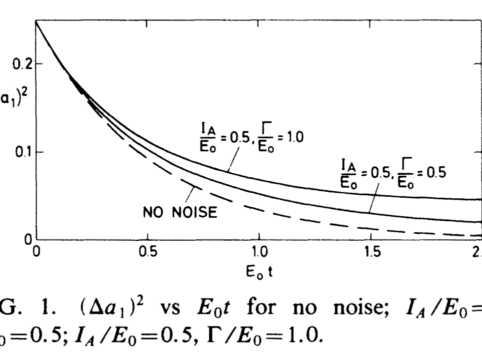

FIG. 1. $(\Delta a_1)^2$ vs $E_0 t$ for no noise; $I_A/E_0=0.5$, $\Gamma/E_0=0.5$; $I_A/E_0=0.5$, $\Gamma/E_0=1.0$.

Note that the operator $a^\dagger a + aa^\dagger$ is not coupled directly to a^2 and $(a^\dagger)^2$ as is the case for amplitude fluctuations. In order to calculate the stochastic expectation value of $a^\dagger a + a^\dagger a$ we need to evaluate two auxiliary quantities $\langle a^2 e^{i\phi}\rangle$ and $\langle (a^{\dagger 2})e^{-i\phi}\rangle$. The situation is very similar to the case of laser phase fluctuations of a driving light coupled to a two-level atom.[12] This known and studied example shows that the inversion and the dipole transition operators couple differently to the phase fluctuations of the laser. We shall than use the same techniques as in the case of a two-level system[13] in order to obtain an exact stochastic expectation value of $\langle \underline{\psi}\rangle$ given by Eq. (22). For the fluctuating random phase $\phi(t)$ given by the Wiener-Levy stochastic process it has been shown that the following exact equation is satisfied[14,15]:

$$\langle \dot{\underline{\psi}}\rangle = (\underline{M}_0 - D\underline{M}^2)\langle \underline{\psi}\rangle \tag{24}$$

for arbitrary form of the time-independent matrices $\underline{M}_0$ and $\underline{M}$. This matrix equation specified for $\underline{\psi}$, $\underline{M}_0$ and $\underline{M}$ given by Eqs. (23a) and (23b) can be solved exactly using, for example, the Laplace-transform techniques. For the vacuum initial state of the signal mode, we obtain

$$\langle a^\dagger a + aa^\dagger\rangle = \int_C \frac{dz}{2\pi i}\frac{e^{zt}(z+D)}{z+zD-4E_0^2} . \tag{25}$$

Computing the roots of the algebraic equation in the denominator in Eq. (25) and choosing properly the contour of integration C we find the explicit time evolution as follows:

$$\langle a^\dagger a + aa^\dagger\rangle = \left[\frac{D}{2\beta}\sinh(\beta t) + \cosh(\beta t)\right] e^{-Dt/2} , \tag{26}$$

where

$$\beta = \tfrac{1}{4}D^2 + 4E_0^2 . \tag{27}$$

Finally, the last operator required for the squeezing amplitudes is the stochastic average of $\langle a^2\rangle$. Again from the Heisenberg equations of motion with fluctuating phase we generate a multiplicative stochastic equation of the form given by Eq. (22) with

$$\underline{\psi} = \begin{bmatrix} a^2 \\ e^{-i\phi}(a^\dagger a + aa^\dagger) \\ e^{-2i\phi}(a^\dagger)^2 \end{bmatrix} , \tag{28a}$$

and different forms of $\underline{M}_0$ and $\underline{M}$,

$$\underline{M}_0 = \begin{bmatrix} 0 & -E_0 & 0 \\ -2E_0 & 0 & -2E_0 \\ 0 & -E_0 & 0 \end{bmatrix} , \qquad \underline{M}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} . \tag{28b}$$

As in the previous case the stochastic expectation value of $\langle \underline{\psi}\rangle$ satisfies an exact differential equation (29) with matrices $\underline{M}_0$ and $\underline{M}_1$ given now by expressions (28b). With our specific initial condition, the Laplace-transform solution has the following exact form:

$$\langle a^2\rangle = -\int_C \frac{dz}{2\pi i}\frac{e^{zt}E_0(z+4D)}{[z^3+5Dz^2+(4D^2-4E_0^2)z-8E_0^2 D]} . \tag{29}$$

The exact time dependence, accordingly, has the form

$$\langle a^2\rangle = -\sum_{\substack{i,j,k \\ i\neq j\neq k}} \frac{e^{\lambda_i t}E_0(\lambda_i+4D)}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)} , \tag{30}$$

where λ_i are the roots of the following cubic equation:

$$\lambda^3 + 5D\lambda^2 + 4(D^2-E_0^2)\lambda - 8E_0^2 D = 0 . \tag{31}$$

These roots can be obtained exactly using the Car-

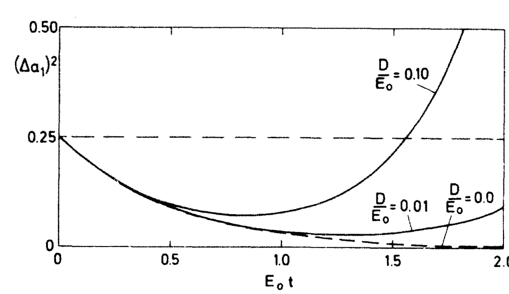

FIG. 2. $(\Delta a_1)^2$ vs $E_0 t$ for $D/E_0 = 0.0$, 0.01, and 0.10.

dano formulas. We can, however, get a reasonable understanding of the physics solving this cubic equation in the realistic limit of small phase fluctuations (as compared with the driving Rabi frequency, i.e., $D \ll E_0$). In this limit,

$$\lambda_1 \simeq -2D \,,$$
$$\lambda_2 \simeq 2E_0 - \tfrac{3}{2}D \,, \qquad (32)$$
$$\lambda_3 \simeq -2E_0 - \tfrac{3}{2}D \,,$$

and accordingly

$$\langle a^2 \rangle \simeq e^{-2Dt} \frac{DE_0}{(2E_0^2 - D^2/8)} - e^{(2E_0 - 3D/2)t} \frac{(2E_0 + 5D/2)}{4(2E_0 + D/2)} + e^{-(2E_0 + 3D/2)t} \frac{(2E_0 - 5D/2)}{4(2E_0 - D/2)} \,. \qquad (33)$$

From Eqs. (26) and (33) we obtain the following formulas for the variance of the Hermitian amplitudes with laser phase fluctuations (with approximated roots λ_i):

$$(\Delta a_1)^2 \simeq \frac{1}{4} \left[e^{-2Dt} \frac{2DE_0}{(2E_0^2 - D^2/8)} - e^{(2E_0 - 3D/2)t} \frac{(2E_0 + 5D/2)}{2(2E_0 + D/2)} + e^{-(2E_0 + 3D/2)t} \frac{(2E_0 - 5D/2)}{2(2E_0 - D/2)} + \frac{e^{-Dt/2} D \sinh(2E_0 t)}{(D^2 + 16E_0^2)^{1/2}} + e^{-Dt/2} \cosh(2E_0 t) \right] , \qquad (34a)$$

$$(\Delta a_2)^2 = -\frac{1}{4} \left[e^{-2Dt} \frac{2DE_0}{2E_0^2 - D^2/8} - e^{(2E_0 - 3D/2)t} \frac{(2E_0 + 5D/2)}{2(2E_0 + D/2)} + e^{-(2E_0 + 3D/2)t} \frac{(2E_0 - 5D/2)}{2(2E_0 - D/2)} - \frac{e^{-Dt/2} D \sinh(2E_0 t)}{(D^2 + 16E_0^2)^{1/2}} - e^{-Dt/2} \cosh(2E_0 t) \right] . \qquad (34b)$$

In Fig. 2 we have plotted $(\Delta a_1)^2$ versus $E_0 t$ for various values of D/E_0. The fluctuations in the amplitude a_1 increase due to the phase fluctuation of the laser field. The fluctuations $(\Delta a_1)^2$ exhibit a minimum which decreases.

Equations (34a) and (34b) simplify considerably in the limit $D \ll t^{-1} \ll E_0$. We then obtain

$$(\Delta a_1)^2 = \tfrac{1}{4} e^{-2E_0 t} + \tfrac{1}{4} e^{2E_0 t} (\tfrac{1}{2} Dt) \,, \qquad (35a)$$

$$(\Delta a_2)^2 = \tfrac{1}{4} e^{2E_0 t} (1 - Dt) \,. \qquad (35b)$$

It is clear from Eq. (35a) that if the pump phase is off by ϕ, then the large uncertainty $\frac{1}{4}\exp(2E_0 t)$ in the amplified quadrature $(\Delta a_2)^2$ is mixed into the uncertainty of the squeezed quadrature $(\Delta a_1)^2$ with phase angle ϕ; Dt is, roughly speaking, the amount by which ϕ random walks in time t.

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