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Long-range interactions in free-free transitions

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The soft-photon approximation for bremsstrahlung derived some time ago by Low and extended by Feshbach and Yennie to allow for resonances is further generalized, in the framework of nonrelativistic scattering theory, to apply to the case where the scattering potential has a long-range Coulomb tail. The derivation is based on an asymptotic evaluation of the matrix element in configuration space in which electric-dipole, magnetic-dipole, and electric-quadrupole components of the particle-field interaction are accounted for. The $M1$ and $E2$ contributions give rise to retardation corrections, of order v/c, to the leading $E1$ term. The Coulomb tail has the effect of introducing certain factors into the approximate bremsstrahlung amplitude which depend logarithmically on the photon frequency. The result obtained here has an immediate application to the problem of Coulomb scattering in a low-frequency laser field and provides a generalization of earlier work based on the electric-dipole approximation.

I. INTRODUCTION

The radiation of a low-energy photon by a charged particle during a collision with another particle is most likely to occur in the region outside the range of the scattering potential. This provides the basis for a soft-photon approximation which allows one to express the bremsstrahlung amplitude in terms of the amplitude for scattering without radiation. The leading contribution to the cross section is of order ω^{-1}, where ω is the angular frequency of the photon. Not only this term, but also a correction term of order ω^0 can be determined from a knowledge of the elastic scattering amplitude and its derivatives. This result, first derived by Low,[1] was later generalized by Feshbach and Yennie[2] to allow for scattering resonances. The improved theory provides the quantum-theoretical basis for the suggestion that the delay time in a nuclear reaction may be determined from a measurement of the bremsstrahlung cross section.

If both target and projectile are charged, the scattering potential will have a long-range Coulomb tail, the effect of which had been neglected in the earlier work of Low and of Feshbach and Yennie. Here we shall derive an extension of the Feshbach-Yennie approximation for the bremsstrahlung amplitude which accounts for the Coulomb tail. When expanded in powers of the frequency we find terms of order $\ln\omega$ and $\omega \ln^2\omega$ in addition to terms of order ω^{-1} and ω^0 present in the short-range case.[3] The error in the approximation is of order $\omega \ln\omega$. Our method is based on an analysis of the matrix element in configuration space with proper account taken of the Coulomb distortion of the asymptotic wave functions. An earlier application of this approach[4] was limited to the use of the electric-dipole approximation for the particle-field interaction, and the effect of target recoil was ignored. Both of these limitations are removed here, and a more complete and systematic derivation is provided. We find that, just as in the case where the potential is of short range,[1] only magnetic-dipole ($M1$) and electric-quadrupole ($E2$) corrections need be included in a nonrelativistic calculation; higher-order multipole terms lead to corrections of order $(v/c)^2$ and smaller. The $M1$ and $E2$ corrections are of order v/c and may be interpreted as arising from the recoil of the charge particle upon emission of a photon.

It should be mentioned that the amplitude for scattering in a low-frequency laser field can be related directly to the single-photon spontaneous bremsstrahlung amplitude.[4] Thus the result derived here for the latter amplitude can be used to obtain a low-frequency approximation for scattering in a laser field in which corrections to the electric-dipole approximation, as well as effects of the Coulomb tail of the scattering potential, are included.

There is an interesting similarity between the theory of soft-photon bremsstrahlung and that of K-shell ionization accompanying a nuclear collision. Both have been proposed as methods for studying time-delay effects with both classical and quantum descriptions provided in each case. An asymptotic evaluation of the K-shell ionization matrix element has recently been presented by Blair and Anholt.[5] Methods developed in the present paper, in which higher-order terms in the asymptotic expansion of the Coulomb wave function are retained, may be useful in extending the Blair-Anholt theory to higher order in the energy-loss parameter.

In Sec. II the bremsstrahlung amplitude is defined, the effect of target recoil is determined, and the relevant terms in the multipole expansion are presented. The problem of evaluating the angular and radial integrals which appear in the matrix element is taken up in Secs. III A and III B. With these results in hand the derivation of the approximate bremsstrahlung amplitude is straightforward and is outlined in Sec. III C. The paper concludes, in Sec. IV, with a discussion of the results obtained.

II. BREMSSTRAHLUNG MATRIX ELEMENT

The scattering system studied here consists of two particles, one (the "projectile") of charge e and mass m and the other (the "target") of charge Ze and mass Am. We ignore the effects of the internal structure of the particles, as well as spin and relativistic effects. The initial momenta of projectile and target are $\vec{p}_1$ and $\vec{p}_2$, respectively. The system is initially in a state of zero total momentum $\vec{p}_1+\vec{p}_2=0$; the total energy is $p^2/2\mu$, where p is the relative momentum and μ is the reduced mass. In the course of the scattering a photon of momentum $\vec{k}$ and frequency ω is emitted, the particles emerging with momenta $\vec{p}\,'_1$ and $\vec{p}\,'_2$. From momentum conservation we have $\vec{p}\,'_1+\vec{p}\,'_2=-\vec{k}$. Accordingly, the energy associated with the center-of-mass motion in the final state is negligible in the nonrelativistic regime, and the total particle energy in the final state is just $p'^2/2\mu$, where $\vec{p}\,'$ is the final relative momentum.

The S matrix element may be expressed (in units where $\hbar=1$) as

$$S=-2\pi i\delta(p'^2/2\mu+\omega-p^2/2\mu)\mathscr{T} \tag{2.1}$$

with

$$\mathscr{T}=-\frac{e}{mc}\vec{\mathscr{A}}\cdot\langle\psi'\,|\,\vec{J}_{\vec{k}}\,|\,\psi\rangle\ . \tag{2.2}$$

We have defined

$$\vec{\mathscr{A}}=\left[\frac{2\pi c^2}{\omega\mathscr{V}}\right]^{1/2}\vec{\lambda}^*\ , \tag{2.3}$$

where $\vec{\lambda}$ is the photon polarization vector and $\mathscr{V}$ is the quantization volume. We also have

$$\vec{J}_{\vec{k}}=-i\left[\vec{\nabla}_{\vec{r}_1}e^{-i\vec{k}\cdot\vec{r}_1}+\frac{Z}{A}\vec{\nabla}_{\vec{r}_2}e^{-i\vec{k}\cdot\vec{r}_2}\right]\ , \tag{2.4}$$

valid in the Coulomb gauge, in which $\vec{k}\cdot\vec{\lambda}^*=0$. The wave function in the initial state is

$$\psi=(2\pi)^{-3/2}e^{i(\vec{p}_1+\vec{p}_2)\cdot\vec{R}}u_{\vec{p}}^{(+)}(\vec{r})\ , \tag{2.5}$$

where $\vec{r}=\vec{r}_1-\vec{r}_2$ and $\vec{R}=(\vec{r}_1+A\vec{r}_2)/(1+A)$. Since we take $\vec{p}_1+\vec{p}_2=0$, we have $\vec{\nabla}_{\vec{r}_1}\psi=-\vec{\nabla}_{\vec{r}_2}\psi=\vec{\nabla}_{\vec{r}}\psi$. The final state is represented as

$$\psi'=(2\pi)^{-3/2}e^{i(\vec{p}\,'_1+\vec{p}\,'_2)\cdot\vec{R}}u_{\vec{p}\,'}^{(-)}(\vec{r})\ . \tag{2.6}$$

The wave function $u_{\vec{p}}^{(+)}(\vec{r})$ satisfies the Schrödinger equation in the center-of-mass frame in the presence of the full scattering potential, a superposition of Coulomb and short-range components. Asymptotically, we have the form

$$u_{\vec{p}}^{(+)}(\vec{r})\sim(2\pi)^{-3/2}(2pr)^{-1}\sum_{l=0}^{\infty}(2l+1)i^{l+1}[u_l^{(+)*}(p,r)-S_l(p)u_l^{(+)}(p,r)]P_l(\hat{r}\cdot\hat{p})\ ,\quad r\to\infty \tag{2.7}$$

where S_l is the partial-wave S-matrix element, and $u_l^{(+)}$ is the outgoing-wave spherical Coulomb function. We note the relation $u_{\vec{p}}^{(-)*}=u_{-\vec{p}}^{(+)}$.

After integration over the center-of-mass coordinate Eq. (2.2) becomes

$$\mathscr{T}=\delta(\vec{p}\,'_1+\vec{p}\,'_2+\vec{k})T(\vec{p}\,',\vec{p})\ ; \tag{2.8}$$

the T matrix may be expressed as

$$T(\vec{p}\,',\vec{p})=\mathscr{M}\left[\vec{p}\,',\vec{p};\frac{A}{1+A}\vec{k}\right]-\frac{Z}{A}\mathscr{M}\left[\vec{p}\,',\vec{p};-\frac{1}{1+A}\vec{k}\right] \tag{2.9}$$

with

$$\mathscr{M}(\vec{p}\,',\vec{p};\vec{k})=\frac{ie}{mc}\vec{\mathscr{A}}\cdot\int u_{-\vec{p}\,'}^{(+)}(\vec{r})\vec{\nabla}_{\vec{r}}e^{-i\vec{k}\cdot\vec{r}}\times u_{\vec{p}}^{(+)}(\vec{r})d^3r\ . \tag{2.10}$$

We confine our study in the following to the function $\mathscr{M}(\vec{p}\,',\vec{p};\vec{k})$, which represents the bremsstrahlung amplitude for scattering from a stationary target. Effects of target recoil are accounted for by making use of Eq. (2.9).

The exponential in Eq. (2.10) may be expanded as $1-i\vec{k}\cdot\vec{r}$ with higher-order terms, which lead to corrections of order $(v/c)^2$, ignored. Following a standard procedure[6] we are led to a truncated multipole expansion $\mathscr{M}\cong\mathscr{M}^{E1}+\mathscr{M}^{M1}+\mathscr{M}^{E2}$. The electric-dipole contribution is

$$\mathscr{M}^{E1}=\frac{ie}{mc}\mu\omega\vec{\mathscr{A}}\cdot\int u^{(+)}_{-\vec{p}'}(\vec{r})\vec{r}u^{(+)}_{\vec{p}}(\vec{r})d^3r \tag{2.11a}$$

with $\omega=p^2/2\mu-p'^2/2\mu$. The magnetic-dipole term may be expressed in terms of the angular momentum operator $\vec{L}_{\hat{r}}=-i\vec{r}\times\vec{\nabla}_{\vec{r}}$ as

$$\mathscr{M}^{M1}=\frac{ie}{2mc}(\vec{k}\times\vec{\mathscr{A}})\cdot\int u^{(+)}_{-\vec{p}'}(\vec{r})\vec{L}_{\hat{r}}u^{(+)}_{\vec{p}}(\vec{r})d^3r\ . \tag{2.11b}$$

The electric-quadrupole term takes the form

$$\mathscr{M}^{E2}=\frac{e}{2mc}\mu\omega\int u^{(+)}_{-\vec{p}'}(\vec{r})(\vec{k}\cdot\vec{r})(\vec{\mathscr{A}}\cdot\vec{r})\times u^{(+)}_{\vec{p}}(\vec{r})d^3r\ . \tag{2.11c}$$

III. ASYMPTOTIC EVALUATION

A. Angular integration

We consider here the angular integrations which arise when the wave functions appearing in Eqs. (2.11a)–(2.11c) are replaced by their partial-wave expansions. These integrations can be performed using formulas of the type

$$\sum_{l'=0}^{\infty}\int d\Omega_{\hat{r}}(2l'+1)P_{l'}(\hat{p}'\cdot\hat{r})Y_{\lambda\nu}(\hat{r})P_l(\hat{p}\cdot\hat{r})=4\pi Y_{\lambda\nu}(\hat{p}')P_l(\hat{p}'\cdot\hat{p})\ . \tag{3.1}$$

To derive this result one first makes use of the spherical harmonic addition formula

$$(2l+1)P_l(\hat{p}\cdot\hat{r})=4\pi\sum_{m=-l}^{l}Y_{lm}(\hat{r})Y^*_{lm}(\hat{p}) \tag{3.2}$$

with a similar expansion for $P_{l'}(\hat{p}'\cdot\hat{r})$. The integrand now will contain a product of three spherical harmonics. This may be reduced using the identity[7]

$$Y_{\lambda\nu}(\hat{r})Y_{lm}(\hat{r})=\sum_J\sum_M\left[\frac{(2\lambda+1)(2l+1)}{4\pi(2J+1)}\right]^{1/2}\langle\lambda l\nu m\,|\,\lambda lJM\rangle\langle\lambda l00\,|\,\lambda lJ0\rangle Y_{JM}(\hat{r})\ . \tag{3.3}$$

(It is only the existence of an expansion of this type which is used; the explicit form of the expansion coefficients plays no role in the derivation.) The angular integration may now be performed using orthonormality. At this stage we have a linear combination of functions $Y_{l'm'}(\hat{p}')$ with expansion coefficients of the form shown in Eq. (3.3); it may therefore be summed to $Y_{\lambda\nu}(\hat{p}')Y_{lm}(\hat{p}')$. Then, continuing to retrace our steps, we use the addition formula (3.2) to arrive at the stated result, Eq. (3.1). As an immediate consequence we have the relations

$$\sum_{l'=0}^{\infty}\int d\Omega_{\hat{r}}(2l'+1)P_{l'}(\hat{p}'\cdot\hat{r})\vec{\mathscr{A}}\cdot\hat{r}P_l(\hat{p}\cdot\hat{r})=4\pi\vec{\mathscr{A}}\cdot\hat{p}'P_l(\hat{p}'\cdot\hat{p}) \tag{3.4}$$

and

$$\sum_{l'=0}^{\infty}\int d\Omega_{\hat{r}}(2l'+1)P_{l'}(\hat{p}'\cdot\hat{r})(\vec{k}\cdot\hat{r})(\vec{\mathscr{A}}\cdot\hat{r})P_l(\hat{p}\cdot\hat{r})=4\pi(\vec{k}\cdot\hat{p}')(\vec{\mathscr{A}}\cdot\hat{p}')P_l(\hat{p}'\cdot\hat{p})\ . \tag{3.5}$$

In a similar way, by expanding $L_{\hat{r}}Y_{lm}(\hat{r})$ as a superposition of spherical harmonics, we find

$$\sum_{l'=0}^{\infty}\int d\Omega_{\hat{r}}(2l'+1)P_{l'}(\hat{p}'\cdot\hat{r})\vec{L}_{\hat{r}}P_l(\hat{p}\cdot\hat{r})=4\pi\vec{L}_{\hat{p}'}P_l(\hat{p}'\cdot\hat{p}) \tag{3.6}$$

with $\vec{L}_{\hat{p}'}=-i\vec{p}'\times\vec{\nabla}_{\vec{p}'}$.

We also encounter integrals which are of the type shown on the left-hand side of Eq. (3.1) but with $Y_{\lambda\nu}(\hat{r})$ replaced by $[L^2_{\hat{r}},Y_{\lambda\nu}(\hat{r})]$. To see how such integrals arise in the calculation consider the matrix element in Eq. (2.11a) with the wave functions replaced by their asymptotic forms, as shown in Eq. (2.7). The products of radial spherical Coulomb functions which appear in the integrand may be approximated with the aid of the well-known asymptotic expansion of $u_l^{(+)}(p,r)$.[8] We have, for example,

$$u_{l'}^{(+)*}(p',r)u_l^{(+)}(p,r)\sim e^{i(p-p')r}i^{(l'-l)}(2pr)^{-in}(2p'r)^{in'}\{1-(2ipr)^{-1}[l'(l'+1)-l(l+1)+2in]\}\ , \tag{3.7}$$

where

$$n=\frac{Ze^2\mu}{p},\quad n'=\frac{Ze^2\mu}{p'}\ . \tag{3.8}$$

In the expression in braces on the right-hand side of Eq. (3.7) we have ignored terms of order r^{-2} and higher and have also ignored the distinction between p and p'. These are the approximations which set the limits on the accuracy of our final result, as discussed further below. When the eigenvalues $l(l+1)$ and $l'(l'+1)$ are replaced by the operator $L_{\hat{r}}^2$ (acting to the right on P_l in one case and to the left on $P_{l'}$ in the other) and the Hermiticity property of this operator is used, we obtain the commutator form indicated above. A formula of the type (3.1), but with $Y_{\lambda\nu}$ replaced by its commutator with L^2, is readily derived; the identity

$$[L_{\hat{r}}^2, Y_{\lambda\nu}(\hat{r})] = \lambda(\lambda+1)Y_{\lambda\nu}(\hat{r}) + 2[\vec{L}_{\hat{r}} Y_{\lambda\nu}(\hat{r})]\cdot\vec{L}_{\hat{r}} \quad (3.9)$$

allows one to reduce this case to those considered above. It is also clear that the spherical harmonic may be replaced by $\vec{\mathscr{A}}\cdot\hat{r}$ on the left-hand side of the integration formula and by $\vec{\mathscr{A}}\cdot\hat{p}'$ on the right. Use of the relation

$$\vec{L}_{\hat{p}'}(\vec{\mathscr{A}}\cdot\hat{p}') = -i\hat{p}'\times\vec{\mathscr{A}} \quad (3.10)$$

along with Eq. (3.9) then leads to the desired form

$$\sum_{l'=0}^{\infty} \int d\Omega_{\hat{r}}(2l'+1)P_{l'}(\hat{p}\cdot\hat{r})[L_{\hat{r}}^2, \vec{\mathscr{A}}\cdot\hat{r}]P_l(\hat{p}\cdot\hat{r}) = 4\pi[2\vec{\mathscr{A}}\cdot\hat{p}' - 2(\hat{p}'\times\vec{\mathscr{A}})\times\vec{p}'\cdot\vec{\nabla}_{\vec{p}'}]P_l(\hat{p}'\cdot\hat{p}) \ . \quad (3.11)$$

A similar formula holds with $\vec{\mathscr{A}}\cdot\hat{r}$ replaced on the left by $(\vec{k}\cdot\hat{r})(\vec{\mathscr{A}}\cdot\hat{r})$. The commutator appearing on the right-hand side reduces to

$$[L_{\hat{p}'}^2, (\vec{k}\cdot\hat{p}')(\vec{\mathscr{A}}\cdot\hat{p}')] = 6(\vec{k}\cdot\hat{p}')(\vec{\mathscr{A}}\cdot\hat{p}') - 2[(\vec{\mathscr{A}}\cdot\hat{p}')(\hat{p}'\times\vec{k})\times\vec{p}'\vec{\nabla}_{\vec{p}'} + (\vec{k}\cdot\hat{p}')(\hat{p}'\times\vec{\mathscr{A}})\times\vec{p}'\cdot\vec{\nabla}_{\vec{p}'}] \ . \quad (3.12)$$

B. Radial integration

In the low-frequency limit the dominant contribution to the bremsstrahlung matrix element comes from the asymptotic domain in configuration space, here defined by the condition $r > r_0$. We require that r_0 be large enough so that the short-range component of the scattering potential may be neglected in constructing the asymptotic solutions. (Of course, the potential affects the scattering parameters which appear in those solutions.) A class of radial integrals which must be considered in the asymptotic evaluation of the matrix element is of the form

$$I(r_0,s) = \int_{r_0}^{\infty} e^{i(p-p')r}(2pr)^{-in}(2p'r)^{in'} r^s dr \ , \quad (3.13)$$

where s is an integer. Another type of integral arises which can be obtained from this by interchange of initial and final momenta, in a manner shown explicitly below. The integral (3.13) is made well defined by the introduction of a convergence factor $e^{-\epsilon r}$ with $\epsilon \to 0$ in the final result. This result is singular in the limit $p' \to p$ (or $\omega \to 0$) for $s \geq -1$. We adopt the rule that those contributions to the radial integration which remain finite for $\omega \to 0$ are to be discarded. In particular, we neglect those terms in which the rapidly oscillating exponential $e^{\pm i(p+p')r}$ appears in place of the factor $e^{i(p-p')r}$.

Consider now the integral $I(r_0,s)$ for $s = 0$, 1, and 2. Since these integrals remain finite when the range of integration is extended down to the origin we may, in accordance with the rule adopted above, set $r_0 = 0$ and make use of the relation

$$\lim_{\epsilon\to 0} \int_0^{\infty} e^{-(\epsilon+ia)r} r^b dr = (ia)^{-1-b}\Gamma(1+b) \ . \quad (3.14)$$

We then find, for $s = 0$,

$$I(0,0) = [-i(p-p')]^{-1}B(p',p) \ , \quad (3.15)$$

where

$$B(p',p) = e^{-|n-n'|\pi/2}\left[\frac{|p-p'|}{2p}\right]^{i(n-n')}\left[\frac{p'}{p}\right]^{in'} \times\Gamma[1-i(n-n')] \ . \quad (3.16)$$

Note that with terms of order $(n-n')^2$ neglected we have $\Gamma[1-i(n-n')] = 1 + i(n-n')\gamma$, where $\gamma = 0.577\,215\,7\ldots$ is the Euler-Mascheroni constant. The remaining integrals may then be expressed as

$$I(0,1) = [-i(p-p')]^{-2}[1-i(n-n')] \times B(p',p) \ , \quad (3.17)$$

$$I(0,2) = [-i(p-p')]^{-3}[2-i(n-n')] \times[1-i(n-n')]B(p',p) \ . \quad (3.18)$$

Integrals with $s > 2$ do not appear in the calculation. For $s = -1$ we have, after an integration by parts,

$$I(r_0,-1) = [i(n-n')]^{-1}e^{i(p-p')r_0} \times(2pr_0)^{-i(n-n')}\left[\frac{p'}{p}\right]^{in'} + \left[\frac{p-p'}{n-n'}\right]I(r_0,0) \ . \quad (3.19)$$

Using Eqs. (3.15) and (3.16), and ignoring terms which remain finite for $p - p' \to 0$, we find

$$I(r_0,-1) \cong [i(n-n')]^{-1} \times\left[1-\left[\frac{|p-p'|}{2p}\right]^{i(n-n')}\right] \cong -\ln\left[\frac{|p-p'|}{2p}\right] \ . \quad (3.20)$$

Terms of this order could be retained. However, they appear with coefficients which involve sums over partial waves, and we have found no simple closed-form expression for these sums. In the following we ignore terms of order $\ln\omega$ in the radial integrals; this introduces errors of order $\omega\ln\omega$ in the matrix elements (2.11). It follows that in an expansion of the expression (3.16) for $B(p',p)$ about $\omega=0$ terms of order $\omega^2\ln\omega$ may be consistently neglected.

C. Low-frequency approximation

We are now prepared to evaluate the matrix element (2.10) in the asymptotic approximation described above. For notational convenience we rewrite Eqs. (2.11a)—(2.11c) in the form

$$\mathcal{M}^{\alpha}(\vec{p}\,',\vec{p};\vec{k})=\frac{ie}{mc}\mu\omega\int u^{(+)}_{-\vec{p}\,'}(\vec{r})V^{\alpha}(\vec{k})\times u^{(+)}_{\vec{p}}(\vec{r})d^3r \tag{3.21a}$$

with $\alpha=E1$, $M1$, or $E2$ and

$$V^{E1}=\vec{\mathscr{A}}\cdot\vec{r}\,, \tag{3.21b}$$

$$V^{M1}=(p^2-p'^2)^{-1}\vec{k}\times\vec{\mathscr{A}}\cdot\vec{L}_{\hat{r}}\,, \tag{3.21c}$$

$$V^{E2}=-\tfrac{1}{2}i(\vec{k}\cdot\vec{r})(\vec{\mathscr{A}}\cdot\vec{r})\,. \tag{3.21d}$$

The asymptotic form of the wave function is shown in Eq. (2.7). Each partial wave is a superposition of an incoming part $u_l^{(+)*}$ and an outgoing part $u_l^{(+)}$. The product of initial- and final-state wave functions contains a part which is outgoing in the initial state and incoming in the final state [see Eq. (3.7)]. Once the contribution from this part is evaluated the contribution from that which is incoming in the initial state and outgoing in the final state may be obtained by making the switch $\vec{p}\leftrightarrow-\vec{p}\,'$ in the momentum variables. The parts which are purely outgoing or purely incoming may be ignored in our approximation. Thus we write

$$\int u^{(+)}_{-\vec{p}\,'}(\vec{r})V^{\alpha}(\vec{k})u^{(+)}_{\vec{p}}(\vec{r})d^3r=M^{\alpha}(\vec{p}\,',\vec{p};\vec{k})+M^{\alpha}(-\vec{p},-\vec{p}\,';\vec{k}) \tag{3.22}$$

and apply the results obtained in Secs. III A and III B to evaluate the functions $M^{\alpha}(\vec{p}\,',\vec{p};\vec{k})$. With uninteresting algebraic details omitted we may state the result for the $E1$ contribution as

$$M^{E1}(\vec{p}\,',\vec{p};\vec{k})=(i\mu/pp')(p-p')^{-2}B(p',p)[\vec{\mathscr{A}}\cdot\vec{p}\,'t(p\hat{p}\,',\vec{p})+(p-p')\vec{p}\,'\times(\vec{\mathscr{A}}\times\hat{p}\,')\cdot\vec{\nabla}_{\vec{p}\,'}t(\vec{p}\,',\vec{p})]\,, \tag{3.23}$$

where

$$t(p\hat{p}\,',\vec{p})=(-4\pi)(2\mu)^{-1}(2\pi)^{-3}\left[(2ip)^{-1}\sum_{l=0}^{\infty}(2l+1)S_l(p)P_l(\hat{p}\,'\cdot\hat{p})\right] \tag{3.24}$$

is the conventionally defined t matrix for scattering by a potential with a long-range Coulomb tail.[8] [The gradient in Eq. (3.23) is evaluated at $\vec{p}\,'=p\hat{p}\,'$.] The $M1$ and $E2$ contributions are

$$M^{M1}(\vec{p}\,',\vec{p};\vec{k})=(i\mu/2pp')(p-p')^{-2}B(p',p)\vec{p}\,'\times(\vec{k}\times\vec{\mathscr{A}})\cdot\vec{\nabla}_{\vec{p}\,'}t(\vec{p}\,',\vec{p})\,, \tag{3.25}$$

$$\begin{aligned}M^{E2}(\vec{p}\,',\vec{p};\vec{k})=&(i\mu/pp')(p-p')^{-2}B(p',p)\\&\times\left[(\vec{\mathscr{A}}\cdot\hat{p}\,')(\vec{k}\cdot\hat{p}\,')\frac{p}{p-p'}\left[1-\frac{3}{2}\frac{p-p'}{p}-i(n-n')\right]t(p\hat{p}\,',\vec{p})\right.\\&\left.+\tfrac{1}{2}(\vec{k}\cdot\hat{p}\,')\vec{p}\,'\times(\vec{\mathscr{A}}\times\hat{p}\,')\cdot\vec{\nabla}_{\vec{p}\,'}t(\vec{p}\,',\vec{p})+\tfrac{1}{2}(\vec{\mathscr{A}}\cdot\hat{p}\,')\vec{p}\,'\times(\vec{k}\times\hat{p}\,')\cdot\vec{\nabla}_{\vec{p}\,'}t(\vec{p}\,',\vec{p})\right]\,.\end{aligned} \tag{3.26}$$

In forming the sum $M=M^{E1}+M^{M1}+M^{E2}$ some simplification is achieved by expanding the triple vector products, taking advantage of certain cancellations, and then regrouping terms. We also make use of the relation (correct to the required accuracy)

$$(\vec{k}\cdot\hat{p}\,')\frac{p}{p-p'}=\frac{\vec{k}\cdot\vec{p}\,'}{\mu\omega}p'\left[1+\frac{3}{2}\frac{p-p'}{p}\right]$$

and observe that terms of second order in $\vec{k}\cdot\vec{p}\,'/\mu\omega$ may be neglected in this nonrelativistic calculation. We find

$$M(\vec{p}\,',\vec{p};\vec{k})=(i\mu/pp')(p-p')^{-2}B(\vec{p}\,',\vec{p};\vec{k})$$
$$\times\left[\frac{\omega}{\omega-\vec{k}\cdot\vec{p}\,'/\mu}\vec{\mathscr{A}}\cdot\vec{p}\,'t(p\hat{p}\,',\vec{p})+(p-p')\vec{p}\,'\times(\vec{\mathscr{A}}\times\hat{p}\,')\cdot\vec{\nabla}_{\vec{p}\,'}t(\vec{p}\,',\vec{p})\right.$$
$$\left.+(\vec{\mathscr{A}}\cdot\hat{p}\,')\vec{p}\,'\times(\vec{k}\times\hat{p}\,')\cdot\vec{\nabla}_{\vec{p}\,'}t(\vec{p}\,',\vec{p})\right], \tag{3.27}$$

where

$$B(\vec{p}\,',\vec{p};\vec{k})=B(p',p)\left[1+i\frac{n}{p}\vec{k}\cdot\hat{p}\,'\right]. \tag{3.28}$$

The bremsstrahlung matrix element is obtained by forming

$$\mathscr{M}(\vec{p}\,',\vec{p};\vec{k})=\frac{ie}{mc}\mu\omega[M(\vec{p}\,',\vec{p};\vec{k})+M(-\vec{p},-\vec{p}\,';\vec{k})]. \tag{3.29}$$

[The time-reversal property $t(-\vec{p},-\vec{p}\,')=t(\vec{p}\,',\vec{p})$ is useful here. Note also that in Eq. (3.27) $\omega=p^2/2\mu-p'^2/2\mu$ changes sign under the momentum interchange while $\vec{k}$ remains fixed.] The result is

$$\mathscr{M}(\vec{p}\,',\vec{p};\vec{k})=-\frac{e}{mc}\left[B(\vec{p}\,',\vec{p};\vec{k})\frac{\vec{\mathscr{A}}\cdot\vec{p}\,'}{\omega-\vec{k}\cdot\vec{p}\,'/\mu}t(p\hat{p}\,',\vec{p})-B(-\vec{p},-\vec{p}\,';\vec{k})\frac{\vec{\mathscr{A}}\cdot\vec{p}}{\omega-\vec{k}\cdot\vec{p}/\mu}t(\vec{p}\,',p'\hat{p})\right.$$
$$+\mu B(\vec{p}\,',\vec{p};\vec{k})\hat{p}\,'\times(\vec{\mathscr{A}}\times\hat{p}\,')\cdot\vec{\nabla}_{\vec{p}\,'}t(\vec{p}\,',\vec{p})+\mu B(-\vec{p},-\vec{p}\,';\vec{k})\hat{p}\times(\vec{\mathscr{A}}\times\hat{p})\cdot\vec{\nabla}_{\vec{p}}t(\vec{p}\,',\vec{p})$$
$$\left.+B(p',p)\frac{\vec{\mathscr{A}}\cdot\hat{p}\,'}{\omega}\vec{p}\,'\times(\vec{k}\times\hat{p}\,')\cdot\vec{\nabla}_{\vec{p}\,'}t(\vec{p}\,',\vec{p})+B(p,p')\frac{\vec{\mathscr{A}}\cdot\hat{p}}{\omega}\vec{p}\times(\vec{k}\times\hat{p})\cdot\vec{\nabla}_{\vec{p}}t(\vec{p}\,',\vec{p})\right]. \tag{3.30}$$

In the case where the emitted radiation is linearly polarized, so that $\vec{\mathscr{A}}$ is real, the result (3.30) may be expressed more concisely in terms of the t matrix evaluated at slightly shifted momenta. Thus we define

$$\vec{p}(\vec{k})=\vec{p}-(\mu\omega-\vec{k}\cdot\vec{p})\frac{\vec{\mathscr{A}}}{\vec{\mathscr{A}}\cdot\vec{p}}-\vec{k}, \tag{3.31a}$$

$$\vec{p}\,'(\vec{k})=\vec{p}\,'+(\mu\omega-\vec{k}\cdot\vec{p}\,')\frac{\vec{\mathscr{A}}}{\vec{\mathscr{A}}\cdot\vec{p}\,'}+\vec{k}. \tag{3.31b}$$

Introducing the relation

$$t(p\hat{p}\,',\vec{p})\cong t(\vec{p}\,',\vec{p})+(p-p')\hat{p}\,'\cdot\vec{\nabla}_{\vec{p}\,'}t(\vec{p}\,',\vec{p}),$$

we find, after a slight rearrangement of terms in Eq. (3.30),

$$\mathscr{M}(\vec{p}\,',\vec{p};\vec{k})=-\frac{e}{mc}\left[B(\vec{p}\,',\vec{p};\vec{k})\frac{\vec{\mathscr{A}}\cdot\vec{p}\,'}{\omega-\vec{k}\cdot\vec{p}\,'/\mu}t[\vec{p}\,'(\vec{k}),\vec{p}]-B(-\vec{p},-\vec{p}\,';\vec{k})\frac{\vec{\mathscr{A}}\cdot\vec{p}}{\omega-\vec{k}\cdot\vec{p}/\mu}t[\vec{p}\,',\vec{p}(\vec{k})]\right]. \tag{3.32}$$

Note that by virtue of the relations

$$p^2(\vec{k})/2\mu=p'^2/2\mu,\quad p'^2(\vec{k})/2\mu=p^2/2\mu, \tag{3.33}$$

the t-matrix elements in Eq. (3.32) are on the energy shell.

IV. SUMMARY AND CONCLUSION

Let us recall that the low-frequency approximation for the bremmstrahlung matrix element $\mathscr{M}(\vec{p}\,',\vec{p};\vec{k})$, either in the version (3.30) or in the form (3.32) to which it reduces when the photon is linearly polarized, must be combined with Eq. (2.9) to account for the effect of the recoil of the target. The result so obtained provides a generalization of the Feshbach-Yennie approximation [see Eq. (21) of Ref. 2], reducing to it when the parameters n and n' are set equal to zero, i.e., when the effect of the long-range Coulomb tail is ignored. A low-frequency approximation for single-photon bremsstrahlung had been derived earlier,[4] but only the electric-dipole contribution was included. Here we have included magnetic-dipole and electric-quadrupole contributions as well. Since higher-order multipoles introduce corrections of order v^2/c^2, they cannot be consistently included in a nonrelativistic formulation of the problem. The effect of the Coulomb interaction is to change the analytic form of the low-frequency expansion, introducing terms which depend logarithmically on the photon frequency. The existence of these "anomalous" terms should be recognized in any attempt to represent experimentally determined bremsstrahlung cross sections, over a range of low frequencies, in some analytic form. By virtue of the inclusion of $M1$ and $E2$ contributions the low-frequency approximation derived here is correct to order v/c.

Perhaps the most interesting feature of the Feshbach-Yennie version of the low-frequency approximation for bremsstrahlung, one which is maintained in the generalization derived here, is that it is expressed in terms of the physical, elastic scattering amplitude at the incident and final energies. This result remains valid even in the presence of narrow resonances in the scattering. Thus the first term in Eq. (3.32) represents the amplitude for scattering followed by the emission of the photon, while the second term accounts for the process in which the photon is radiated prior to the collision. As emphasized by Feshbach and Yennie,[2] the interference between these two amplitudes contains information about the scattering which is not available from a study of the radiationless process alone. Specifically, it allows for a separate determination of both the nonresonant contribution to the cross section associated with the direct reaction process and the resonant contribution arising from the formation of a compound intermediate state. There have been attempts recently to test the validity of the Feshbach-Yennie approximation by comparison with experimentally determined cross sections.[9] It would be interesting to see how inclusion of the Coulomb modifications derived here affects the comparison between theory and experiment. These modifications are contained in the B functions defined in Eqs. (3.16) and (3.28), and their deviation from unity provides a measure of the importance of the Coulomb effect.

In earlier work on Coulomb scattering in the presence of a laser field[10] a low-frequency approximation for the transition amplitude was obtained. It took the form of a product of two factors, one representing the effect of the laser field in initial and final states and the other, given by an expression similar to that shown in Eq. (2.10) above, representing the emission or absorption of a single laser photon during the collision. We were able at that time to obtain only the leading term in an expansion of the single-photon bremsstrahlung amplitude in powers of the frequency. (The standard method of derivation based on the nonrelativistic Lippmann-Schwinger equation provides a first-order correction term but is valid only for potentials of short range. Relativistic methods, which make use of gauge invariance and analyticity properties,[1] are similarly restricted since the analyticity assumptions break down for Coulomb scattering.) The method of asymptotic evaluation in configuration space, introduced subsequently in Ref. 4 and improved on in the present work, provides a more effective calculational procedure than that adopted earlier[10] and has allowed for the determination of higher-order correction terms. By combining the results of the present work with the theory developed in Ref. 4—specifically the approximate representation of the transition amplitude appearing in Eq. (2.28) of Ref. 4—we arrive at an estimate of the stimulated bremsstrahlung amplitude which includes the $M1$ and $E2$ contributions that had been neglected previously. It has been pointed out[11,12] that resonance studies of the type suggested by Feshbach and Yennie may be considerably more effective when applied to stimulated bremsstrahlung processes since one has a measure of control over the intensity, frequency, and polarization properties of the external field. The more accurate approximation derived here may prove useful in such studies.

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