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Classical limit of an induced harmonic oscillation with radiation damping

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The previously proposed method to obtain classical limits of quantized fields using von Neumann—lattice coherent states as basis states for the projection form of the Schrödinger equation is modified in order to take into account the reaction of the radiation field to the charge in a compact manner. The modification is based on the work of Glauber on the time evolution of coherent states. To illustrate the method, a charged harmonic oscillator with and without initial external electromagnetic field is investigated.

I. INTRODUCTION

In our previous paper[1] (hereafter referred to as I) we have formulated a method to obtain classical limits of quantized fields using von Neumann—lattice coherent states (VNLC) as basis states for the projection form of the Schrödinger equation. We applied the method to derive the semiclassical Schrödinger equation for a charged particle, with the electromagnetic field treated as a c-number field, from the fully quantized theory.

It is the purpose of the present paper to modify the previous method in order to take into account the reaction of the radiation field to the charged particle in a compact manner. The modification is based on the work of Glauber[2,3] on the time evolution of coherent states.

We shall apply this method to show how one can derive the classical description of a charged harmonic oscillator with radiation damping from the fully quantized description of the system. Although this is only a specific example to illustrate the method, this example itself is of great interest.

As is well known, the "lifetime" of a quantum-mechanical charged harmonic oscillator in the nth excited state due to the electromagnetic radiation damping is

$$\tau_n = \frac{1}{n\Gamma} \quad \text{with} \quad \Gamma = \frac{2e^2\nu^2}{3mc^3}\,, \tag{1.1}$$

where m is the mass, ν is the frequency, and e is the charge of the oscillator. This means that the lifetime decreases proportionally as $1/n$ when the excitation energy is increased. In apparent contradiction to this, the lifetime of the corresponding classical oscillator is

$$\tau = \frac{1}{\Gamma}\,, \tag{1.2}$$

which is independent of the energy of the oscillator. Such an apparent discrepancy is, of course, due to the different definitions of lifetime in the two cases. In the quantum-mechanical case, the lifetime is defined as the decay time of the nth excited state to the $(n-1)$th excited state. In the classical case, the lifetime is defined as the time in which the excitation energy of the oscillator is reduced approximately to a half of the initial value. That is, qualitatively a highly excited harmonic oscillator (classical limit case) has to make transitions approximately n times until its original excitation energy is reduced to half of the initial value. This can be shown, i.e., Eq. (1.2), also more quantitatively by a statistical consideration.

But another discrepancy remains which cannot be resolved so simply. If one makes a Fourier analysis of the electromagnetic radiation of a damped classical harmonic oscillator, the frequency width of this radiation is given by

$$\Delta\omega \approx \frac{\Gamma}{2\pi} \tag{1.3}$$

instead of

$$\Delta\omega \approx n\frac{\Gamma}{2\pi}\,. \tag{1.4}$$

Relation (1.4) would follow from a simple application of the energy-lifetime uncertainty principle; that is, as already mentioned above, the average lifetime of a quantum-mechanical harmonic oscillator state is $1/n\Gamma$ and therefore its energy width is

$$\Delta E = \Delta\omega\hbar \approx n\Gamma\hbar \ .$$

That the electromagnetic radiation of a damped classical charged oscillator has such a reduced frequency width is connected with the fact that in the classical limit many different oscillator states radiate coherently in contrast to the radiation of low quantum oscillator excitations. A detailed understanding of this reduced frequency width can only be obtained from a consistent quantum-mechanical derivation of the classical limit.

The problem of the quantum-mechanical charged oscillator interacting with its radiation field has been investigated by many authors,[4] but we think that our method, which we shall present here, is much more compact and can be straightforwardly extended to the case where an initial external electromagnetic field excites the charged oscillator. In this way the external field can also be included in the correct quantum-mechanical description of the total system.

In Sec. II, the formalism is briefly outlined avoiding overlaps with the previous paper and clarifying the new points based on the work of Glauber. Also in Sec. II, the Hamiltonian of the system is discussed and a set of equations for the time-dependent complex eigenvalues for the basis states is derived from that Hamiltonian by making use of the projection method. Starting with the set of equations for the complex eigenvalues, in Sec. III the radiation damping of an initially displaced charged oscillator is treated. It is explicitly shown how the corresponding classical description can be obtained starting from the fully quantized treatment. In Sec. IV, an induced radiation of the charged oscillator is treated. That is, initially the charged oscillator is in its ground state and a nonvanishing (expectation value) electromagnetic field exists. For this case our method shows a superiority to other methods like Wigner-Weisskopf type approximations. The appearance of two damping modes are examined. In Sec. V the results are discussed and concluding remarks are given.

II. DERIVATION OF THE EQUATION OF MOTION

As in the previous paper (I), we start with the time-dependent Schrödinger equation formulated as a projection equation

$$\left\langle \delta\psi \left| \left[H_{\text{int}} + \frac{\hbar}{i}\frac{\partial}{\partial t} \right] \right| \psi \right\rangle = 0 \ . \tag{2.1}$$

This equation is written in the interaction picture with regard to both the charged harmonic oscillator and the radiation field in contrast to the mixed picture used in I.

For simplicity, we consider a charged harmonic oscillator moving only in the z direction. Nothing essential will be lost by such a simplification. It will only make the formulas simpler. The interaction Hamiltonian H_{int} given in Eq. (2.3) of I then assumes the form

$$H_{\text{int}}(t) = \frac{-e}{mc}\vec{A}(t)\cdot\vec{p}(t) = \frac{-e}{mc}A_z(t)p_z(t) \ , \tag{2.2}$$

where the field operator $A_z(t)$ is the z component of the electromagnetic vector potential $\vec{A}(t)$ and the operator $p_z(t)$ is the z component of the momentum of the particle with the charge e and the mass m. Here we have neglected $\vec{A}^2$ terms. In the following discussions we shall consider only the lowest-order term in the coupling constant, because we are interested in the radiation damping effects, where the lowest-order term plays an essential role. Introducing the operators B and $B^\dagger$, which satisfy the commutation relation

$$[B,B^\dagger] = 1 \ , \tag{2.3}$$

we can write $p_z(t)$ as

$$p_z(t) = i\left[\frac{m\hbar\nu}{2}\right]^{1/2}(B^+e^{i\nu t} - Be^{-i\nu t}) \ , \tag{2.4}$$

where ν is the frequency of the charged oscillator. For the following considerations we further assume that the frequency is so small that the wavelength of the emitted radiation field is much larger than the spatial dimensions of the harmonic oscillator.[5] This means that only dipole waves are emitted and $\vec{A}(\vec{r},t)$ can be described as a superposition of dipole waves with different frequencies ω_n[6]:

$$\vec{A}(\vec{r},t) = \sum_{n=1}^{\infty} c\left[\frac{3\hbar}{2\omega_n L}\right]^{1/2}(a_n e^{-i\omega_n t} + a_n^\dagger e^{i\omega_n t}) \times \text{trans}\left[\vec{\epsilon}_n \frac{\sin(k_n r)}{r}\right] \ , \tag{2.5}$$

where ω_n are the frequencies of dipole waves and the operators a_n and $a_n^\dagger$ satisfy the commutation relations

$$[a_n, a_{n'}^{\dagger}] = \delta_{n,n'} \ . \tag{2.6}$$

The polarization vectors $\vec{\epsilon}_n$ in Eq. (2.5) are the unit vectors lying in the z direction due to the oscillation direction of the harmonic oscillator. The notation trans($\cdots$) means to take the transverse part of the vector field ($\cdots$) and can be be written explicitly as

$$\text{trans}[\vec{\epsilon} j_0(kr)] = k\left[\vec{\epsilon} + \frac{1}{k^2}(\vec{\epsilon}\cdot\vec{\nabla})\vec{\nabla}\right] j_0(kr) = \vec{\epsilon}\left[j_0(kr) - \frac{1}{kr}j_1(kr)\right] - \vec{e}_r\epsilon_r j_2(kr) \ , \tag{2.7}$$

where j_0, j_1, and j_2 are spherical Bessel functions and $\vec{e}_r\epsilon_r$ is the radial component of $\vec{\epsilon}$. In Eq. (2.5), L is the radius of the normalization sphere and the wave number k_n takes the values

$$k_n = \frac{\omega_n}{c} = \frac{\pi n}{L} \quad (n = 1, 2, \ldots) \ . \tag{2.8}$$

The norm factor $c\sqrt{3\hbar/(2\omega_n L)}$ causes the free Hamiltonian of the electromagnetic field to become

$$H_{\text{el}} = \frac{1}{8\pi}\int (\vec{\mathrm{E}}^2 + \vec{\mathrm{B}}^2) d^3x = \frac{1}{8\pi}\int \left[\frac{1}{c^2}\left[\frac{\partial}{\partial t}\vec{\mathrm{A}}\right]^2 + (\vec{\nabla}\times\vec{\mathrm{A}})^2\right] d^3x = \sum_{n=1}^{\infty} \hbar\omega_n (a_n^{\dagger} a_n + \tfrac{1}{2}) \tag{2.9}$$

as it must be due to the commutation relations (2.6). For small r values, Eq. (2.7) has the form

$$\text{trans}\left[\vec{\epsilon}_n \frac{\sin(k_n r)}{r}\right] = \tfrac{2}{3} k_n \vec{\epsilon}_n + O(k_n r) \ . \tag{2.10}$$

Thus, using (2.10), the long wavelength approximation discussed at the beginning of this section gives the interaction Hamiltonian

$$H_{\text{int}}(t) = \left[\frac{-e}{mc}\right] i \left[\frac{m\hbar v}{2}\right]^2 \sum_{n=1}^{\infty} \left[\frac{2\hbar\omega_n}{3}\right]^{1/2} (a_n e^{-i\omega_n t} + a_n^{\dagger} e^{i\omega_n t})(B^{\dagger} e^{ivt} - B e^{-ivt}) \ . \tag{2.11}$$

This $H_{\text{int}}(t)$ can be further simplified by introducing the rotating-wave approximation,[7] namely, ignoring the fast oscillating terms with the phase factors $\exp[\pm i(\omega_n + v)t]$ and retaining only the terms with the phase factors $\exp[\pm i(\omega_n - v)t]$. With this approximation we find

$$H_{\text{int}}(t) \approx -i\hbar\epsilon \sum_{n=1}^{\infty} \sqrt{\omega_n}\{a_n B^{\dagger}\exp[-i(\omega_n - v)t] - a_n^{\dagger} B \exp[i(\omega_n - v)t]\} \ , \tag{2.12}$$

where

$$\epsilon = \frac{e}{c}\left[\frac{v}{3mL}\right]^{1/2} . \tag{2.13}$$

Next we must determine suitable basis states for the projection equation (2.1). As already discussed in I, the von Neumann–lattice coherent states (VNLC) are suitable as basis states for the electromagnetic field. In the present case we introduce such a VNLC basis also for the charged harmonic oscillator. The properties of VNLC are listed in Sec. III of I and here we repeat only the properties that we need for the present considerations. The VNLC for the nth mode electromagnetic field are denoted as $|d_n\rangle$ which has the properties

$$a_n|d_n\rangle = d_n|d_n\rangle, \quad \langle d_n|a_n^{\dagger} = \langle d_n|d_n^*, \quad d_n = \sqrt{\pi}(l_n + im_n) \ , \quad l_n, m_n = 0, \pm 1, \pm 2, \ldots \tag{2.14}$$

and we define

$$|\{d\}\rangle \equiv \prod_{n=1}^{\infty} |d_n\rangle \ . \tag{2.15}$$

Similarly, the VNLC for the charged harmonic oscillator are denoted as $|\Lambda\rangle$ which has the properties

$$B|\Lambda\rangle = \Lambda|\Lambda\rangle, \quad \langle\Lambda|B^{\dagger} = \langle\Lambda|\Lambda^*, \quad \Lambda = \sqrt{\pi}(S + iM) \ , \quad S, M = 0, \pm 1, \pm 2, \ldots \tag{2.16}$$

and we also define $|\Lambda,\{d\}\rangle$ as

$$|\Lambda,\{d\}\rangle \equiv |\Lambda\rangle\,|\{d\}\rangle \;. \tag{2.17}$$

With basis states (2.17) for the system we can make the following ansatz for the state $|\psi\rangle$ in the projection equation (2.1):

$$|\psi(t)\rangle = \sum_{\Lambda}\sum_{\{d\}} f(\Lambda,\{d\};t)\,|\Lambda,\{d\}\rangle \;, \tag{2.18}$$

where the sum notations are defined as

$$\sum_{\Lambda} \equiv \sum_{S=-\infty}^{\infty}\sum_{M=-\infty}^{\infty}\;,\quad \sum_{\{d\}} \equiv \sum_{d_1}\sum_{d_2}\cdots \tag{2.19}$$

with

$$\sum_{d_n} \equiv \sum_{l_n=-\infty}^{\infty}\sum_{m_n=-\infty}^{\infty}\;.$$

The variation $\langle\delta\psi|$ in Eq. (2.1) becomes

$$\langle\delta\psi| = \sum_{\Lambda'}\sum_{\{d'\}} \langle\{d'\},\Lambda'|\,\delta f^*(\Lambda',\{d'\};t)\;. \tag{2.20}$$

Substituting Eqs. (2.18) and (2.20) into Eq. (2.1) and taking into account that δf^* is a completely arbitrary variation, we can obtain the set of coupled equations

$$\sum_{\Lambda}\sum_{\{d'\}} \langle\{d'\},\Lambda'|H_{\rm int}(t)|\Lambda,\{d\}\rangle f(\Lambda,\{d\};t) + \sum_{\Lambda}\sum_{\{d\}}\langle\{d'\},\Lambda'|\Lambda,\{d\}\rangle\frac{\hbar}{i}\frac{\partial}{\partial t}f(\Lambda,\{d\};t)=0\;, \tag{2.21}$$

which must be satisifed for arbitrary $\Lambda'=\sqrt{\pi}(S'+iM')$ and $d_n'=\sqrt{\pi}(l_n'+im_n')$, where S',M',l_n', and m_n' are integers.

Equations (2.21) can be greatly simplified as follows: It is known that if a state, which is initially a coherent state $|\{d\}\rangle$, remains as a coherent state during the subsequent time evolution, then the Hamiltonian must have the form[2]

$$H=\sum_{j,k} h_{jk}(t)a_j^\dagger a_k + \sum_k [g_k(t)a_k^\dagger + g_k^*(t)a_k] + \beta(t) \tag{2.22}$$

with

$$h_{jk}=h_{kj}^*,\quad g_k=\text{CF},\quad\text{and}\quad \beta=\text{RF}\;, \tag{2.23}$$

where CF represents complex function and RF represents real function. Of course, in general, the above Hamiltonian does not guarantee the existence of such a coherent state.[3] A special case, which allows such a coherent state solution, is

$$H=\sum_{j,k} h_{jk}(t)a_j^\dagger a_k,\quad h_{jk}=h_{kj}^* \tag{2.24}$$

as we shall show explicitly at the end of this section.

As one can see from Eqs. (2.9) and (2.12), our total Hamiltonian

$$H_{\rm tot}=H_{\rm el}+H_{\rm int}+H_{\rm osc}$$

with

$$H_{\rm osc}=\hbar\nu(B^\dagger B+\tfrac{1}{2}) \tag{2.25}$$

is just a specific case of the Hamiltonian (2.24). Therefore, in order to solve the set of coupled equations (2.21), we consider at first a situation where for $t=0$ the system is described by a specific coherent state $|\Lambda(0),\{d(0)\}\rangle$. Here $\Lambda(t)$ and $\{d(t)\}$ are time-dependent eigenvalues of the coherent states. For such an initial state, we can write $|\psi(t)\rangle$ in Eq. (2.18) as

$$|\psi(t)\rangle=\sum_{\Lambda}\sum_{\{d\}} f(\Lambda,\{d\};t)\,|\Lambda,\{d\}\rangle = |\Lambda(t),\{d(t)\}\rangle\;, \tag{2.26}$$

which can represent an exact time evolution of the state with the Hamiltonian (2.25). If we find the time-dependent coherent states $|\Lambda(t),\{d(t)\}\rangle$ for arbitrary initial conditions $|\Lambda(0),\{d(0)\}\rangle$, then we can construct any solution for Eqs. (2.21) by linearly superposing those $|\Lambda(t),\{d(t)\}\rangle$.

Substituting (2.26) into (2.21) and using Eqs. (2.12), (2.14), and (2.16), we find the first term in the left-hand side (LHS) of Eq. (2.21) to be

$$-i\hbar\epsilon\sum_{n=1}^{\infty}\sqrt{\omega_n}\{d_n(t)\Lambda'^*\exp[-i(\omega_n-\nu)t]-d_n'^*\Lambda(t)\exp[i(\omega_n-\nu)t]\}\langle\Lambda',\{d'\}|\Lambda(t),\{d(t)\}\rangle \tag{2.27}$$

and the second term to be

$$i\hbar\left[\frac{-1}{2}\frac{\partial}{\partial t}|\Lambda(t)|^2+\Lambda^*\frac{\partial}{\partial t}\Lambda(t)+\sum_{n=1}^{\infty}\left[\frac{-1}{2}\frac{\partial}{\partial t}|d_n(t)|^2+d_n'^*\frac{\partial}{\partial t}d_n(t)\right]\right]\langle\Lambda',\{d'\}|\Lambda(t),\{d(t)\}\rangle\;. \tag{2.28}$$

We introduce (2.27) and (2.28) into the LHS of Eq. (2.21) and organize them as polynomials of Λ'^* and $d_n'^*$. Since the equation must hold for arbitrary values of these variables, each coefficient must vanish.

Thus we obtain

$$\frac{-i\hbar}{2}\frac{\partial}{\partial t}|\Lambda(t)|^2-\frac{i\hbar}{2}\sum_{n=1}^{\infty}\frac{\partial}{\partial t}|d_n(t)|^2=0\ , \tag{2.29}$$

$$\frac{\partial}{\partial t}\Lambda(t)=-\epsilon\sum_{n=1}^{\infty}\sqrt{\omega_n}d_n(t)\exp[-i(\omega_n-\nu)t]\ , \tag{2.30}$$

$$\frac{\partial}{\partial t}d_n(t)=\epsilon\sqrt{\omega_n}\Lambda(t)\exp[i(\omega_n-\nu)t]\ . \tag{2.31}$$

It is straightforward to show that Eq. (2.29) can be derived from Eqs. (2.30) and (2.31). Therefore, we have to consider only Eqs. (2.30) and (2.31). That Eq. (2.29) follows from the other equations is, in the present case, the direct proof for the fact that a coherent state $|\Lambda,\{d\}\rangle$ remains a coherent state during the time evolution with the Hamiltonian (2.25), because it shows that these equations do not contradict each other.

III. HARMONIC OSCILLATOR WITH RADIATION DAMPING

If we solve Eqs. (2.30) and (2.31) to find Λ and d_n as functions of time, then we can construct the time-dependent coherent states that describe the time evolution of the system and can also calculate the time-dependent expectation values for any observables of the system. In this section we consider the oscillation of a charged harmonic oscillator under the influence of its own radiation damping. For such a case, we have the following initial conditions at $t=0$:

$$\Lambda(0)=\Lambda_0\equiv|\Lambda_0|e^{i\phi}\ , \tag{3.1a}$$

$$d_n(0)=0\ . \tag{3.1b}$$

These conditions mean that, as far as the expectation values are concerned, at $t=0$ the charged harmonic oscillator is excited with its energy proportional to $|\Lambda_0|^2$ and no external electromagnetic field is present. We shall see this later again by calculating the expectation values for the relevant observables.

First, we eliminate $d_n(t)$ by integrating (2.31) and inserting the result into Eq. (2.30). With

$$d_n(t)=\epsilon\sqrt{\omega_n}\int_0^t\Lambda(\tau)\exp[i(\omega_n-\nu)\tau]d\tau \tag{3.2}$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\Lambda(\tau)&=-\epsilon^2\sum_{n=1}^{\infty}\omega_n\int_0^t\Lambda(\tau)\exp[i(\omega_n-\nu)(\tau-t)]d\tau\ ,\\ &=-\epsilon^2\sum_{n=1}^{\infty}\omega_n\int_{-t}^0\Lambda(\tau'+t)\exp[i(\omega_n-\nu)\tau']d\tau'\ .\end{aligned} \tag{3.3}$$

In order to evaluate the integral in the right-hand side (RHS) of (3.3), we use the fact that the coupling constant ϵ is so small that $\Lambda(t)$ changes very slowly with that t compared to $\exp[i(\omega_n-\nu)t]$. With this in mind and also introducing the convergence factor $e^{\beta\tau'}(\beta\to 0^+)$, which physically means that adiabatic switching on the electric charge of the oscillator at $t=0$, we obtain, for $\nu t\gg 1$,

$$\frac{\partial}{\partial t}\Lambda(t)\approx-\epsilon^2\Lambda(t)\sum_{n=1}^{\infty}\frac{\omega_n}{i(\omega_n-\nu)+\beta}\ . \tag{3.4}$$

The discrete sum $\sum_n$ in (3.4), as usual, can be substituted by a continuous integral as

$$\sum_{n=1}^{\infty}\to\frac{L}{\pi c}\int_0^{\infty}d\omega\ . \tag{3.5}$$

Then we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\omega_n}{i(\omega_n-\nu)+\beta}&=\frac{L}{\pi c}\int_0^{\infty}d\omega\frac{\omega}{i(\omega-\nu)+\beta}\\ &=\frac{L\nu}{c}-\frac{iL}{\pi c}\mathscr{P}\int_0^{\infty}d\omega\frac{\omega}{\omega-\nu}\ ,\end{aligned} \tag{3.6}$$

where $\mathscr{P}$ means the principal value of the integral.

Of course the above sum diverges. But the important fact is that the sum can be split into a finite real part and a diverging imaginary part in a unique manner. The imaginary part of (3.6) is proportional to the frequency shift and contributes for instance to the very small Lamb shift. The real part of (3.6) is proportional to the radiation damping.

In the present work we are interested in the radiation damping of the charged harmonic oscillator. Therefore we simply assume the frequency shift

$$\Delta\nu=\frac{e^2}{3mc^2}\frac{\nu}{\pi}\mathscr{P}\int_0^{\infty}d\omega\frac{\omega}{\omega-\nu} \tag{3.7}$$

is very small. Introducing

$$\Gamma \equiv \frac{2e^2\nu^2}{3mc^3} , \tag{3.8}$$

we can write (3.4) in the following form:

$$\frac{\partial}{\partial t}\Lambda(t) = \frac{-\Gamma}{2}\Lambda(t) + i\Delta\nu\Lambda(t) . \tag{3.9}$$

Here it should be noted that this is essentially the Wigner-Weisskopf approximation.[8] This equation can be solved immediately and we obtain

$$\Lambda(t) = \Lambda_0 \exp\left[-\frac{\Gamma}{2}t + i\Delta\nu t\right] , \tag{3.10}$$

where Λ_0 is given by the initial condition (3.1a). Inserting (3.10) into (2.29) and integrating it we find

$$d_n(t) = \frac{e}{c}\left[\frac{\nu\omega_n}{3mL}\right]^{1/2}\Lambda_0 \times \frac{e^{-\Gamma/2t}\exp[i(\omega_n - \nu + \Delta\nu)t] - 1}{i(\omega_n - \nu + \Delta\nu) - \Gamma/2} , \tag{3.11}$$

where we have used the initial condition (3.1b).

Using (3.10) and (3.11), we can now construct explicitly the time-dependent coherent state $|\Lambda(t),\{d(t)\}\rangle$ of (2.26) and evaluate the expectation values for various observables, which determine the physical properties of the charged harmonic oscillator with radiation damping. The expectation value for the oscillator amplitude,

$$R_z(t) = \left[\frac{\hbar}{2m\nu}\right]^{1/2}(B^\dagger e^{i\nu t} + Be^{-i\nu t}) \tag{3.12}$$

can be calculated by using (2.16), (2.26), (3.10), and (3.1a)

$$\langle\{d(t)\},\Lambda(t)\,|\,R_z(t)\,|\,\{d(t)\},\Lambda(t)\rangle = de^{(-\Gamma/2)t}\cos(\nu t + \phi) , \tag{3.13}$$

where we have introduced

$$d \equiv \left[\frac{2\hbar}{m\nu}\right]^{1/2} |\Lambda_0| \tag{3.14}$$

and neglected the frequency shift $\Delta\nu$ for simplicity. Such a neglection of $\Delta\nu$ does not change the essential structure of our formulas. We shall also put $\phi = 0$ from now on in this section to simplify the equations.

From (3.13) we can see that the expectation value $\langle R_z\rangle$ shows the motion of the classical charged harmonic oscillator with the classical value of the radiation damping factor given by (3.8).[9] As already discussed in I, for very high excitations of the oscillator, i.e., $|\Delta_0| \gg 1$, the root-mean-square relative deviations ΔR_z of $\langle R_z\rangle$ become very small. In this limit, the fully quantum-mechanical description of the radiation damping of the charged harmonic oscillator approaches the classical description. We shall come back to this point later.

The expectation value for the electromagnetic potential $\vec{A}$ can be obtained from (2.5) and (2.26) as

$$\langle\vec{A}(\vec{r},t)\rangle = \sum_{n=1}^{\infty} c\left[\frac{3\hbar}{2\omega_n L}\right]^{1/2} \text{trans}\left[\vec{\epsilon}_n \frac{\sin(k_n r)}{r}\right]\langle\{d(t)\},\Lambda(t)\,|\,(a_n e^{-i\omega_n t} + a_n^\dagger e^{i\omega_n t})\,|\,\Lambda(t),\{d(t)\}\rangle . \tag{3.15}$$

Using (2.14), (2.15), and (3.11) and replacing the discrete sum by the corresponding integral, we find

$$\langle\vec{A}\rangle = \frac{e}{\pi c}\left[\frac{\hbar\nu}{2m}\right]^{1/2}\Lambda_0\int_0^\infty d\omega \frac{\exp\left[-\frac{\Gamma}{2}t - i\nu t\right] - e^{-i\omega t}}{-\frac{\Gamma}{2} + i(\omega - \nu)} \text{trans}\left[\vec{\epsilon}_z \frac{\sin[(\omega/c)r]}{r}\right] + \text{c.c.}, \tag{3.16}$$

where the polarization unit vectors $\vec{\epsilon}_z$ are the unit vectors in the z direction [see remark to Eq. (2.5)]. We are interested in the expectation value $\langle\vec{A}(\vec{r},t)\rangle$ for large r and for very large t for which all switching-on effects have died down, i.e., $[t - (r/c)]\omega \gg 2\pi$. For that purpose, we need the asymptotic behavior of

$$\text{trans}\left[\vec{\epsilon}_z \frac{\sin[(\omega/c)r]}{r}\right]$$

for $r \to \infty$. From (2.7), by retaining only the $1/r$ terms, we get

$$\text{trans}\left[\vec{\epsilon}_z \frac{\sin[(\omega/c)r]}{r}\right] \approx \left[\vec{\epsilon}_z - \frac{\vec{r}}{r}\cos\theta\right] \frac{\sin[(\omega/c)r]}{r} , \tag{3.17}$$

where $r\cos\theta \equiv \vec{r}\cdot\vec{\epsilon}_z$. Taking into account that $[t-(r/c)]\omega \gg 2\pi$, $\Gamma \ll \nu$, and $r \ll ct$, the ω integral in (3.16) with (3.17) can be asymptotically evaluated by integration in the complex ω plane:

$$\langle \vec{A}(\vec{r},t)\rangle = \frac{-\nu P}{c}\exp\left[-\frac{\Gamma}{2}\left[t-\frac{r}{c}\right]\right]\frac{\sin[\nu(t-r/c)r]}{r}\left[\vec{\epsilon}_z - \frac{\vec{r}}{r}\cos\theta\right] + O\left[\frac{1}{r^2}\right] , \tag{3.18}$$

where

$$P \equiv ed . \tag{3.19}$$

The expectation value $\langle \vec{A}\rangle$ has the usual form of the classical electromagnetic dipole-radiation field created by a damped charged oscillator.

It should be noted that (3.16) contains the incoming wave terms with the factor $\exp[\pm i\omega(t+r/c)]$, which cancel each other in the integration over ω. This must be so, because the initial conditions (3.1) allow only outgoing electromagnetic waves. Furthermore, again by integration in the complex ω plane, it can be shown that for $t-r/c<0$ the expectation value $\langle \vec{A}(\vec{r},t)\rangle$ vanishes. This is also obvious from the initial conditions (3.1), which state that at $t=0$ no electromagnetic field is present.

Next, we examine how the expectation value for the energy of the harmonic oscillator, as well as the radiation field, changes with regard to large times, i.e., $t \gg 1/\Gamma$. In order to find the expectation value for the energy of the oscillator as a function of time, we have to calculate

$$\langle H_{\text{osc}}\rangle = \nu\hbar\langle \Lambda(t),\{d(t)\} \,|\, (B^\dagger B + \tfrac{1}{2}) \,|\, \Lambda(t),\{d(t)\}\rangle . \tag{3.20}$$

Using the properties of coherent states given by (2.16) and the explicit form of $\Lambda(t)$ given by (3.10) and neglecting the zero-point energy, we obtain

$$\langle H_{\text{osc}}\rangle = \nu\hbar |\Lambda_0|^2 e^{-\Gamma t} = \frac{m\nu^2 d^2}{2} e^{-\Gamma t} , \tag{3.21}$$

from which we find the energy decrease per second,

$$\frac{d}{dt}\langle H_{\text{osc}}\rangle = -\Gamma\langle H_{\text{osc}}\rangle = -\frac{e^2\nu^4 d^2}{3c^3} e^{-\Gamma t} . \tag{3.22}$$

The expectation value of the energy of the nth mode radiation field, whose Hamiltonian is

$$H_n = \hbar\omega_n (a_n^\dagger a_n + \tfrac{1}{2}) ,$$

can also be calculated similarly. Neglecting the zero-point energy and using (2.14) and the explicit form of $d_n(t)$ given by (3.11), we find

$$\langle H_n\rangle = \hbar\omega_n |d_n(t)|^2$$

$$= \frac{e^2\nu\hbar\omega_n^2 |\Lambda_0|^2}{3mc^2 L}\left[\frac{1+e^{-\Gamma t} - e^{-(\Gamma/2)t}\{\exp[i(\omega_n-\nu)t] + \exp[-i(\omega_n-\nu)t]\}}{(\omega_n-\nu)^2 + \Gamma^2/4}\right] . \tag{3.23}$$

For large t, for which all switching-on effects disappear, the energies $\hbar\omega_n$ of the different modes can be put approximately equal to $\hbar\nu$. Of course, if we do not replace $\hbar\omega_n$ in the overall factor of (3.23) by $\hbar\nu$, the sum over n diverges. This divergence, which appears in higher-order terms in power of the coupling constant, has the same origin as the divergence discussed earlier [see remarks to Eqs. (3.6) and (3.7)] and therefore can be disregarded again. For such a case we can calculate the expectation value for the entire energy of the radiation field as a function of t. By making use of complex ω integration, we obtain

$$\langle H_{\rm el}\rangle=\sum_{n=1}^{\infty}\langle H_n\rangle=\frac{L}{\pi c}\int_0^{\infty}d\omega\langle H(\omega)\rangle\approx\nu\hbar|\Lambda_0|^2(1-e^{-\Gamma t})=\frac{m\nu^2d^2}{2}(1-e^{-\Gamma t}) , \tag{3.24}$$

from which we immediately find

$$\frac{d}{dt}\langle H_{\rm el}\rangle=\frac{e^2\nu^4d^2}{3c^3}e^{-\Gamma t} . \tag{3.25}$$

From (3.22) and (3.25), we see that the expectation value for the total energy is conserved, as it should be:

$$\langle H_{\rm tot}\rangle\approx\langle H_{\rm osc}\rangle+\langle H_{\rm el}\rangle\approx{\rm const} . \tag{3.26}$$

Furthermore, we can see that for large t, for which the energies $\hbar\omega_n$ of the different modes can be put approximately equal to $\hbar\nu$, the energy-conservation law in the form of $dE/dt=0$ becomes equivalent to the validity of (2.29).

The expectation value for the average radiation energy per second can also be calculated by using the Poynting vector $\vec{S}$. Using (3.18) and (3.24) we find for large oscillation amplitudes, where contributions from the noncommutability of the operators $\vec{E}$ and $\vec{B}$ can be neglected,

$$\left[\frac{d}{dt}\langle H_{\rm el}\rangle\right]_{\rm time\ av}=\int[\langle\vec{S}\rangle]_{\rm time\ av}\cdot\vec{n}d\Omega_R\approx\frac{c}{4\pi}\int[\langle\vec{E}\rangle\times\langle\vec{B}\rangle]_{\rm time\ av}\cdot\vec{n}\,d\Omega_R$$

$$\approx\frac{e^2\nu^4d^2}{3c^3}\exp\left[-\Gamma\left[t-\frac{R}{c}\right]\right] . \tag{3.27}$$

This is in agreement with (3.24) except for the factor $e^{(\Gamma/c)R}$. This factor describes the retardation of the radiation field at large radius $R\gg2\pi c/\omega$, where the surface integration is carried out.

Now let us consider some essential properties of the electromagnetic radiation field. From (3.18) and the discussion following it, we find the time space part $X(r,t)$, i.e., the part that has $(t-r/c)$ dependence, of the electromagnetic potential for large t as

$$X(r,t)=\begin{cases}\exp\left[-\frac{\Gamma}{2}\left[t-\frac{r}{c}\right]\right]\cos\left[\nu\left[t-\frac{r}{c}\right]\right] & \text{for } r<ct\\ 0 \text{ for } r>ct ,\end{cases} \tag{3.28}$$

where the switching-on effects have been neglected. A Fourier analysis of this $X(r,t)$ immediately gives the frequency width of

$$\Delta\omega=\Gamma , \tag{3.29}$$

which is independent of the excitation amplitude of the oscillator.

It is interesting to calculate the number of photons which are emitted during one oscillation of the charged oscillator. Through the use of (3.22) such a number N can be calculated as

$$N=\frac{1}{\nu\hbar}\left[\frac{2\pi}{\nu}\right]\left|\frac{d}{dt}\langle H_{\rm osc}(t)\rangle\right|$$

$$=\frac{2\pi}{\hbar\nu^2}\Gamma\langle H_{\rm osc}(t)\rangle . \tag{3.30}$$

Through the use of (3.21), this can be written as

$$N\approx\frac{4\pi}{3}\frac{e^2}{\hbar c}\frac{1}{2}\left[\frac{\nu d}{c}\right]^2,\quad e^{-\Gamma t}\approx1 . \tag{3.31}$$

Of course, this result can also be obtained from the classical Larmor's formula. From (3.31) it is obvious that for a harmonic oscillator with the change e (electron charge) it is necessary to make many oscillations to emit one photon even if the amplitude of the oscillator is very large. In other words, if we consider an atomic electron, even a highly excited electron must make many circuits in order to emit one photon. That is, even for such a highly excited electron, the kinetic energy is so small that a quantum-mechanical treatment is necessary to describe the electromagnetic emission process. Only for electrons with large relativistic energies, such as electrons in synchrocyclotrons, is an appropriate classical treatment allowed. In the nonrelativistic region, the classical limit can be reached for a charge larger than $\sim 50\, e$.

As we have seen in this section, the difference between the classical and the quantum-mechanical description of the radiation field including the intermediate description can be jointly expressed by the coherent states $|\{d\}\rangle$. In the classical limit the corresponding coherent states $|\{d\}\rangle$ with large $\{d\}$ values, i.e., $\sum_n |d_n|^2 \gg 1$, describes (relatively well-localized classical) wave packets with relatively well-defined light quanta number for each mode, which are therefore relatively little influenced by a measurement of the electromagnetic field. On the other hand, for small $\{d\}$ values, i.e., $\sum_n |d_n|^2 \sim 1$, the corresponding coherent states have relative root-mean-square deviations for the field expectation values which are of the same order or even larger than these expectation values themselves. Consequently, all coherence effects vanish. In other words, a measurement of the electromagnetic field influences this field very strongly. This shows that in such a case, i.e., the quantum-mechanical case, the electromagnetic radiation can only be described by probability distributions of the emitted light quanta.

IV. INDUCED HARMONIC OSCILLATOR WITH RADIATION DAMPING

In this section we consider the charged harmonic oscillator under the influence of an external electromagnetic field. We assume that at $t=0$ the expectation value for the amplitude of the oscillator is zero and the expectation value for the mode of the electromagnetic potential $\vec{A}$ has a nonvanishing value. We assume that the expectation values for all the other modes of $\vec{A}$ are zero at $t=0$. Thus, as in Sec. III, the corresponding time evolution of the state of the system can be obtained by solving Eqs. (2.30) and (2.31) to find $\Lambda(t)$ and the $d_n(t)$'s. Then, the only differences from Sec. III are the initial conditions

$$\Lambda(0)=0\ , \tag{4.1}$$

$$d_j(0)=d_0\neq 0\ , \tag{4.2}$$

$$d_n(0)=0,\quad n\neq j\ . \tag{4.3}$$

In order to solve Eqs. (2.30) and (2.31) for the initial condition (4.1)–(4.3) we modify the approximation scheme introduced in Sec. III as follows: First, we treat the jth mode separately from other modes. Second, we assume that all other modes $(n\neq j)$ contribute to the damping factor Γ defined by (3.8). The second step can be justified because the omission of a single mode d_j from the sum (3.6) does not affect Γ. Thus, we have the following set of equations:

$$\frac{\partial}{\partial t}\Lambda(t)=-\frac{\Gamma}{2}\Lambda(t)-\epsilon\sqrt{\omega}\,d_j(t)\exp[-i(\omega-\nu)t],\quad \omega\equiv\omega_j \tag{4.4}$$

$$\frac{\partial}{\partial t}d_j(t)=\epsilon\sqrt{\omega}\,\Lambda(t)\exp[i(\omega-\nu)t]\ . \tag{4.5}$$

In Eq. (4.4) we have already eliminated the modes $d_{n\neq j}$, analogously to (3.9), which satisfy the following equation [see Eq. (2.31)]:

$$\frac{\partial}{\partial t}d_n(t)=\epsilon\sqrt{\omega_n}\,\Lambda(t)\exp[i(\omega_n-\nu)t]\ . \tag{4.6}$$

The solutions of these equations for the initial condition (4.1)–(4.3) can be easily obtained:

$$d_j(t)\equiv d(t)\approx d_0\left[\frac{\kappa}{-\Gamma/2+i(\omega-\nu)}\exp\left[-\frac{\Gamma}{2}t+i(\omega-\nu)t\right]+e^{-\kappa t}\right], \tag{4.7}$$

$$\Lambda(t)\approx\frac{\kappa d_0}{\epsilon\sqrt{\omega}}\{e^{-(\Gamma/2)t}-\exp[-\kappa t-i(\omega-\nu)t]\}\ , \tag{4.8}$$

where

$$\kappa \equiv \frac{\epsilon^2 \omega}{\Gamma/2 - i(\omega - \nu)} \qquad (4.9)$$

with

$$|\kappa| \ll \Gamma$$

The results (4.7) and (4.8) show the appearance of two different damping modes. One is characterized by Γ and another is characterized by κ. We shall call them the Γ mode and the κ mode, respectively. The Γ mode comes from the interaction between the charged harmonic oscillator and its own radiation field. The κ mode comes essentially from the interaction between the charged harmonic oscillator and the external electromagnetic field, whose energy is stored in the normalization volume with the radius L. In contrast to Γ, which is independent of L, the damping factor κ depends on L. For large L values, κ becomes very small. If L is increased keeping the energy density of the external field constant, then the "relative damping" of the external field energy decreases proportionally to L^{-3}. Here the relative damping is the ratio of the absorbed field energy per second to its total energy. This is understandable because in the case of constant energy density the energy of the external field increases with L^3.

For the case where the stored energy in the external field is very large and where all switching-on effects disappeared, i.e., $t \gg 1/\Gamma$, we can neglect the Γ mode in (4.7) and (4.8) to find

$$d(t) \approx d_0 e^{-\kappa t} \qquad (4.10)$$

and

$$\Lambda(t) \approx -\frac{\kappa d_0}{\epsilon\sqrt{\omega}} \exp[-\kappa t - i(\omega - \nu)t] \ . \qquad (4.11)$$

Using the results (4.7) and (4.8) or (4.10) and (4.11) for $\Lambda(t)$ and $d(t)$, we can construct the corresponding state vector of the system in the form of (2.26), from which expectation values for various observables can be evaluated.

First, we examine the relation between the expectation value for the displacement of the charged oscillator and the expectation value of the electric field for $r=0$. For that purpose we have to consider only the complex electromagnetic potential component $A_z^{\text{compl}}(r=0,t)$ and the complex position operator $R_z^{\text{compl}}(t)$ that are defined as

$$A_z^{\text{compl}}(r=0,t) \equiv \left[\frac{2\hbar\omega}{3L}\right]^{1/2} a^\dagger e^{i\omega t}, \quad a^\dagger \equiv a_j^\dagger \qquad (4.12)$$

$$R_z^{\text{compl}}(t) \equiv \left[\frac{\hbar}{2m\nu}\right]^{1/2} B^\dagger e^{i\nu t} \ , \qquad (4.13)$$

where we consider only the jth mode of A_z. The expression (4.12) can be obtained from (2.5) using (2.10). The expectation values of these complex operators correspond to the absolute values and the phase shifts of the classical observables involved.

For $\Gamma t \gg 1$, using (4.10) and (4.11) for $d(t)$ and $\Lambda(t)$, we find the expectation values for the operators A_z^{compl} and R_z^{compl} as

$$\langle A_z^{\text{compl}}(r=0,t)\rangle \approx \left[\frac{2\hbar\omega}{3L}\right]^{1/2} d_0^* \exp[-(\kappa^* - i\omega)t] \qquad (4.14)$$

and

$$\langle R_z^{\text{compl}}(t)\rangle \approx -P_0 \exp[-(\kappa^* - i\omega)t] \ , \qquad (4.15)$$

where

$$P_0 \equiv \left[\frac{\hbar}{2m\nu}\right]^{1/2} \frac{\kappa^* d_0^*}{\epsilon\sqrt{\omega}}$$

$$= \epsilon\sqrt{\omega} \left[\frac{\hbar}{2m\nu}\right]^{1/2} \frac{d_0^*}{\Gamma/2 + i(\omega - \nu)} \ . \qquad (4.16)$$

The expectation value for the complex electric field is then

$$\langle E_z^{\text{compl}}(r=0,t)\rangle = \frac{-1}{c}\left\langle \frac{\partial}{\partial t} A_z^{\text{compl}}(r=0,t)\right\rangle \approx \frac{-1}{c}\left[\frac{2\hbar\omega}{3L}\right]^{1/2} d_0^*(i\omega - \kappa^*) \exp[-(\kappa^* - i\omega)t] \ . \qquad (4.17)$$

From (4.15) and (4.17) we find

$$\frac{\langle R_z^{\text{compl}}(t)\rangle}{\langle E_z^{\text{compl}}(t)\rangle} \approx \frac{e}{2m\omega} \frac{1}{[(\nu - \omega) + i\Gamma/2]} \ . \qquad (4.18)$$

This is just the result obtained in the classical treatment of the charged harmonic oscillator excited by an external field and with radiation damping.

In order to evaluate the expectation value $\langle \vec{A}(\vec{r},t)\rangle$ of the radiation field it is necessary to find explicitly $d_n(t)$ $[n\neq j]$ which satisfy (4.6). Substituting the explicit form of $\Lambda(t)$ given by (4.8) into (4.6), we readily find (here again we neglect $\Delta\nu$)

$$d_n(t)=\frac{\epsilon^2 d_0\sqrt{\omega_n\omega}}{\Gamma/2-i(\omega-\nu)}\left[\frac{\exp\left[-\frac{\Gamma}{2}t+i(\omega_n-\nu)t\right]-1}{-\frac{\Gamma}{2}+i(\omega_n-\nu)}+\frac{\exp[-\kappa t+i(\omega_n-\omega)t]-1}{\kappa-i(\omega_n-\omega)}\right]$$

$$\approx d_0\kappa\left[\frac{\omega_n}{\omega}\right]^{1/2}\left[\frac{1}{\Gamma/2-i(\omega_n-\nu)}-\frac{1}{\kappa+i(\omega-\omega_n)}+\frac{\exp[-\kappa t-i(\omega-\omega_n)t]}{\kappa+i(\omega-\omega_n)}\right] \tag{4.19}$$

for $t\gg 1/\Gamma$.

Now the asymptotic behavior of $\langle \vec{A}^{\,\mathrm{compl}}(\vec{r},t)\rangle_{\mathrm{rad}}$ for the radiation field can be evaluated through the use of (3.15) and (4.19) as

$$\langle \vec{A}^{\mathrm{compl}}(\vec{r},t)\rangle_{\mathrm{rad}}=\left\langle \sum_{n=1}^{\infty} c\left[\frac{3\hbar}{2L\omega_n}\right]^{1/2} d_n^*(t)e^{i\omega_n t}\frac{\sin[(\omega_n/c)r]}{r}\right\rangle\left[\vec{\epsilon}_z-\frac{\vec{r}}{r}\cos\theta\right]$$

$$=-id_0^*\kappa^*L\left[\frac{3\hbar}{2L\omega}\right]^{1/2}\frac{\exp[-(\kappa^*-i\omega)(t-r/c)]}{r}\sigma\left[t-\frac{r}{c}\right]\left[\vec{\epsilon}_z-\frac{\vec{r}}{r}\cos\theta\right] \tag{4.20}$$

for $r\gg c/\omega$ and $t\gg 1/\Gamma$. In obtaining this, the terms of order κ/P are neglected. The function $\sigma(t-r/c)$ is defined as

$$\sigma\left[t-\frac{r}{c}\right]=\begin{cases}1, & \text{for } t>\frac{r}{c}\\ 0, & \text{for } t<\frac{r}{c}\,.\end{cases} \tag{4.21}$$

Through the use of (4.15), (4.20) can be written as

$$\langle \vec{A}^{\mathrm{compl}}(\vec{r},t)\rangle_{\mathrm{rad}}=\frac{ie\nu}{cr}\left\langle R_z^{\mathrm{compl}}\left[t-\frac{r}{c}\right]\right\rangle \times\sigma\left[t-\frac{r}{c}\right]\left[\vec{\epsilon}_z-\frac{\vec{r}}{r}\cos\theta\right] \tag{4.22}$$

which is nothing else but the classical result.

Next, let us examine the energy transfer per second from the initially stored electromagnetic field, i.e., the external field, of the frequency ω to the scattered radiation field produced by the oscillator for $t\gg 1/\Gamma$. Through the use of (4.10) and (4.19), the time derivation of the energy expectation value of the radiation field can be found as

$$\frac{d}{dt}\langle H_{\mathrm{rad}}\rangle=\frac{d}{dt}\sum_{n=1}^{\infty}\hbar\omega_n\,|d_n(t)|^2 \approx\frac{-2\hbar}{\omega}|\kappa d_0|^2\exp[-2(\mathrm{Re}\kappa)t] \times\sum_{n=1}^{\infty}\frac{(\mathrm{Re}\kappa)\omega_n^2}{|\kappa+i(\omega-\omega_n)|^2}\,, \tag{4.23}$$

where the summation over n diverges. Here again we neglect the divergence terms which appear in the higher-order terms in power of the coupling constant [see the remark to (3.6)] and obtain

$$\frac{d}{dt}\langle H_{\mathrm{rad}}\rangle\approx-\frac{2\hbar\omega L}{c}|\kappa d_0|^2\exp[-2(\mathrm{Re}\kappa)t]\,. \tag{4.24}$$

For the charged oscillator, using (4.11) we immediately find

$$\frac{d}{dt}\langle H_{\rm osc}\rangle=\frac{d}{dt}\hbar\nu\,|\Lambda(t)|^2$$
$$\approx-\frac{2({\rm Re}\kappa)\hbar\nu}{\epsilon^2\omega}|\kappa d_0|^2\exp[-2({\rm Re}\kappa)t]\ . \tag{4.25}$$

From (4.24) and (4.25) we find that $d/dt\langle H_{\rm osc}\rangle$ and $d/dt\langle H_{\rm rad}\rangle$ are connected through the relation

$$\frac{d}{dt}\langle H_{\rm osc}\rangle=\frac{\frac{\nu}{\omega}\left[\frac{\Gamma}{2}\right]^2}{\left[\frac{\Gamma}{2}\right]^2+(\omega-\nu)^2}\frac{1}{\Gamma/2}\frac{c}{L}\frac{d}{dt}\langle H_{\rm rad}\rangle\ . \tag{4.26}$$

Because $L/c\gg 1/\Gamma$, this means that the energy change of the charged oscillator is negligibly small. It is also interesting to observe that the classical equation of motion for the charged oscillator can be derived straightforwardly from (4.4) and (4.5) by expressing $\Lambda(t)$ and $d(t)$ in terms of $\langle R_z(t)\rangle$ and $\langle E_z(t)\rangle$. One obtains

$$\frac{d^2}{dt^2}\langle R_z(t)\rangle+\Gamma\frac{d}{dt}\langle R_z(t)\rangle+\nu^2\langle R_z(t)\rangle\approx\frac{e}{m}\langle E_z(r=0,t)\rangle\ . \tag{4.27}$$

V. DISCUSSION AND CONCLUDING REMARKS

We have shown explicitly how the limiting case of the classical description of the charged harmonic oscillator with radiation damping can be obtained from the fully quantized theory. The present formalism is based on the Schrödinger equation written in projection form using VNLC as basis states for the oscillator and the electromagnetic field. This makes it possible to solve the problem in a very comprehensive manner. Furthermore, the expressions obtained in terms of time-dependent coherent states describe a continuous transition from the quantum-mechanical region to the classical region, although the physical interpretations for the absorption and radiation processes in the two limit regions are completely different.

The rotating-wave approximation (2.12) to the interaction Hamiltonian (2.11) is essential to obtain the basic equations (2.29)–(2.31). That is, the approximated interaction Hamiltonian $H_{\rm int}\approx H_{\rm rot}$ is a special case of the form (2.24) that allows a state, which is initially a coherent state, to remain a coherent state during the subsequent time evolution. Corrections to such an approximation may be evaluated by treating $\lambda H'\equiv H_{\rm int}-H_{\rm rot}$ perturbationally, with λ as a dimensionless expansion parameter. If we write the state vector for $H_{\rm int}=H_{\rm rot}+\lambda H'$ as

$$|\psi(t)\rangle=\sum_{\Lambda(0)}\sum_{\{d(0)\}}|\Lambda(t),\{d(t)\}\rangle f(\Lambda(0),\{d(0)\};t)\ ,$$

where the indices $\Lambda(0)$ and $\{d(0)\}$ denote the initial condition of the superposed coherent states and $|\Lambda(t),\{d(t)\}\rangle$ satisfies

$$i\hbar\frac{\partial}{\partial t}|\Lambda(t),\{d(t)\}\rangle=H_{\rm rot}|\Lambda(t),\{d(t)\}\rangle\ ,$$

then we can immediately find from the projection equation (2.1)

$$\sum_{\Lambda(0)}\sum_{\{d(0)\}}\{\langle\phi_s|\lambda H'|\Lambda(t),\{d(t)\}\rangle f(\Lambda(0),\{d(0)\};t)-i\hbar\langle\phi_s|\Lambda(t),\{d(t)\}\rangle\frac{\partial}{\partial t}f(\Lambda(0),\{d(0)\};t)\}=0$$

which must be satisified for arbitrary $|\phi_s\rangle$'s. This equation may be perturbationally solved by expanding

$$f(\Lambda(0),\{d(0)\};t)$$

in powers of λ. For $|\phi_s\rangle$'s any complete sets of vectors in the Hilbert space can be used.

Concerning the nontrivial discrepancy of the frequency width of the radiation pointed out in Sec. I, we have shown explicitly in Secs. III and IV that the correct treatment of the classical limit of the fully quantized theory gives the right frequency width

$$\Delta\omega=\Gamma\ .$$

It should be noted that these coherence effects are similar to the coherence effects appearing in laser systems.[10] Although we have treated a well-known system to illustrate the method, it should be remarked that the method is general and can be applied also to more complex problems.

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