PHYSICAL REVIEW A VOLUME 27, NUMBER 3 MARCH 1983

Chaotic dynamics of an impact oscillator

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(Received 11 August 1982)

An impact oscillator is shown to exhibit complex dynamics. Its resonance response contains regions where, after an infinite cascade of period-doubling bifurcations, chaotic motion typical of a strange attractor is observed. The regions are bounded on both sides by subharmonic resonances. Quantitative agreements are obtained with the Feigenbaum scenario of chaos. This novel feature of a substantial marine technology program may be of general cross-disciplinary interest to mathematicians and physicists.

There is much current interest in physics in chaotic motions governed by strange attractors. We present a new example of a system exhibiting such phenomena, and make quantitative comparisons with a scenario of period doubling as a route to chaos.[1-4] There is good agreement, supporting the hope that the behavior of many nonlinear dynamical systems can be understood by means of a universal model.

The present work arose in marine engineering, where the slackening of a mooring can introduce a discontinuity in stiffness of, for example, an articulated oil-loading tower.[5] Such systems can sometimes be modeled as an impact oscillator, that rebounds elastically whenever the displacement X drops to zero, subjected to sinusoidal forcing. This simple system is of wide interest beyond marine technology.

The nondimensionalized equation of motion of our impact oscillator can be written as[5]

$$\ddot{X}+\frac{2\zeta}{\eta}\dot{X}+\frac{1}{4\eta^2}X=\frac{1}{\eta^2}\sin\tau,\quad X>0\ ,$$

where a dot denotes differentiation with respect to time τ, ζ is the damping ratio defined with respect to the effective natural frequency of the unforced and undamped rebounding system, and η is the ratio of the forcing frequency to this effective frequency. Defining y as half the maximum value of X, the resonance response curve has been determined as $y(\eta)$ in Fig. 1 for the fixed damping ratio shown. Well-defined resonant peaks are seen, corresponding to a fundamental response ($n=1$) at $\eta\approx 1$ and subharmonic resonances of order $n=2, 3, 4, \ldots$ at $\eta\approx n$.

The areas shown in Fig. 1 correspond to solutions with one impact per response cycle, but between the peaks more than one impact is observed. A precise digital computer program[5] has been used to explore carefully the region between two adjacent peaks, using the Poincaré mapping points $(X,\dot{X})$ at τ equal to multiples of 2π. A cascade of period-doubling bifurcations is observed (Fig. 2) leading to a chaotic solution at $\eta=4.5$. Notice that the chaos is located between an $n=4$ cascade deriving from the $n=4$ resonance, and an $n=5$ cascade originating from the $n=5$ resonance.

This approach to chaos via period doubling is in line with the Feigenbaum scenario[6] which draws on the behavior of a universal quadratic map. The bifurcations of this map have the property that the consecutive control parameter intervals tend to a fixed ratio of $\delta=4.6692\ldots$ as an accummulation point is approached.

In our problem, we define η_n as the range of η over which the subharmonic of order n is observed. We then find from a refined version of Fig. 2 that $\eta_8/\eta_{16}=4.56$, $\eta_{16}/\eta_{32}=4.69$, and $\eta_{32}/\eta_{64}=4.64$. These agree very well with Feigenbaum's number, which relates strictly to the limit as n tends to infinity.

A standard test for the chaotic motions of a strange attractor is that solutions from adjacent starts should diverge exponentially until they become completely uncorrelated. This behavior for the impact oscillator

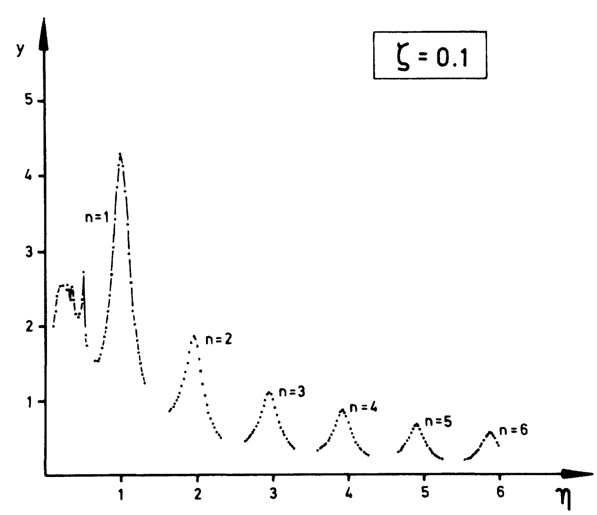

FIG. 1. Resonance response curve for the impact oscillator.

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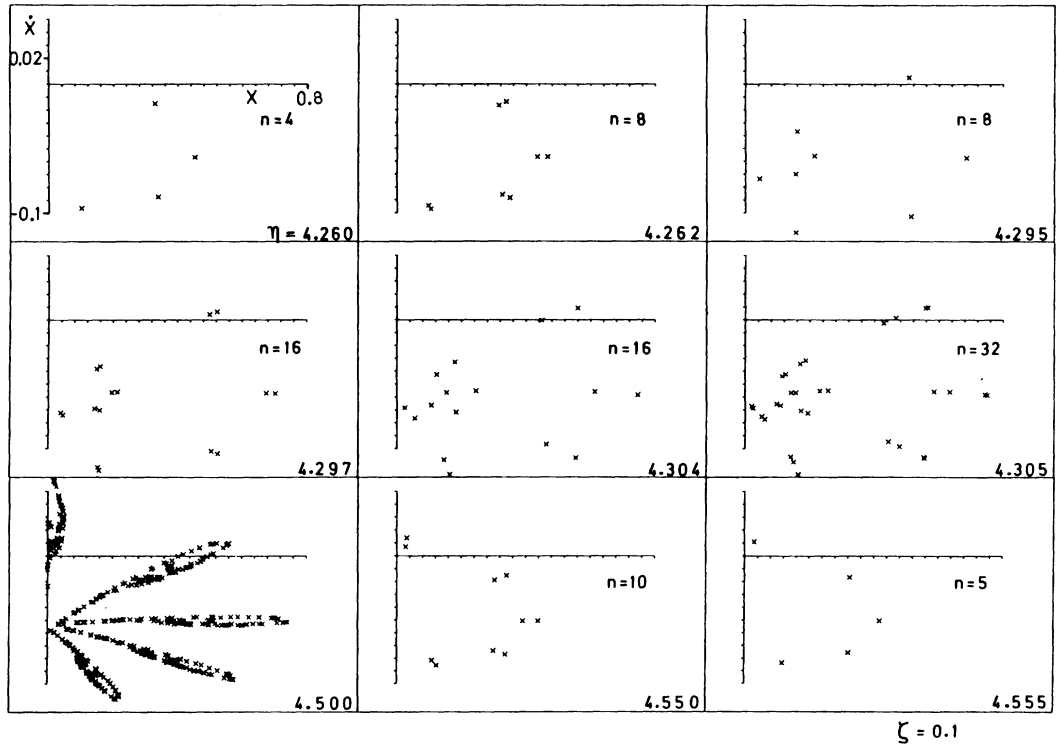

FIG. 2. Sequence of steady-state Poincaré maps, showing period-doubling bifurcations leading to chaos at $\eta = 4.5$.

is shown in Fig. 3 for three starting separations on a plot of $-\log_{10} R$ against steps of $\Delta\tau = 2\pi$. The noisy straight lines confirm that R varies as $R_0 N^i$, where i is the number of steps and N is the Liapunov number.

From a set of graphs similar to Fig. 3, we estimate the value of $N = 1.17$ for our impact oscillator. This compares with $N = 1.26$ for regions of the quadratic map, and $N = 1.52$ for the Hénon strange attractor.[7–11]

We note, finally, that we have observed the coexistence of an $n = 10$ subharmonic with a presumed strange attractor at a value of $\eta = 4.55$ as shown in Fig. 4. The solution obtained here depends only on the starting conditions of the time integration.

This discovery of period-doubling bifurcations, chaos, and strange attractors in the resonance of a simple impact oscillator may be of interest to engineers, physicists, and mathematicians alike. Full details of the marine study are planned to be published elsewhere.[5,12]

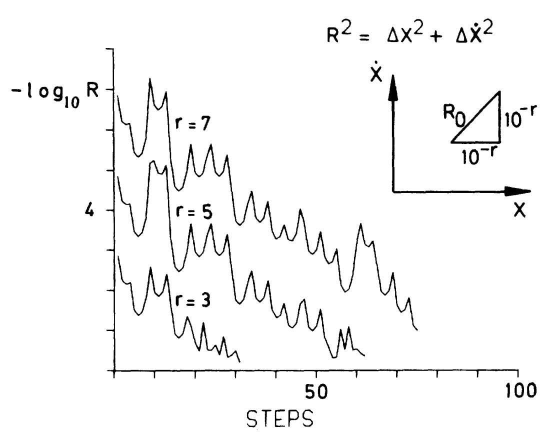

FIG. 3. Divergence study showing a noisy exponential growth of the separation between close starts.

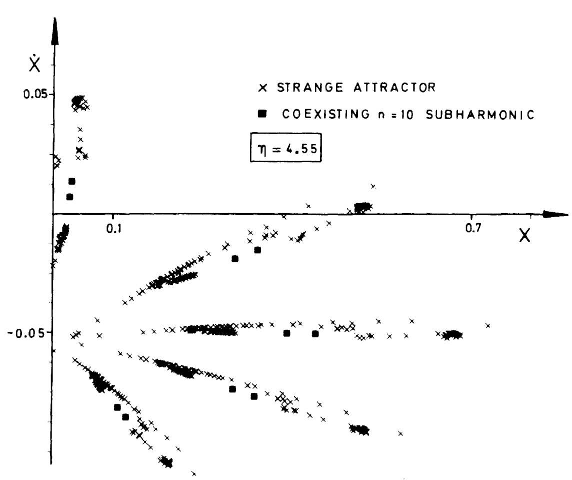

FIG. 4. Attractors of two coexisting multiple solutions for a fixed system. The attractor observed depends on the starting conditions.

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