Restricted multiple three-wave interactions: Painlevé analysis

C. R. Menyuk, H. H. Chen, and Y. C. Lee
Laboratory for Plasma and Fusion Energy Studies, University of Maryland, College Park, Maryland 20742
(Received 22 June 1982)

Restricted multiple three-wave interactions, in which a set of wave triads interact through one shared wave, are discussed. The integrability of this system is explored through the use of Painlevé analysis. Numerical results in the special case where there are only two triads are also reported. The results are consistent with the Painlevé analysis.

I. GENERAL INTRODUCTION

Three-wave interactions, because they are the lowest-order nonlinear wave coupling possible, play an important role in every branch of physics where nonlinear wave phenomena can occur. The simplest possible interaction of this sort is a conservative single-triad interaction, where the system consists of only three interacting waves. This system is completely integrable, a result which was apparently first derived in the context of optical phenomena,[1] but has also been rederived in the context of fluid dynamics[2] and plasma physics.[3] A dissipative, single-triad interaction is already sufficiently complicated to exhibit a wide variety of sophisticated behavior, including strange attractors,[4,5] and is still being actively investigated.

More complicated three-wave interactions, involving two, three,[6,7] or sometimes even more triads have also been studied but, generally, when one considers multiple three-wave interactions, it is simplest to go to the limit of many waves and use the random-phase approximation[8] or some other related statistical assumption.[9] Unfortunately, in many if not most cases, it is not at all clear from the literature that such assumptions are valid, and this question clearly requires closer examination.

In order for most such assumptions to be valid, the system must be "mixing" on the time scale of interest, so that it rapidly loses all knowledge of its initial condition. It immediately follows that the system must be nonintegrable. If the system were integrable, then its trajectories would be bound to specific hypersurfaces in phase space and its time history, even averaged, could depend sensitively on which hypersurface it was located initially. Hence, it is of great importance to determine under what conditions three-wave systems are integrable. Ideally, we could find an analog to the theorem of Bruns[10] which states that for an N-body gravitational or Coulomb interaction ($N \geq 3$), the only algebraic integrals of the motion are the classical integrals, irrespective of the masses of the interacting bodies, and, as a result, the N-body system is almost certainly nonintegrable for almost all choices of the masses. However, different types of three-wave interactions can have different constants of the motion, and such a global statement does not appear possible. Instead, one must focus in turn on each type of three-wave interaction.

In addressing the question of when three-wave systems may be treated statistically, and more particularly of when they are integrable, we focus attention here on a test wave system which appears to be the simplest possible multiply interacting, three-wave system with an arbitrary number of waves. In this system, one wave is common to all the triads, which are otherwise noninteracting, as shown schematically in Fig. 1. While this system is not the most important three-wave system from the point of view of applications, it has the great virtue of bring sufficiently simple to analyze in detail for an arbitrarily large number of waves, and hence provides a useful starting point from which to begin the analysis of more complicated systems. To our knowledge, this system was first studied by Watson,

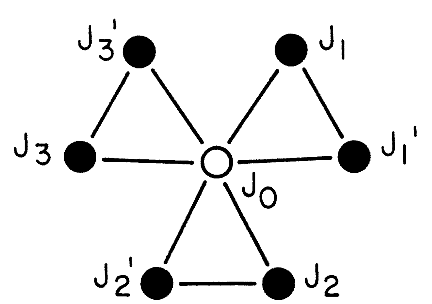

FIG. 1. Schematic illustration of a three-triad interaction. Only the waves connected by solid lines interact directly.

West, and Cohen,[11] who used it to model the growth of a low-frequency internal ocean wave due to the interaction of a spectrum of higher-frequency surface waves. In this case, the shared wave is a daughter wave in each triad. Meiss[12] has studied this system using a variety of statistical assumptions and proposed it as a model for both ocean wave and plasma turbulence. Paradoxically, he has also suggested that this system may be completely integrable.[13] Our results indicate that the system is in general nonintegrable, but, at the same time, the statistical assumptions are not generally valid.

Since the previously mentioned authors[11–13] were primarily interested in the case where the shared wave is a daughter wave in each triad, they have focused attention on the Hamiltonian, written in action-angle variables

$$H=\omega_0 J_0+\sum_{n=1}^{N}(\omega_n J_n+\omega_n' J_n') - \sum_{n=1}^{N}\epsilon_n (J_0 J_n J_n')^{1/2}\cos(\theta_n-\theta_n'-\theta_0) , \quad (1)$$

where J_0 and θ_0 refer to the shared wave; J_n, θ_n, J_n', and θ_n' refer to the other members of the nth triad besides the shared wave; and N is the total number of triads. One can also imagine situations in which the shared wave is a pump wave in each triad, in which case the Hamiltonian becomes

$$H=\omega_0 J_0+\sum_{n=1}^{N}(\omega_n J_n+\omega_n' J_n') - \sum_{n=1}^{N}\epsilon_n (J_0 J_n J_n')^{1/2}\cos(\theta_0-\theta_n-\theta_n') . \quad (2)$$

Letting

$$b_0\equiv(J_0)^{1/2}\exp(-i\theta_0) ,$$
$$b_n\equiv(J_n)^{1/2}\exp(-i\theta_n) ,$$

and

$$b_n'\equiv(J_n')^{1/2}\exp(-i\theta_n') ,$$

Eq. (1) generates the equations of motion

$$\dot b_0=\frac{i}{2}\sum_{n=1}^{N}\epsilon_n b_n b_n'^{*}-i\omega_0 b_0 ,$$
$$\dot b_n=\frac{i}{2}\epsilon_n b_0 b_n'-i\omega_n b_n , \quad 1\le n\le N \quad (3a)$$
$$\dot b_n'=\frac{i}{2}\epsilon_n b_0^{*} b_n-i\omega_n' b_n' , \quad 1\le n\le N$$

as well as the complex-conjugate equations

$$\dot b_0^{*}=-\frac{i}{2}\sum_{n=1}^{N}\epsilon_n b_n^{*} b_n'+i\omega_0 b_0^{*} ,$$
$$\dot b_n^{*}=-\frac{i}{2}\epsilon_n b_0^{*} b_n'^{*}+i\omega_n b_n^{*} , \quad 1\le n\le N \quad (3b)$$
$$\dot b_n'^{*}=-\frac{i}{2}\epsilon_n b_0 b_n^{*}+i\omega_n' b_n'^{*} , \quad 1\le n\le N .$$

Similarly, letting

$$a_0\equiv(J_0)^{1/2}\exp(-i\theta_0) ,$$
$$a_n\equiv(J_n)^{1/2}\exp(-i\theta_n) ,$$

and

$$a_n'\equiv(J_n')^{1/2}\exp(-i\theta_n') ,$$

Eq. (2) generates the equations of motion

$$\dot a_0=\frac{i}{2}\sum_{n=1}^{N}\epsilon_n a_n a_n'-i\omega_0 a_0 ,$$
$$\dot a_n=\frac{i}{2}\epsilon_n a_0 a_n'^{*}-i\omega_n a_n , \quad 1\le n\le N \quad (4a)$$
$$\dot a_n'=\frac{i}{2}\epsilon_n a_0 a_n^{*}-i\omega_n' a_n' , \quad 1\le n\le N$$

and

$$\dot a_0^{*}=-\frac{i}{2}\sum_{n=1}^{N}\epsilon_n a_n^{*} a_n'^{*}+i\omega_0 a_0^{*} ,$$
$$\dot a_n^{*}=-\frac{i}{2}\epsilon_n a_0^{*} a_n'+i\omega_n a_n^{*} , \quad 1\le n\le N \quad (4b)$$
$$\dot a_n'^{*}=-\frac{i}{2}\epsilon_n a_0^{*} a_n+i\omega_n' a_n'^{*} , \quad 1\le n\le N .$$

We consider both possibilities here.

This paper is the first in a series of three papers devoted to studying the restricted multiple—three-wave—interaction system just described. This system is "restricted" in the sense that the only interaction between the wave triads is through a single wave which they all share.

The first two papers of this series are concerned with determining the integrability of this system. This first paper consists of two essentially independent parts. In the first part (Sec. II) we examine the analytic structure of the equations of motion, Eqs. (3) and (4), in the complex time plane. In particular, we wish to determine for what values of the coupling coefficients ϵ_n and the frequencies ω_0, ω_n, and ω_n', all movable singularities in the complex plane consist of simple poles. This property, called the Painlevé property, has proved to be an excellent indication of when simple Hamiltonian systems are integrable.[14–18] Our system turns out to have this property when all the coupling coefficients are equal, $\epsilon_1=\epsilon_2=\cdots=\epsilon_N$, regardless of the frequencies or when $\epsilon_1=\epsilon_2=\cdots=\epsilon_M=2\epsilon_{M+1}=2\epsilon_{M+2}=\cdots=2\epsilon_N$ and $\Delta_1=\Delta_2=\cdots\Delta_M=2\Delta_{M+1}=2\Delta_{M+2}=\cdots=2\Delta_N$, where $\Delta_n=\omega_n-\omega_n'-\omega_0$ in the case of Eq. (3) and $\Delta_n=\omega_0-\omega_n-\omega_n'$ in the case of Eq. (4). When the coupling coefficients are not equal, the singularities are algebraic except in the case where $\epsilon_1=\epsilon_2=\cdots=\epsilon_M=2\epsilon_{M+1}=\cdots=2\epsilon_N$, where they are logarithmic if they are not simple.

In the second part (Sec. III) of this paper, we specialize to the case when $N=2$ and present numerical evidence which indicates that the system is in general nonintegrable when $\epsilon_1\neq\epsilon_2$ and $\epsilon_1\neq 2\epsilon_2$. The procedure we use is to first reduce the system to two degrees of freedom using the quadratic constants of the motion (Manley-Rowe relations), and then make surface-of-section plots for the reduced system. These plots generally show the stochasticity and higher-order island structure indicative of nonintegrability. In one case, we have calculated the Lyapunov exponent in a stochastic region and shown that it converges to a definite positive value.

In the second planned paper of this series we demonstrate that when $\epsilon_1=\epsilon_2=\cdots=\epsilon_N$, our system is integrable, and we also obtain a number of related results. The procedure we use is to first turn our set of ordinary differential equations, Eqs. (3) and (4), into partial differential equations. We then find the Lax pair for the partial differential equations and from it determine the constants of the motion for the original ordinary differential equations. We further show that identical results can be extracted from a formalism developed by Ablowitz and Haberman.[19] We then use their formalism to show that the system of equations, Eqs. (3) and (4), can be made integrable for arbitrary coupling coefficients by including further waves in the system. We also use their formalism to show that a more general class of three-wave interaction systems, of which the restricted system under consideration here is a special case, can be made integrable by an appropriate choice of coupling coefficients. Finally, we show how to reduce our system to quadratures of elliptic functions in the special case where the shared wave is a daughter wave in all triads and $\Delta_1=\Delta_2=\cdots=\Delta_N$, where $\Delta_n=(\omega_n-\omega_n'-\omega_0)$.

In the third planned paper of this series, we examine the validity of standard statistical assumptions for our system, in particular the random-phase approximation and the microcanonical ensemble approach. These two approaches will not yield identical results since our system possesses a phase-dependent constant, namely, the Hamiltonian. In the first part of this paper we establish certain minimum conditions which must hold for the statistical assumptions to be valid. In the second part of this paper we present numerical evidence indicating that these conditions are not met, except possibly on a very long time scale for the microcanonical ensemble approach.

II. PAINLEVÉ ANALYSIS

A. Introduction

No deductive procedure exists at the present date for determining when a set of ordinary differential equations is integrable and when not. This vexing and difficult problem has remained unsolved (and is perhaps insoluble in general) despite the continuous effort that has been devoted to it for close to a hundred years by mathematicians and mathematical physicists. As a result, the demonstration that a given system is integrable or nonintegrable remains something of a hit-or-miss proposition. In this situation, any procedure which provides a reliable means of "guessing" when a system will be integrable is more than welcome.

For certain simple, yet important systems Painlevé analysis has proved remarkably successful in identifying integrable cases. In this analysis, one determines when the movable singularities of the equations of motion in the complex time plane consist of only simple poles. It is these cases which one identifies as integrable. This procedure was first used by Kovalevskaya[14] in the 1880's to study rigid-body motion in a gravitational field. She recovered all the cases then known to be integrable for arbitrary initial conditions, as well as one more; no other cases which are integrable for arbitrary initial conditions have ever been found. Recently, this procedure has been applied by Tabor and Weiss[15] to the Lorenz system, by Bountis, Segur, and Vivaldi to the Toda lattice, coupled quartic oscillators, and the Hénon-Heiles system,[16] and by Chang and coworkers[17,18] to a variety of systems including several of those already mentioned. In all cases, this procedure has succeeded in identifying the previously known integrable cases as well as several new ones.

All these systems have polynomial equations of motion. It is not known at this point whether that is just coincidence or not. It is known that for some slightly more complicated systems, notably the Kepler problem, this method fails.[20] It is also known that even for polynomial systems, this method does not necessarily succeed in identifying *all* the integrable cases, since integrable systems have been found which have the "weak Painlevé" property of possessing rational singularities with no logarithmic terms.[21]

The system which we are considering, Eqs. (3) and (4), has polynomial equations of motion, and it is therefore reasonable to expect that the Painlevé analysis should successfully identify integrable cases. This expectation has been confirmed by the numerical work to be presented in Sec. III of this paper and the analytical work of paper II in this series.

An unusual feature of this calculation is that the number of waves allowed in the system, and hence, the number of degrees of freedom, is arbitrarily large, instead of 2 or 3 as has been the case in almost all previous applications of the Painlevé analysis. This difference leads to technical difficulties, e.g., the inversion of arbitrarily large matrices, but necessitates no procedural changes.

B. Preliminaries

To determine the integrability of Eq. (4) for a given set of coupling coefficients and frequencies, it is sufficient to determine the integrability of a system in which the shared wave is a daughter wave in all triads, for the same set of coupling coefficients and frequencies transformed such that ω_0 becomes $-\omega_0$, ω_n becomes $-\omega_n$, and ω_n' stays the same. To demonstrate this result, it is sufficient to note that Eq. (4) can be brought into the same form as Eq. (3) by making the transformation $a_0=-b_0^*$, $a_0^*=b_0$, $a_n=b_n^*$, $a_n^*=-b_n$, $a_n'=b_n'$, $a_n'^*=b_n'^*$, and transforming the frequencies as just described. Hence, any constant of the motion for Eq. (3) can be made into a constant of the motion for Eq. (4) by suitably transforming variables and vice versa. It should be noted that this transformation is equivalent to extending the real and imaginary parts of b_0 and b_n into the complex plane.

In the Painlevé analysis, one needs to extend the real and imaginary parts of b_0, b_n, and b_n' into the complex plane. That can be done most conveniently by treating Eqs. (3a) and (3b) as independent. Hereafter, the asterisk will be taken to refer to the now-independent variables, b_0^*, b_n^*, and $b_n'^*$ and *not* to indicate complex conjugation.

We now put the equations of motion in a useful canonical form. Using the generating function

$$F=-\omega_0 J_0 t-\tfrac{1}{2}\sum_{n=1}^{N}[(\omega_n+\omega_n'+\omega_0)J_n t+(\omega_n+\omega_n'-\omega_0)J_n' t]\ , \qquad (5)$$

Eq. (1) becomes

$$H=\sum_{n=1}^{N}\left[\frac{\Delta_n}{2}J_n-\frac{\Delta_n}{2}J_n'-\epsilon_n(J_0J_nJ_n')^{1/2}\cos(\theta_n-\theta_n'-\theta_0)\right]. \qquad (6)$$

The coupling coefficients may be considered positive, since, if any of them are not, they can be made so by adding π to the appropriate phases. Equation (6) generates the equations of motion

$$\dot b_0=\frac{i}{2}\sum_{n=1}^{N}\epsilon_n b_n b_n'^*\ , \qquad (7a)$$

$$\dot b_0^*=-\frac{i}{2}\sum_{n=1}^{N}\epsilon_n b_n^* b_n'\ , \qquad (7b)$$

$$\dot b_n=\frac{i}{2}\epsilon_n b_0 b_n'-\frac{i}{2}\Delta_n b_n\ ,\quad 1\le n\le N \qquad (7c)$$

$$\dot b_n^*=-\frac{i}{2}\epsilon_n b_0^* b_n'^*+\frac{i}{2}\Delta_n b_n^*\ ,\quad 1\le n\le N \qquad (7d)$$

$$\dot b_n'=\frac{i}{2}\epsilon_n b_0^* b_n+\frac{i}{2}\Delta_n b_n'\ ,\quad 1\le n\le N \qquad (7e)$$

$$\dot b_n'^*=-\frac{i}{2}\epsilon_n b_0 b_n^*-\frac{i}{2}\Delta_n b_n'^*\ ,\quad 1\le n\le N\ . \qquad (7f)$$

A version of the Painlevé analysis has been discussed in some detail by Ablowitz, Ramani, and Segur,[21] and the procedure is shown schematically in Fig. 2. Essentially, one attempts to determine the solution to the equations of motion as a power series in the neighborhood of each singularity in the complex time plane and show that each variable b can be written in the form

$$b^m=\sum_{j=0}^{\infty}a_j(z-z_0)^{p+j}\ ,$$

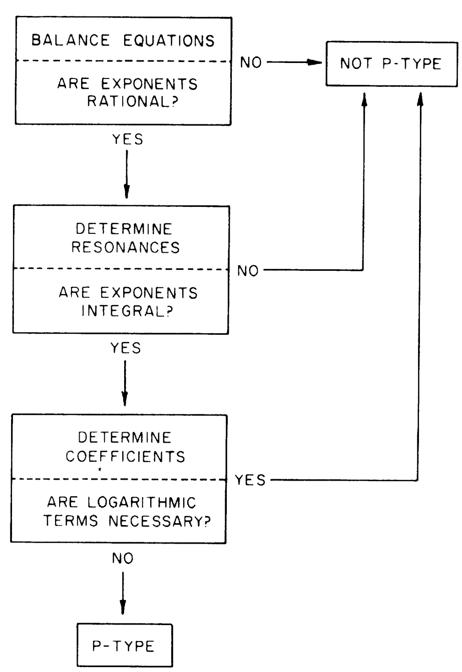

FIG. 2. Flow diagram of the Painlevé analysis.

where z is complex time, z_0 is the singularity location, and m and p are any integers. In the first step of the process, one must show that m and p are integral or, equivalently, that there exists an expression for b,

$$b=\sum_{j=0}^{\infty} a_j(z-z_0)^{p+j} ,$$

where p is rational. Second, one must show that when one adds up the number of arbitrary coefficients in the expansions of all the variables of the problem, the number of arbitrary coefficients must equal the number of variables. Third, one must show that no logarithmic terms enter the expansion. In general, this procedure breaks down at some point indicating that the singularity is algebraic or logarithmic. This procedure does not identify essential singularities; however, that is not a serious drawback since these singularities appear not to be present in simple systems of the sort which we are considering here.

C. Balancing the equations of motion

Designating complex time by z and any particular singularity by z_0, we shall attempt to expand b_0, b_0^*, b_n, b_n^*, b_n', and $b_n'^*$ in a power series about the point $z=z_0$, which leads to the expression

$$\begin{aligned}
b_0&=\alpha_0(z-z_0)^{p_0}+\sum_{j=1}^{\infty} a_{0j}(z-z_0)^{p_0+j} ,\\
b_0^*&=\alpha_0^*(z-z_0)^{p_0^*}+\sum_{j=1}^{\infty} a_{0j}^*(z-z_0)^{p_0^*+j} ,\\
b_n&=\alpha_n(z-z_0)^{p_n}+\sum_{j=1}^{\infty} a_{nj}(z-z_0)^{p_n+j} ,\\
b_n^*&=\alpha_n^*(z-z_0)^{p_n^*}+\sum_{j=1}^{\infty} a_{nj}^*(z-z_0)^{p_n^*+j} ,\\
b_n'&=\alpha_n'(z-z_0)^{p_n'}+\sum_{j=1}^{\infty} a_{nj}'(z-z_0)^{p_n'+j} ,\\
b_n'^*&=\alpha_n'^*(z-z_0)^{p_n'^*}+\sum_{j=1}^{\infty} a_{nj}'^*(z-z_0)^{p_n'^*+j} ,
\end{aligned} \tag{8}$$

where we recall that the asterisk does *not* indicate complex conjugation, and that $1\le n\le N$, which from now on will be understood for all the following equations except where some other restriction on n is specifically noted.

Equations (8) may be written compactly in the form

$$\begin{aligned}
\underline{b}_0&=\underline{\alpha}_0(z-z_0)^{\underline{p}_0}+\sum_{j=1}^{\infty} \underline{a}_{0j}(z-z_0)^{\underline{p}_0+j} ,\\
\underline{b}_n&=\underline{\alpha}_n(z-z_0)^{\underline{p}_n}+\sum_{j=1}^{\infty} \underline{a}_{nj}(z-z_0)^{\underline{p}_n+j} ,
\end{aligned} \tag{8'}$$

where we have defined the 2×1 column matrix $\underline{b}_0\equiv(b_0,b_0^*)$, the 4×1 column matrices $\underline{b}_n\equiv(b_n,b_n',b_n^*,b_n'^*)$, and $\underline{\alpha}_0$, $\underline{\alpha}_n$, $\underline{a}_{0j}$, $\underline{a}_{nj}$, $\underline{p}_0$, and $\underline{p}_n$ are analogously defined, so that

$$\underline{a}_{0j}(z-z_0)^{\underline{p}_0+j}\equiv[a_{0j}(z-z_0)^{p_0+j},a_{0j}^*(z-z_0)^{p_0^*+j}]$$

and

$$\begin{aligned}
&\underline{a}_{nj}(z-z_0)^{\underline{p}_n+j}\\
&\quad\equiv[a_{nj}(z-z_0)^{p_n+j},a_{nj}'(z-z_0)^{p_n'+j} ,\\
&\qquad\qquad a_{nj}^*(z-z_0)^{p_n^*+j} ,a_{nj}'^*(z-z_0)^{p_n'^*+j}] .
\end{aligned}$$

Equations (8) may be even more compactly written

$$\underline{b}=\underline{\alpha}(z-z_0)^{\underline{p}}+\sum_{j=1}^{\infty} \underline{a}_j(z-z_0)^{\underline{p}+j} , \tag{8''}$$

where $\underline{b}\equiv(\underline{b}_0,\underline{b}_1,\ldots,\underline{b}_N)$ is now a $(4N+2)\times1$ column matrix; and $\underline{\alpha}$, $\underline{a}_j$, and $\underline{p}$ are analogously defined.

To determine $\underline{p}$, we first note that sufficiently close to the singular point z_0,

$$\underline{b}=\underline{\alpha}\zeta^{\underline{p}} , \tag{9}$$

where $\zeta\equiv z-z_0$. We then substitute Eq. (9) into Eq. (7) and balance the most singular terms.

Since $\dot{b}_n$ is always one order more singular than b_n, $\dot{b}_n$ must be balanced by the term $(i/2)\epsilon_n b_0 b_n'$ in Eq. (7c). Similar conclusions hold for Eqs. (7d), (7e), and (7f), leading to the relations

$$\begin{aligned}
&p_n-1=p_0+p_n' , \quad p_n^*-1=p_0^*+p_n'^* ,\\
&p_n'-1=p_0^*+p_n , \quad p_n'^*-1=p_0+p_n^* ,
\end{aligned} \tag{10}$$

from which it follows immediately that

$$\begin{aligned}
&p_0=-1-x , \quad p_0^*=-1+x ;\\
&p_n'-p_n=x , \quad p_n^*-p_n'^*=x ,
\end{aligned} \tag{11}$$

where x is arbitrary. From Eq. (11), it then follows that whatever terms balance in Eq. (7a), the corresponding terms will also balance in Eq. (7b).

There are evidently a large number of different ways to balance the terms in Eqs. (7a) and (7b). It turns out that the only singularities which are realizable are those for which $\dot{b}_0$ is included among the terms which balance in Eq. (7a) and, consequently, $\dot{b}_0^*$ is included among the terms which balance in Eq. (7b). The proof of this assertion is contained in Sec. II F. Letting $n=1,2,\ldots,M$ designate those terms which balance with $\dot{b}_0$ and $\dot{b}_0^*$ and $n=M+1,M+2,\ldots,N$ designate those terms which do not, we find relations

$$-p_0-1=p_n+p_n'^* \ , \quad 1\le n\le M$$
$$-p_0-1<p_n+p_n'^* \ , \quad M+1\le n\le N \tag{12}$$

which, using Eq. (11), becomes

$$-2=p_n+p_n^*=p_n'+p_n'^* \ , \quad 1\le n\le M$$
$$-2<p_n+p_n^*=p_n'+p_n'^* \ , \quad M+1\le n\le N \ . \tag{13}$$

From Eq. (13), it immediately follows that

$$p_n=-1-y_n \ , \quad p_n^*=-1+y_n \ , \quad 1\le n\le M$$
$$p_n'=-1-y_n+x \ , \quad p_n'^*=-1+y_n-x \ , \quad 1\le n\le M \tag{14}$$

and

$$p_n=-u_n-y_n \ , \quad p_n^*=-u_n+y_n \ , \quad M+1\le n\le N$$
$$p_n'=-u_n-y_n+x \ , \quad p_n'^*=-u_n+y_n-x \ , \quad M+1\le n\le N \tag{15}$$

where y_n is arbitrary and $u_n<1$.

To further specify these exponents we need to examine the coefficients of the most singular terms. These are

$$p_0\alpha_0=\frac{i}{2}\sum_{n=1}^{M}\epsilon_n\alpha_n\alpha_n'^* \ , \tag{16a}$$

$$p_0^*\alpha_0^*=-\frac{i}{2}\sum_{n=1}^{M}\epsilon_n\alpha_n'\alpha_n^* \ , \tag{16b}$$

$$p_n\alpha_n=\frac{i}{2}\epsilon_n\alpha_0\alpha_n' \ , \tag{16c}$$

$$p_n^*\alpha_n^*=-\frac{i}{2}\epsilon_n\alpha_0^*\alpha_n'^* \ , \tag{16d}$$

$$p_n'\alpha_n'=\frac{i}{2}\epsilon_n\alpha_0^*\alpha_n \ , \tag{16e}$$

$$p_n'^*\alpha_n'^*=-\frac{i}{2}\epsilon_n\alpha_0\alpha_n^* \ . \tag{16f}$$

Multiplying (16c) by (16e) and (16d) by (16f), we find

$$p_np_n'=p_n^*p_n'^*=-\frac{\epsilon_n^2}{4}\alpha_0\alpha_0^* \ . \tag{17}$$

Using Eq. (14), it follows, for $1\le n\le M$, that $y_n=x/2$. Hence, we find

$$p_n=p_n'^*=-1-\frac{x}{2} \ , \quad 1\le n\le M$$
$$p_n'=p_n^*=-1+\frac{x}{2} \ , \quad 1\le n\le M \tag{18}$$

and also that $\epsilon_1=\epsilon_2=\cdots=\epsilon_M\equiv\epsilon$. From Eqs. (16c)–(16f), we obtain the further relations $\alpha_i\alpha_j'=\alpha_j\alpha_i'$ and $\alpha_i^*\alpha_j'^*=\alpha_j^*\alpha_i'^*$ $(1\le i\le M;1\le j\le M)$. Multiplying Eq. (16a) by Eq. (16c) and Eq. (16b) by Eq. (16d), and using these relations, it follows that

$$p_0p_n=p_0^*p_n^*=-\frac{\epsilon^2}{4}\sum_{\ell=1}^{M}\alpha_\ell'\alpha_\ell'^* \ , \tag{19}$$

from which we conclude $x=0$, so that $p_0=p_0^*=p_n=p_n^*=p_n'=p_n'^*=-1$ $(1\le n\le M)$.

To determine the exponents when $N\ge n>M$, we first note from Eq. (15) that $p_n=p_n'=-u_n-y_n$ and $p_n^*=p_n'^*=-u_n+y_n$. From Eq. (17), it then follows that either $u_n=0$ or $y_n=0$. Dividing Eq. (17) by the relation $1=-(\epsilon^2/4)\alpha_0\alpha_0^*$, we find that when $\epsilon_n<\epsilon$ $(M+1\le n\le N)$, four possibilities exist,

$$p_n=p_n'=p_n^*=p_n'^*=\epsilon_n/\epsilon \ , \tag{20a}$$

$$p_n=p_n'=p_n^*=p_n'^*=-\epsilon_n/\epsilon \ , \tag{20b}$$

$$p_n=p_n'=-p_n^*=-p_n'^*=\epsilon_n/\epsilon \ , \tag{20c}$$

$$p_n=p_n'=-p_n^*=-p_n'^*=-\epsilon_n/\epsilon \ , \quad M+1\le n\le N \ . \tag{20d}$$

When $\epsilon_n\ge\epsilon$ $(M+1\le n\le N)$, the second possibility (20b) is eliminated, because it implies $u_n\ge 1$.

D. Determining the resonances

In order for the possibilities described in the last section to be realizable, it must be possible to continue the expansion in such a way that there are as many arbitrary coefficients as there are variables. If the number of arbitrary coefficients is smaller than the number of variables, then the expansion corresponds to a special choice of initial conditions and is not realizable in practice.

To investigate this question, we first determine how many of the coefficients α_0, α_0^*, α_n, α_n^*, α_n', and $\alpha_n'^*$ may be chosen arbitrarily. From Eq. (17), it follows that α_0 determines α_0^*. Similarly, from Eqs. (16c)–(16f), it follows that α_n determines α_n^* and α_n' determines $\alpha_n'^*$. Finally, from Eq. (16a) or Eq. (16b), it follows that one of the α_n' $(1\le n\le M)$, which we may choose to be α_1', is determined from the other α_n' $(1\le n\le M)$ and the α_n $(1\le n\le M)$. Another arbitrary coefficient comes from our freedom to pick the real part of z_0, corresponding to the origin of time. Hence, we have found a total of $2N+1$ arbitrary coefficients.

The system we are considering has $4N+2$ variables, and the other $2N+1$ arbitrary coefficients, if they exist, must come from the expansion of the solution. As one expands and equates terms of equal order, one finds that at each order one obtains a system of linear equations for the new coefficients in terms of nonlinear combinations of the old coefficients. If the determinant of the linear, homogeneous part of this system of equations is nonzero, then a unique solution exists for the new coefficients. However, if the determinant is zero, then at least one of the new coefficients is arbitrary. To determine this linear system at any given order r, it is sufficient to take the combination

$$\underline{b} \sim \underline{\alpha}\zeta^{\ell} + \underline{\beta}_r \zeta^{\ell+r} , \tag{21}$$

substitute it into those terms of Eq. (7) which were balanced in the previous section, and extract the combinations linear in $\underline{\beta}_r$. Since the terms we balanced in the previous section were the lowest-order terms, any other combinations involving $\underline{\beta}_r$ will be of higher order than those which we are keeping.

This procedure yields the system of equations

$$\underline{M}_r \underline{\beta}_r = \begin{bmatrix} \underline{R}_r & \underline{H}_1 & \underline{H}_2 & \underline{H}_3 & \cdots & \underline{H}_{N-1} & \underline{H}_N \\ \underline{V}_1 & \underline{D}_{1r} & \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{0} \\ \underline{V}_2 & \underline{0} & \underline{D}_{2r} & \underline{0} & \cdots & \underline{0} & \underline{0} \\ \underline{V}_3 & \underline{0} & \underline{0} & \underline{D}_{3r} & & \vdots & \vdots \\ & & & & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & & \underline{D}_{N-1,r} & \underline{0} \\ \underline{V}_N & \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{D}_{Nr} \end{bmatrix} \begin{bmatrix} \underline{\beta}_{0r} \\ \underline{\beta}_{1r} \\ \underline{\beta}_{2r} \\ \vdots \\ \\ \underline{\beta}_{N-1,r} \\ \underline{\beta}_{Nr} \end{bmatrix} = \underline{0} , \tag{22}$$

where

$$\underline{R}_r \equiv \begin{bmatrix} 1-r & 0 \\ 0 & 1-r \end{bmatrix} ; \quad \underline{V}_n \equiv \begin{bmatrix} \frac{i}{2}\epsilon_n \alpha_n' & 0 \\ 0 & \frac{i}{2}\epsilon_n\alpha_n \\ 0 & -\frac{i}{2}\epsilon_n \alpha_n'^{*} \\ -\frac{i}{2}\epsilon_n\alpha_n^{*} & 0 \end{bmatrix} , \quad \underline{D}_{nr} = \begin{bmatrix} -p_n - r & \frac{i}{2}\epsilon_n\alpha_0 & 0 & 0 \\ \frac{i}{2}\epsilon_n\alpha_0^{*} & p_n - r & 0 & 0 \\ 0 & 0 & p_n^{*} - r & -\frac{i}{2}\epsilon_n \alpha_0^{*} \\ 0 & 0 & -\frac{i}{2}\epsilon_n\alpha_0 & p_n^{*} - r \end{bmatrix} ,$$

$$\underline{H}_n = \begin{bmatrix} \frac{i}{2}\epsilon\alpha_n'^{*} & 0 & 0 & \frac{i}{2}\epsilon\alpha_n \\ 0 & -\frac{i}{2}\epsilon\alpha_n^{*} & -\frac{i}{2}\epsilon\alpha_n' & 0 \end{bmatrix} , \quad 1 \le n \le M$$

$$\underline{H}_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \quad M+1 \le n \le N ;$$

$$\underline{\beta}_{0r} = \begin{bmatrix} \beta_{0r} \\ \beta_{0r}^{*} \end{bmatrix} , \quad \underline{\beta}_{nr} = \begin{bmatrix} \beta_{nr} \\ \beta_{nr}' \\ \beta_{nr}^{*} \\ \beta_{nr}'^{*} \end{bmatrix} . \tag{23}$$

Evaluating $\det \underline{M}_r$, one finds

$$\det \underline{M}_r = (r+1) r^{2N} (r-2)^{2M} (r-3) \prod_{n=M+1}^{N} [(r+p_n)^2 - p_n^2][(r+p_n^{*})^2 - (p_n^{*})^2] . \tag{24}$$

The evaluation of this determinant, while tedious, is essentially straightforward as long as one keeps in mind that the number of factors from $\underline{H}_n$ must equal the number of factors from $\underline{V}_n$ in any given term. We have also made use of the relations

$$-1=\frac{\epsilon^2}{4}\alpha_0\alpha_0^* , \tag{25a}$$

$$-1=\frac{\epsilon^2}{4}\sum_{n=1}^{M}\alpha_n'\alpha_n'^* , \tag{25b}$$

$$1=\frac{\epsilon^2}{4}\sum_{n=1}^{M}\alpha_n\alpha_n^* . \tag{25c}$$

Equation (25a) corresponds to Eq. (17), Eq. (25b) corresponds to Eq. (19), and Eq. (25c) can be derived in the same way as was Eq. (25b).

From Eq. (24), we find one root at $r=-1$, corresponding to the arbitrariness in the real part of z_0; we find $2N$ roots at $r=0$, corresponding to the number of the coefficients α_0, α_0^*, α_n, α_n^*, α_n', and $\alpha_n'^*$ which may be chosen arbitrarily; we find $2M$ roots at $r=2$ and one root at $r=3$; and, finally, we find roots at $r=-2p_n$ and $-2p_n^*$ $(M+1\le n\le N)$. Since we are expanding in increasing powers of ζ, we can only have resonances when $r>0$. It immediately follows that in order to have the full complement of $4N+2$ arbitrary coefficients, it must be the case that $p_n<0$ and $p_n^*<0$ $(M+1\le n\le N)$. Hence, the only realizable singularities are those which correspond to Eq. (20b), which in turn implies that $\epsilon_n/\epsilon<1$ $(M+1\le n\le N)$. We conclude that all singularities in the complex time plane are of the same type. The term or terms with the largest coupling coefficient balance with $\dot{b}_0$ and $\dot{b}_0^*$ in Eqs. (7a) and (7b).

Two special cases emerge from this analysis. The first is when all the coupling coefficients are equal. In this case, if there are no logarithmic terms, a point which will be investigated in the next section, then the solution to the equations of motion, Eq. (7), can be expanded in the form

$$\underline{b}=\zeta^{-1}\sum_{j=0}^{\infty}\underline{a}_j\zeta^j \tag{26}$$

in the neighborhood of any singularity, and is evidently of P type (i.e., possesses the Painlevé property). The second case is $\epsilon_n=\epsilon$ $(1\le n\le M)$, $\epsilon_n=\epsilon/2$ $(M+1\le n\le N)$. In this case, if there are no logarithmic terms, a point which will also be investigated in the next section, then the solution in the neighborhood of any singularity can be expanded in the form

$$\begin{aligned}
&\underline{b}_0=\zeta^{-1}\sum_{j=0}^{\infty}\underline{a}_{0j}\zeta^j ;\\
&\underline{b}_n=\zeta^{-1}\sum_{j=0}^{\infty}\underline{a}_{nj}\zeta^j , \quad 1\le n\le M\\
&\underline{b}_n=\zeta^{-1/2}\sum_{j=0}^{\infty}\underline{a}_{nj}\zeta^j , \quad M+1\le n\le N .
\end{aligned} \tag{27}$$

While the singularities appear at first to be algebraic because of the $\zeta^{-1/2}$ dependence of the leading-order contribution to $\underline{b}_n$ $(M+1\le n\le N)$, this dependence can be easily removed by making the replacement $\underline{u}_n=\underline{b}_n^2\equiv(b_n^2,b_n'^2,b_n^{*2},b_n'^{*2})$ $(M+1\le n\le N)$. The expansion of these variables, excluding for the moment the possibility of logarithmic terms, is evidently of P type.

In all other cases, the singularities are generically algebraic. For example, if we consider the case $\epsilon_n=\epsilon$ $(1\le n\le M)$, $\epsilon_n=\epsilon/3$ $(M+1\le n\le N)$, even if there were no logarithmic terms, the expansion would be

$$\begin{aligned}
&\underline{b}_0=\zeta^{-1}\sum_{j=1}^{\infty}\underline{a}_{0j}\zeta^{-j/3} ,\\
&\underline{b}_n=\zeta^{-1}\sum_{j=1}^{\infty}\underline{a}_{nj}\zeta^{-j/3} , \quad 1\le n\le M\\
&\underline{b}_n=\zeta^{-1/3}\sum_{j=1}^{\infty}\underline{a}_{nj}\zeta^{-j/3} , \quad M+1\le n\le N .
\end{aligned} \tag{28}$$

This expansion cannot be made P type by cubing the variables or taking them to any other power. One might be tempted to remove the algebraic nature of the singularity by making the replacement $\tau\equiv\zeta^{1/3}$. However, it must be recalled that there are an infinite number of singularities in the complex plane, and such a transformation can only make one of them simple. To digress for a moment into generalities, a system with a rational leading-order root and no logarithmic terms can be made P type only if the resonances are integral.

E. Determining low-order coefficients

As stated earlier, when we attempt to expand our solution in the form of Eq. (8), we find at each order an equation of the form

$$\underline{M}_j\underline{A}_j=\underline{S}_j(\underline{A}_1,\underline{A}_2,\ldots,\underline{A}_{j-1}) , \tag{29}$$

where $\underline{M}_j$ is the matrix defined in Eq. (22) and $\underline{A}_j$ is the jth order coefficient in the expansion. When $\det\underline{M}_j$ is not equal to zero, $\underline{A}_j$ has a unique solution

$$\underline{A}_j=\underline{M}_j^{-1}\underline{S}_j(\underline{A}_1,\underline{A}_2,\ldots,\underline{A}_{j-1}) . \tag{30}$$

However, at the resonances when $\det\underline{M}_j=0$, Eq. (30) may not have a solution. The criterion for Eq. (30)

to have a solution is as follows: Let $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_N$ be the eigenvectors of $\underline{M}_j$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ the corresponding eigenvalues. We suppose that $\lambda_n = 0$ $(1 \le n \le R)$ and $\lambda_n \ne 0$ $(R+1 \le n \le N)$. We then write

$$\underline{S}_j = \sum_{n=1}^{N} c_n \underline{v}_n \ , \tag{31}$$

where the c_n are constants. In order for a solution to exist, it must be the case that $c_n = 0$ $(1 \le n \le R)$.

If this criterion is not met at every resonance, then logarithmic terms must be added to the expansion in order to satisfy the original set of ordinary differential equations. Specifically, if we let r be the first resonance where this criterion is not met, then the expansion at that order has the form

$$\underline{A}_j = \sum_{n=1}^{R} c_n \underline{v}_n \zeta^{\underline{p}+j} \ln\zeta + \sum_{n=R+1}^{N} \frac{c_n}{\lambda_n} \underline{v}_n \zeta^{\underline{p}+j} \ . \tag{32}$$

Since the matrices $\underline{M}_j$ which we are considering here are non-Hermitian, the eigenvectors are nonorthogonal and in determining the existence of a solution, it is simplest to resolve the matrix equation, Eq. (29), at each order where resonances exist. Doing so, one obtains a number of solubility conditions equal to the number of resonances.

As noted in Sec. I, there are two cases of interest here. The first is when all coupling coefficients are equal to some constant ϵ. The second is when $\epsilon_n = \epsilon$ $(1 \le n \le M)$ and $\epsilon_n = \epsilon/2$ $(M+1 \le n \le N)$. If we allow $M = N$, then the first case can be treated as a special instance of the second case. Since resonances exist at $r = 1$, 2, and 3, we must carry out the expansion to third order to determine if a solution exists. Writing out the expansion to third order, we have

$$\begin{aligned} \underline{b}_0 &= \underline{\alpha}_0 \zeta^{-1} + (\underline{\beta}_{01} + \underline{a}_{01}) + (\underline{\beta}_{02} + \underline{a}_{02})\zeta \\ &\quad + (\underline{\beta}_{03} + \underline{a}_{03})\zeta^2 \ , \\ \underline{b}_n &= \underline{\alpha}_n \zeta^{-\underline{p}_n} + (\underline{\beta}_{n1} + \underline{a}_{n1})\zeta^{-\underline{p}_n+1} \\ &\quad + (\underline{\beta}_{n2} + \underline{a}_{n2})\zeta^{-\underline{p}_n+2} + (\underline{\beta}_{n3} + \underline{a}_{n3})\zeta^{-\underline{p}_n+3} \ , \end{aligned} \tag{33}$$

where $\underline{p}_n = \mathbf{1}$ if $1 \le n \le M$ and $\underline{p}_n = \frac{1}{2}\mathbf{1}$ if $M+1 \le n \le N$; $\underline{\beta}_j$ is the solution to the homogeneous equation, Eq. (22); and $\underline{a}_j$ is a particular solution of the inhomogeneous equation, Eq. (29). The complete solution $\underline{A}_j$ of Eq. (29) is just $\underline{a}_j + \underline{\beta}_j$. To determine $\underline{S}_j$ $(j = 1,2,3)$, the inhomogeneous term on the right-hand side of Eq. (29), one substitutes Eq. (33) into the equation of motion Eq. (7) and collects terms of the jth order.

It is now possible to resolve Eq. (29). To do so, one begins by eliminating the first two rows of $\underline{M}_j$, to the extent possible, using the elements of $\underline{D}_{nj}$ $(1 \le n \le M)$. Having completed that, one then continues with the elimination of the first two columns and the off-diagonal elements of $\underline{D}_{nj}$. Following this procedure, the resolution of Eq. (29) is essentially straightforward, although quite lengthy due to the rapid proliferation of terms. At $j = 1$, if $M < N$, one finds the solubility condition

$$\Delta_n = -\frac{i\epsilon}{4\alpha_0} \sum_{\ell=1}^{M} \Delta_\ell \alpha_\ell \alpha_\ell'^* \ , \quad M+1 \le n \le N \ . \tag{34}$$

This condition cannot be satisfied in general for arbitrary initial conditions. However, in the case where $\Delta_1 = \Delta_2 = \cdots = \Delta_M \equiv \Delta$, Eq. (34) becomes

$$\Delta_n = \frac{\Delta}{2} \ , \quad M+1 \le n \le N \tag{35}$$

so that in the special case where

$$\Delta_1 = \Delta_2 = \cdots = \Delta_M = 2\Delta_{M+1} = \cdots = 2\Delta_N \tag{36}$$

Eq. (34) is satisfied for arbitrary initial conditions. At $j = 2$, one explicitly finds the condition, when $M < N$, that all the mismatches for $M+1 \le n \le N$ are equal. At $j = 3$, no new conditions emerge. If $M = N$, so that all the coupling coefficients are equal, all the solubility conditions are identically zero, and there are no restrictions on the mismatches. The solution may be found explicitly written out in the Appendix.

F. Demonstration that $\dot{b}_0$ and $\dot{b}_0^*$ must be included among the terms which balance

The results of the Sec. II E complete our determination of the conditions under which our equations of motion, Eq. (7), can be P type, subject to our demonstration that $\dot{b}_0$ and $\dot{b}_0^*$ must be included among the terms which balance in their respective equations.

Suppose that these terms are not included among the terms which balance. Let $1 \le n \le M$ indicate those terms which balance and $M+1 \le n \le N$ those terms which do not. As before, Eq. (11) holds and

$$p_0 = -1 - x \ , \quad p_0^* = -1 + x \ . \tag{37}$$

Balancing exponents and following much the same reasoning as in Sec. II B, we find

$$\begin{aligned} p_n &= -u - y_n \ , \quad p_n^* = -u + y_n \ , \quad 1 \le n \le M \\ p_n' &= -u - y_n + x \ , \quad p_n'^* = -u + y_n - x \ , \\ &\qquad 1 \le n \le M \end{aligned} \tag{38}$$

where $u > 1$ and y_n is arbitrary; and also

$$p_n = -v_n - y_n \ , \ \ p_n^* = -v_n + y_n \ , \ \ M+1 \le n \le N$$
$$p_n' = -v_n - y_n + x \ , \ \ p_n'^* = v_n + y_n - x \ , \qquad M+1 \le n \le N \tag{39}$$

where $v_n < u$. Likewise, balancing coefficients and following much the same reasoning as in Sec. II C, we conclude $y_n = x/2$ $(1 \le n \le M)$, $\epsilon_n \equiv \epsilon$ $(1 \le n \le M)$, and either $v_n = 0$ or $y_n = x/2$ $(M+1 \le n \le N)$. We also obtain the relations

$$\sum_{n=1}^{M} \alpha_n \alpha_n' = \sum_{n=1}^{M} \alpha_n^* \alpha_n'^* = 0 \ . \tag{40}$$

Moving on to determining the resonances, we find that $\underline{M}_r$ is the same as in Eq. (22), except that

$$\underline{R}_r = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \ . \tag{41}$$

Resolving the matrix equation $\underline{M}_r \underline{\beta}_r = 0$, using Eq. (40), one finds that the determinant is identically zero for arbitrary r, indicating that an expansion of the sort we constructed in the previous sections is impossible.

III. NUMERICAL STUDIES OF THE CASE $N=2$

A. Introduction

The N-triad system has $2N-1$ degrees of freedom. It also has $N+1$ quadratic constants of the motion, as we shall show shortly. It immediately follows that the two-triad system can be reduced to a system with two degrees of freedom, and the method of Poincaré surfaces of section[10] can be used to investigate the integrability of this system. In Sec. III B the reduced coordinate system is determined and the Hamiltonian appropriate to this system is obtained. Numerical evidence is displayed, indicating that when $\epsilon_1 \neq \epsilon_2$ and $\epsilon_1 \neq 0.5\epsilon_2$, the motion is nonintegrable. When $\epsilon_1 = 0.5\epsilon_2$, numerical evidence is displayed indicating that when $\Delta_1 = 0.5\Delta_2$, the motion is integrable and that when $\Delta_1 \neq 0.5\Delta_2$, it is not.

Nearby nonintegrable trajectories diverge exponentially from one another.[23] That is to say, the distance between the trajectories is, in the limit $t \to \infty$,

$$d = d_0 \exp(\mu_L t) \ , \tag{42}$$

where μ_L is referred to as the Lyapunov exponent. To provide further evidence of nonintegrability, we demonstrate in Sec. III C that such a divergence occurs in one of the stochastic regions of the case $\epsilon_1 = 0.8\epsilon_2$, $\Delta_1 = \Delta_2 = 0$, and determine the Lyapunov number.

B. Surface-of-section plots

Because Hamiltonian systems with one degree of freedom are trivially integrable, systems with two degrees of freedom are the lowest-order systems which can be nonintegrable. We may label the canonical coordinates of any system with two degrees of freedom q_1, p_1, q_2, and p_2. Suppose we fix p_2 and the Hamiltonian H and consider a range of values for q_1 and p_1, with q_2 being automatically fixed through the relation $H(q_1,p_1,q_2,p_2) = h$, where h is some constant. Suppose further that we solve the equations of motion for each choice of (q_1,p_1) and plot the result whenever p_2 returns to zero. If a second constant of the motion, F, exists for this choice of initial coordination, then all the coordinate pairs (q_1,p_1) resulting from this mapping, are constrained to remain on a one-dimensional curve, which is the intersection of the hypersurfaces $H(q_1,p_1,q_2,p_2) = h$, $F(q_1,p_1,q_2,p_2) = f$, where f is some constant, and $p_2 = 0$. By contrast, if F does not exist, the coordinate pairs resulting from this mapping will cover densely some two-dimensional region in a seemingly random fashion.

If the system is integrable, then F is a global invariant, and *all* trajectories (coordinate pairs resulting from the mapping of a single coordinate pair) lie on one-dimensional curves. If the system is nonintegrable, then F can only be a local invariant. Moreover, given a trajectory which possesses an invariant F, there must be another trajectory arbitrarily close which does not; otherwise the constant F could be analytically continued throughout the space. Hence, nonintegrable systems have complicated structures in which regular trajectories (those possessing local invariants) and stochastic trajectories (those which do not) are pathologically interwoven.

The two-dimensional surfaces which are determined by the intersection of $H(q_1,p_1,q_2,p_2) = h$ and $p_2 = 0$ are referred to as surfaces of section, and the set of trajectories which one obtains numerically are referred to as surface-of-section plots. These surface-of-section plots are a powerful tool for determining the integrability of a given system, because they allow one to obtain a feeling for the entire phase-space dynamics at a single glance. Unfortunately, no similar tool exists for systems with more than two degrees of freedom.

The system we are considering here has $2N+1$ degrees of freedom. However, it also has $N+1$ quadratic constants

$$I_0 = J_0 + \sum_{n=1}^{N} J_n ,$$
$$I_n = J_n + J_n' , \quad 1 \le n \le N \tag{43}$$

which do not include the Hamiltonian and can be used to reduce the system to one with N degrees of freedom. To do so, we operate on the Hamiltonian, Eq. (6), with the generating function

$$F = I_0\theta_0 + \sum_{n=1}^{N} J_n(\theta_n - \theta_n' - \theta_0) + \sum_{n=1}^{N} I_n \theta_n' , \tag{44}$$

and we find, up to an additive constant,

$$H = \sum_{n=1}^{N} (\Delta_n J_n - \epsilon_n V_n \cos\psi_n) , \tag{45}$$

where

$$V_n \equiv \left[J_n(I_n - J_n) \left[I_0 - \sum_{l=1}^{N} J_l \right] \right]^{1/2} ,$$
$$\psi_n \equiv \theta_n - \theta_n' - \theta_0 , \quad 1 \le n \le N . \tag{46}$$

The variables J_n and ψ_n are canonical. We may convert to rectangular canonical coordinates by letting

$$p_n = -(2J_n)^{1/2} \sin\psi_n , \quad 1 \le n \le N$$
$$q_n = (2J_n)^{1/2} \cos\psi_n , \quad 1 \le n \le N . \tag{47}$$

It should be pointed out that the integrability of this reduced system does not always imply the integrability of the original system (except for special choices of the initial conditions). Variables in the original system appear as parameters in the new system, and the new system can be and is[12] integrable for certain special choices of these parameters. This situation corresponds to the original system being locally integrable on a hypersurface of dimensionality lower than the phase space. By contrast, the converse holds true. If the reduced system is nonintegrable, so is the original system.

Even restricting ourselves to the two-triad case, the parameter space is vast, containing eight dimensions H, I_0, I_1, I_2, ϵ_1, ϵ_2, Δ_1, and Δ_2. Moreover, a single surface-of-section plot for just one choice of these parameters can manifest an enormous amount of structure. Given this situation, we have concentrated on varying the parameters which are relevant for determining the integrability of the original system ϵ_1, ϵ_2, Δ_1, and Δ_2 and further concentrated on the cases of interest suggested by the Painlevé analysis.

To integrate the equations of motion we used the Gear-Hindmarsh algorithm.[24] Our data was subjected to a number of numerical checks, including the following: The Hamiltonian was determined to

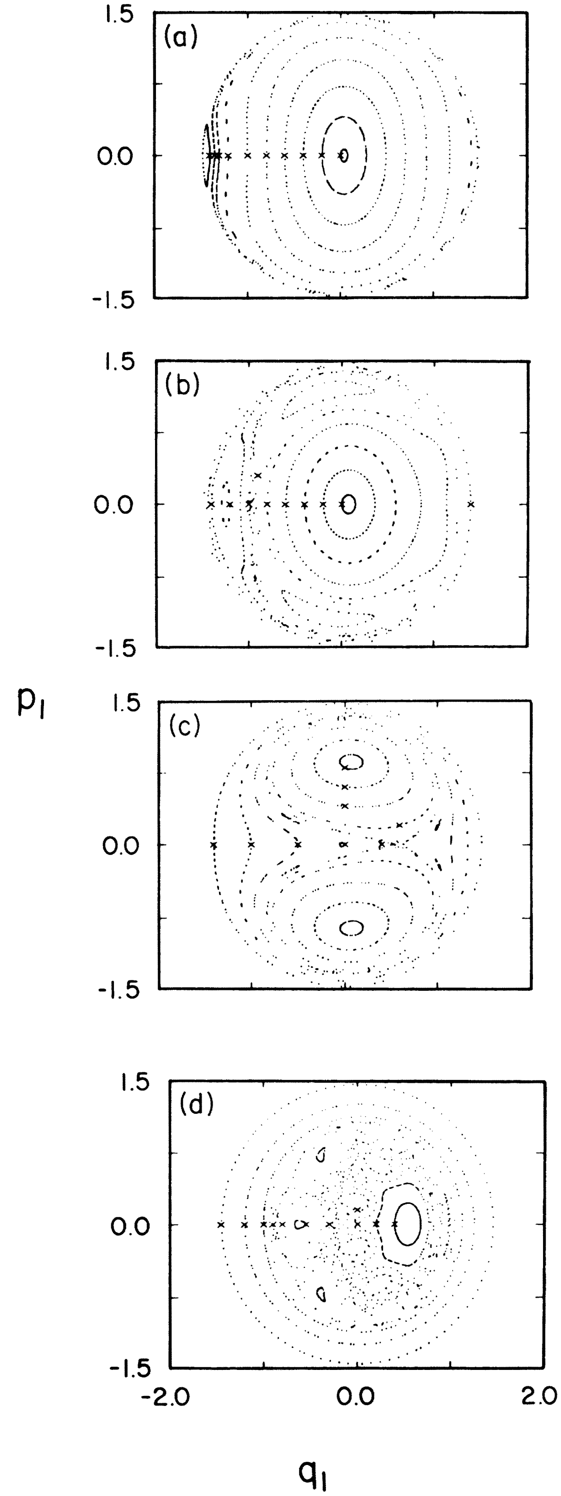

FIG. 3. Surface-of-section plots, varying ϵ_1. Each trajectory contains 100 points and the initial value is marked with a cross. Parameter values are $H = -0.1$, $I_0 = 2.01$, $I_1 = 1.1$, $I_2 = 1.6$, $\Delta_1 = 0.0$, $\Delta_2 = 0.0$, $\epsilon_2 = 1.0$; and (a) $\epsilon_1 = 1.0$, (b) $\epsilon_1 = 0.8$, (c) $\epsilon_1 = 0.4$, (d) $\epsilon_1 = 0.1$.

be constant to within one part in 10^7, the equations were integrated forward and then backward in time and were found to be quite well reproduced, and in conjunction with writing a different numerical algorithm to solve the equations of motion for the third planned paper in this series, we determined that it reproduced the results obtained from the Gear-Hindmarsh algorithm.

Shown in Fig. 3 are results for $\Delta_1=\Delta_2=0$, $\epsilon_2=1.0$, and varying ϵ_1. One hundred surface-of-section points are plotted. There is evidence of nonintegrability in all cases shown except when $\epsilon_1=1.0$. We pursue no further in this paper the case when coupling coefficients are equal, since we explicitly show this case to be integrable in the second paper in this series.

Shown in Fig. 4 is the case $\epsilon_1=0.8$, $\epsilon_2=1.0$, $\Delta_1=\Delta_2=0.0$ of Fig. 3(b) once again. One thousand surface-of-section points are shown in Fig. 4(a) for two trajectories, one regular and the other stochastic. The regular trajectory remains on a one-dimensional curve, while the stochastic trajectory fills a two-dimensional region densely. In Fig. 4(b), the first 50 surface-of-section points of the stochastic trajectory are shown. Over this time scale, the trajectory looks like an eighth-order island. In effect, the trajectory starts near an eighth-order island which is a "sticky point" for the system. This result makes clear the necessity of following a weakly nonintegrable system, like the one that we are considering, on a long time scale in order for its true behavior to emerge.

In Fig. 5(a), we show the case $\epsilon_1=0.5$, $\epsilon_2=1.0$, and $\Delta_1=\Delta_2=0.0$. There is no evidence of nonintegrability. In Fig. 5(b), we blow up a region surrounding a hyperbolic fixed point. If the system were nonintegrable, then this region would be the best place to look for evidence, since a stochastic layer would be expected to form there. Instead, all trajectories look regular. We have blown up this region as much as the fundamental accuracy of the computer we used would allow (14 digits). We only gained two orders of magnitude near the hyperbolic point because the trajectories shown approach each

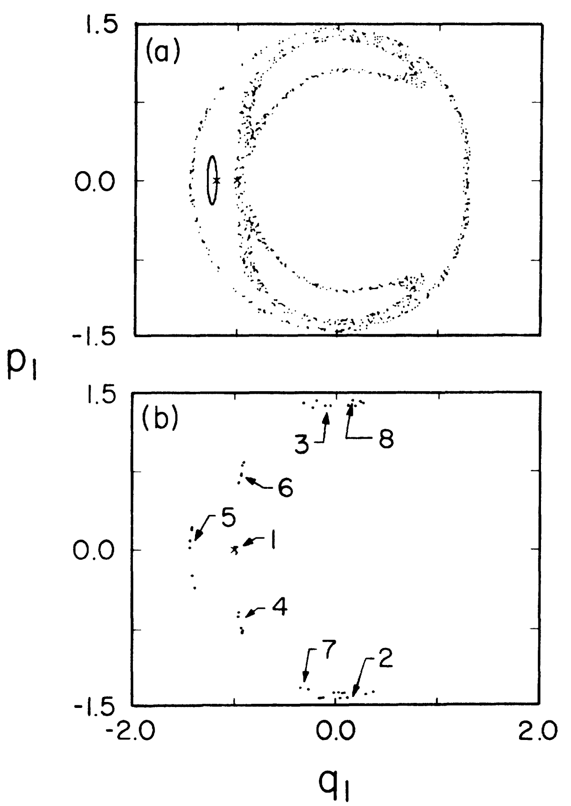

FIG. 4. Surface-of-section plots, with parameters as in Fig. 3(b). In (a), 1000 points are shown for both a regular and a stochastic trajectory. In (b), the first 50 points of the stochastic trajectory are shown. On this time scale, the trajectory appears to describe an eighth-order island. Numbers indicate the order in which the trajectory goes from island to island.

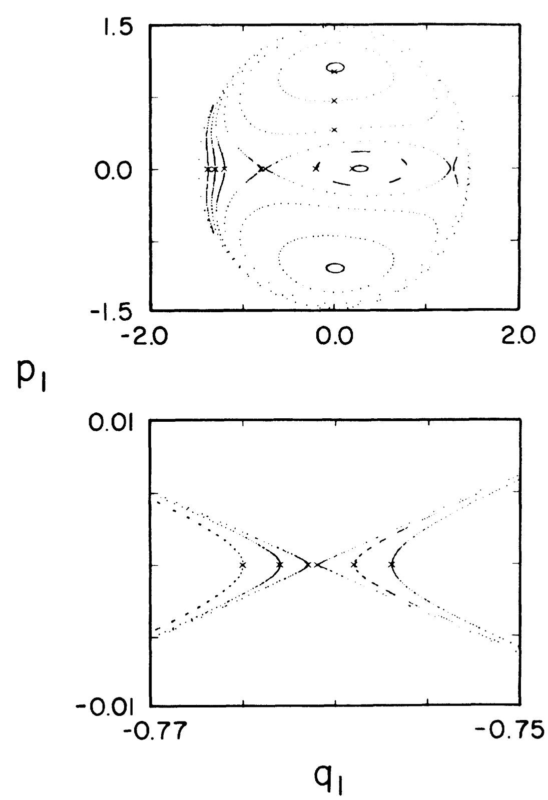

FIG. 5. Surface-of-section plots with $\epsilon_1=0.5$ and the other parameters as in Fig. 3. In (a), a global view of the surface of section is shown. In (b), a blowup of the region surrounding a hyperbolic fixed point is shown. No evidence of stochasticity is visible.

other exponentially once they leave the immediate neighborhood of the fixed point, and a large amount of accuracy is needed to separate them.

In Fig. 6, we show more cases where $\epsilon_1=0.5$ and $\epsilon_2=1.0$ with various values of Δ_1 and Δ_2. When $\Delta_1=\Delta_2/2$, the system appears integrable, and otherwise it does not, consistent with the Painlevé analysis.

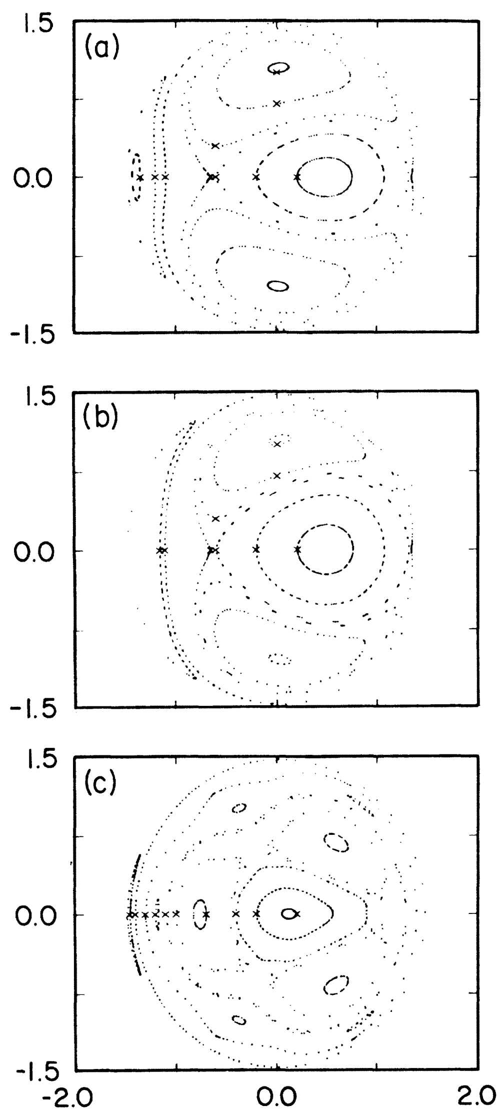

FIG. 6. Surface-of-section plots with $\epsilon_1=0.5$ and varying frequency mismatch. Parameters are $H=-0.1$, $I_0=2.01$, $I_1=1.1$, $I_2=1.6$, $\epsilon_1=0.5$, $\epsilon_2=1.0$; and (a) $\Delta_1=0.1$, $\Delta_2=0.2$, (b) $\Delta_1=0.2$, $\Delta_2=0.4$, (c) $\Delta_1=0.2$, $\Delta_2=0.0$. In (a) and (b), no signs of stochasticity are visible. In (c), stochasticity is visible.

C. Lyapunov exponent

If we linearize the equations of motion of a system with N degrees of freedom, we obtain N new equations. We may solve the $2N$ equations simultaneously, obtaining d, the distance from the origin of the linearized coordinates, as well as the particle trajectory. If the trajectory is stochastic, then, as $t\to\infty$, d should increase exponentially in time; whereas, if the trajectory is regular, then as $t\to\infty$, d should only increase linearly. Hence, letting

$$\mu_L \equiv \lim_{t\to\infty} \frac{1}{t}\ln d \ , \tag{48}$$

we see that μ_L, the Lyapunov exponent, is positive if the trajectory is stochastic and is zero if the trajectory is regular.

For a weakly nonintegrable system of the sort that we are considering here, numerical computation of the Lyapunov exponent is not a trivial matter. The exponentiation can be so weak that it is completely overwhelmed on the time scale of the computation by the linear separation.

To eliminate this difficulty to the extent possible for the problem which we are considering here, we chose the initial vector of the linearized coordinates to be close to (but not exactly in) the direction of the initial vector of the time derivatives of the standard coordinates. We averaged over a time equal to 1000 surface-of-section points after eliminating a time segment equal to the first 200 surface-of-section points.

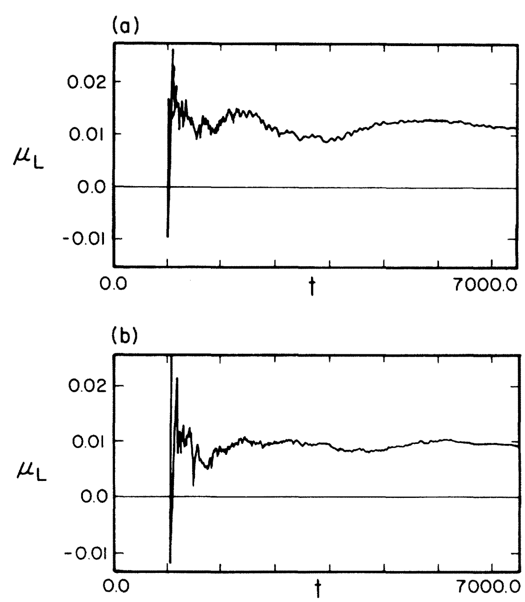

FIG. 7. Lyapunov exponent as a function of time. The Lyapunov exponent is shown for two different trajectories starting in the stochastic region of Fig. 4(a).

The results of this procedure are shown in Fig. 7 for two trajectories in the stochastic region of Fig. 4(a). The first, which is started at $q_1=-1.0$, $p_1=0.0$ has a final value $\mu_L=0.0115\pm0.0028$ and the second, which is started at $q_1=-0.9$, $p_1=0.3$ has a final value $\mu_L=0.0091\pm0.0022$. The error is computed by dividing the raw standard deviation by $(J-1)^{1/2}$, where J is the number of data points. This procedure is correct as long as the data represents a stationary, random process, and the assumption that the process is stationary is reasonable since after 1000 surface-of-section points the stochastic region is fairly well covered. The two final values agree with each other to within their error, which, the reader will note, is quite large ($\simeq 20\%$) even after the care we have taken to reduce it as far as possible.

Because it is difficult to compute accurately and because it gives no feeling for the global properties of phase space, the Lyapunov exponent is a far less powerful tool for studying the integrability of weakly nonintegrable systems than are the surface-of-section plots. It does, however, have the advantage that it can be extended to systems with more degrees of freedom than two, while surface-of-section plots cannot.

IV. SUMMARY

This paper is the first in a series of three papers, devoted to studying under what circumstances restricted multiple three-wave interactions may be treated statistically. We have focused attention on this system, because it appears to be the simplest possible multiply interacting three-wave system with an arbitrarily large number of waves, and can be analyzed in detail.

In this paper, we have used Painlevé analysis to investigate the integrability of this system. Nonintegrability is a necessary (but certainly not sufficient) condition for a system to be treated statistically. The Painlevé analysis indicates that the system is integrable in two special cases. In the first case, all the coupling coefficients are equal and the frequency mismatches arbitrary. The second case is when some of the coupling coefficients equal one-half the others and the corresponding frequency mismatches equal one-half the others.

Specializing to the case where there are only two triads, we have presented numerical evidence indicating that the system is indeed integrable in the cases which Painlevé analysis indicates ought to be integrable and is otherwise nonintegrable.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation under Contract No. PHY-80-14189 and the U. S. Office of Naval Research under Contract No. N000-1477-C-0590.

APPENDIX

Recalling that the general solution of Eq. (29) $\underline{A}_j$ may be written $\underline{\beta}_j+\underline{a}_j$, where $\underline{\beta}_j$ is the solution to Eq. (22) and $\underline{a}_j$ is a particular solution of Eq. (29), we find, through $j=2$, that

$$\beta_{01}=\beta_{01}^*=0\ ,$$

$$\beta_{n1}=\beta'_{n1}=\beta^*_{n1}=\beta'^*_{n1}=0\ ,\quad 1\le n\le M \qquad \text{(A1a)}$$

$$\frac{\beta'_{n1}}{\alpha'_n}=-\frac{\beta_{n1}}{\alpha_n}\ ,\quad \frac{\beta'^*_{n1}}{\alpha'^*_n}=-\frac{\beta^*_{n1}}{\alpha^*_n}\ ,\quad M+1\le n\le M\ ;$$

$$\frac{a_{01}}{\alpha_0}=-\frac{i\epsilon}{8}\frac{\Sigma_0}{\alpha_0}+\frac{i\epsilon}{2}\frac{D_0}{\alpha_0}\ ,$$

$$\frac{a^*_{01}}{\alpha^*_0}=-\frac{i\epsilon}{8}\frac{\Sigma_0}{\alpha_0}-\frac{i\epsilon}{2}\frac{D_0}{\alpha_0}\ ,$$

$$\frac{a_{n1}}{\alpha_n}=\frac{a'^*_{n1}}{\alpha'^*_n}=\frac{i\epsilon}{8}\frac{\Sigma_0}{\alpha_0}+\frac{i\epsilon}{2}\left[\frac{D_0}{\alpha_0}+\frac{\Delta_n}{\epsilon}\right]\ ,\quad 1\le n\le M$$

$$\frac{a'_{n1}}{\alpha'_n}=\frac{a^*_{n1}}{\alpha^*_n}=\frac{i\epsilon}{8}\frac{\Sigma_0}{\alpha_0}-\frac{i\epsilon}{2}\left[\frac{D_0}{\alpha_0}+\frac{\Delta_n}{\epsilon}\right]\ ,\quad 1\le n\le M \qquad \text{(A1b)}$$

$$\frac{a_{n1}}{\alpha_n}=\frac{a^*_{n1}}{\alpha^*_n}=\frac{i\epsilon}{8}\frac{\Sigma_0}{\alpha_0}\ ,\quad a'_{n1}=a'^*_{n1}=0\ ,\quad M+1\le n\le N\ ;$$

$$\frac{\beta^*_{02}}{\alpha^*_0}=\frac{\beta_{02}}{\alpha_0}\ ,\quad \sum_{n=1}^{M}(\alpha^*_n\beta_{n2}-\alpha_n\beta^*_{n2})=0\ ,$$

$$\frac{\beta'_{n2}}{\alpha'_n}=-\frac{\beta_{n2}}{\alpha_n}-\frac{\beta_{02}}{\alpha_0}\ ,\quad \frac{\beta'^*_{n2}}{\alpha'^*_n}=-\frac{\beta^*_{n2}}{\alpha^*_n}-\frac{\beta_{02}}{\alpha_0}\ ,\quad 1\le n\le M \qquad \text{(A1c)}$$

$$\frac{\beta_{n2}}{\alpha_n}=\frac{\beta^*_{n2}}{\alpha^*_n}=\frac{\beta'_{n2}}{\alpha'_n}=\frac{\beta'^*_{n2}}{\alpha'^*_n}=-\frac{1}{4}\frac{\beta_{02}}{\alpha_0}\ ,\quad M+1\le n\le N\ ;$$

$$a_{02}=a^*_{02}=0\ ,$$

$$a_{n2}=0\ ,\quad \frac{a^*_{n2}}{\alpha^*_n}=\frac{1}{2}\frac{\Sigma_1}{\alpha_1\alpha'^*_1}\delta_{n1}\ ,\quad 1\le n\le M$$

$$\frac{a'_{n2}}{\alpha'_n}=-\frac{\epsilon^2}{64}\frac{\Sigma_0^2}{\alpha_0^2}-\frac{\epsilon^2}{4}\left[\frac{D_0^2}{\alpha_0^2}-\frac{\Delta_n^2}{\epsilon^2}\right]\ ,\quad 1\le n\le M$$

$$\frac{a'^*_{n2}}{\alpha'^*_n}=-\frac{1}{2}\frac{\Sigma_1}{\alpha_1\alpha'^*_1}\delta_{n1}-\frac{\epsilon^2}{64}\frac{\Sigma_0^2}{\alpha_0^2}-\frac{\epsilon^2}{4}\left[\frac{D_0^2}{\alpha_0^2}-\frac{\Delta_n^2}{\epsilon^2}\right]\ ,\quad 1\le n\le M \qquad \text{(A1d)}$$

$$\frac{\alpha_{n2}}{\alpha_n}=-\frac{i\epsilon}{16}\frac{\Sigma_0}{\alpha_0}\frac{\beta_{n1}}{\alpha_n}+\frac{i\epsilon}{4}\frac{D_0}{\alpha_0}\frac{\beta_{n1}}{\alpha_n}+\frac{\epsilon^2}{512}\frac{\Sigma_0^2}{\alpha_0^2}-\frac{\epsilon^2}{64}\frac{\Sigma_0}{\alpha_0}\frac{D_0}{\alpha_0}\,,\quad M+1\le n\le N$$

$$\frac{\alpha^*_{n2}}{\alpha^*_n}=-\frac{i\epsilon}{16}\frac{\Sigma_0}{\alpha_0}\frac{\beta^*_{n1}}{\alpha^*_n}-\frac{i\epsilon}{4}\frac{D_0}{\alpha_0}\frac{\beta^*_{n1}}{\alpha^*_n}+\frac{\epsilon^2}{512}\frac{\Sigma_0^2}{\alpha_0^2}+\frac{\epsilon^2}{64}\frac{\Sigma_0}{\alpha_0}\frac{D_0}{\alpha_0}\,,\quad M+1\le n\le N$$

$$\frac{\alpha'_{n2}}{\alpha'_n}=\frac{i\epsilon}{16}\frac{\Sigma_0}{\alpha_0}\frac{\beta_{n1}}{\alpha_n}+\frac{i\epsilon}{4}\frac{D_0}{\alpha_0}\frac{\beta_{n1}}{\alpha_n}-\frac{3\epsilon^2}{512}\frac{\Sigma_0^2}{\alpha_0^2}-\frac{\epsilon^2}{64}\frac{\Sigma_0}{\alpha_0}\frac{D_0}{\alpha_0}\,,\quad M+1\le n\le N$$

$$\frac{\alpha'^*_{n2}}{\alpha'^*_n}=\frac{i\epsilon}{16}\frac{\Sigma_0}{\alpha_0}\frac{\beta^*_{n1}}{\alpha^*_n}-\frac{i\epsilon}{4}\frac{D_0}{\alpha_0}\frac{\beta^*_{n1}}{\alpha^*_n}-\frac{3\epsilon^2}{512}\frac{\Sigma_0^2}{\alpha_0^2}+\frac{\epsilon^2}{64}\frac{\Sigma_0}{\alpha_0}\frac{D_0}{\alpha_0}\,,\quad M+1\le n\le N\,;$$

where

$$\Sigma_0=\sum_{n=M+1}^{N}\alpha_n\alpha'^*_n\,,$$

$$\Sigma_1=\sum_{n=M+1}^{N}(\alpha_n\beta'^*_{n1}+\alpha'^*_n\beta_{n1})\,, \tag{A2}$$

$$D_0=\sum_{n=1}^{M}\left[\frac{i\Delta_n}{2}\right]\alpha_n\alpha'^*_n\,.$$

In the case where $M=N$, Σ_0 and Σ_1 are zero. In the case where $M<N$, the quantities enclosed in large parentheses,

$$\left[\frac{D_0}{\alpha_0}+\frac{\Delta_n}{\epsilon}\right]\ \text{and}\ \left[\frac{D_0^2}{\alpha_0^2}-\frac{\Delta_n^2}{\epsilon^2}\right]\,, \tag{A3}$$

are zero. The solution at $j=3$ is not written down because it is quite lengthy. Actually, it is not necessary to determine this solution, but only to determine that the solubility condition

$$S_{03}-\frac{\alpha_0}{\alpha_0^*}S^*_{03}+\frac{i\epsilon}{2}\sum_{n=1}^{M}(\alpha_nS'^*_{n3}+\alpha'^*_nS_{n3})+\frac{i\epsilon}{2}\frac{\alpha_0}{\alpha_0^*}\sum_{n=1}^{M}(\alpha'_nS^*_{n3}+\alpha^*_nS'_{n3})=0 \tag{A4}$$

is met.

[1]J. A. Armstrong, N. Bloembergen, J. Ducing, and P. S. Pershan, Phys. Rev. 127, 1918 (1962).
[2]L. F. McGoldrick, J. Fluid Mech. 21, 305 (1965).
[3]B. Coppi, M. N. Rosenbluth, and R. N. Sudan, Ann. Phys. (N.Y.) 55, 207 (1969).
[4]M. I. Rabinovich and A. L. Fabrikant, Zh. Eksp. Teor. Fiz. 77, 617 (1979) [Sov. Phys.—JETP 50, 311 (1979)].
[5]J.-M. Wersinger, J. M. Finn, and E. Ott, Phys. Rev. Lett. 44, 453 (1980).
[6]See A. G. Sitenko and H. Wilhelmsson, Phys. Scr. 7, 189 (1973), and other articles in this issue.
[7]O. H. Hald, Phys. Fluids 19, 914 (1976).
[8]See, e.g., R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969), Chap. 1.4 and references cited therein.
[9]See, e.g., R. H. Kraichnan and D. Montgomery, Rep. Prog. Phys. 43, 547 (1980) and references cited therein.
[10]H. Poincaré, *Les méthodes nouvelles de la mécanique céleste* (Gauthier-Villars, Paris, 1892), Vol. I.
[11]K. M. Watson, B. J. West, and B. I. Cohen, J. Fluid Mech. 77, 185 (1976).
[12]J. D. Meiss, N. Pomphrey, and K. M. Watson, Proc. Natl. Acad. Sci. (U.S.A.) 76, 2109 (1979); J. D. Meiss, Ph.D. thesis, Berkeley, 1980 (unpublished).
[13]J. D. Meiss, Phys. Rev. A 19, 1780 (1979).
[14]See, e.g., V. V. Golubev, *Lectures on Integration of the Equations of Motion of a Rigid Body About a Fixed Point* (Gosizdat, Moscow, 1953).
[15]M. Tabor and J. Weiss, Phys. Rev. A 24, 2157 (1981).
[16]T. Bountis, H. Segur, and F. Vivaldi, Phys. Rev. A 25, 1257 (1982).
[17]Y. F. Chang, M. Tabor, and J. Weiss, Phys. Lett. 85A, 211 (1981).
[18]Y. F. Chang, M. Tabor, and J. Weiss, J. Math. Phys. 23, 531 (1982); Y. F. Chang, J. M. Greene, M. Tabor, and J. Weiss (unpublished).
[19]M. J. Ablowitz and R. Haberman, J. Math. Phys. 16, 2301 (1975).
[20]This example was brought to our attention by A. Dragt.
[21]A. Ramani, B. Dorizzi, and B. Grammaticos, Phys. Rev. Lett. 49, 1539 (1982).
[22]M. J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys. 21, 715 (1980).
[23]G. Bennetin, C. Froeschle, and J. P. Scheideker, Phys. Rev. A 19, 2454 (1979).
[24]J. L. Hindmarsh, "TV80LIB Manual," Lawrence Livermore Laboratory, University of California (unpublished).