RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 27, NUMBER 2 FEBRUARY 1983

Periodic perturbation on a period-doubling system

H. Horner
Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 19, D-6900 Heidelberg, West Germany
(Received 24 August 1982)

The effect of a periodic perturbation on a nonlinear dynamic system undergoing a sequence of period doublings is investigated. The results obtained from linear response theory and from numerical calculations resemble the observations made by Giglio *et al.* on Rayleigh-Benard convection.

In a recent Letter,[1] Giglio, Musazzi, and Périni have reported experiments on a period-doubling sequence of transitions in a Rayleigh-Benard cell with low aspect ratio. Their data are presented in the form of power spectra taken at various values of the Rayleigh number R. As R is increased the spectra show first the fundamental frequency f_0 and then in addition $f_1 = f_0/2$ and $f_2 = f_0/4$ as lines as sharp as expected. Increasing R further near the expected $f_3 = f_0/8$ a doublet appears with a separation $\delta f_3 \approx f_0/38$. At still higher values of R they observed other doublets around the expected but not present signals due to $f_4 = f_0/16$ with a spacing $\delta f_4 \approx f_0/19$. At this value of R the signals due to f_3 are found as singlets. The frequency f_4 or odd multiples could never be detected as sharp lines and signals due to further period-doubling bifurcations were also absent.

A possible partial explanation for this unexpected feature might be found in the following. Assume the degrees of freedom responsible for the period doubling are coupled to a weak external periodic perturbation of unknown origin or to other oscillating internal degrees of freedom which are unobserved otherwise. This may give rise to the phenomena observed.

In order to demonstrate this I have studied the one-dimensional mapping

$$x(t+\tau) = f_a(x(\tau)) + \Delta h(\Omega t) \quad , \tag{1}$$

with $f_a(x) = 4ax(1-x)$. This models a discrete nonlinear dynamical system and for $\Delta = 0$ exhibits a period-doubling sequence of bifurcations[2] with Feigenbaum scaling. $h(\Omega t)$ is a periodic function,

$$h(\Omega t + 2\pi) = h(\Omega t) \equiv \sum_\nu h_\nu \exp(i\nu\Omega t) \quad , \tag{2}$$

where ν is integer. The frequency Ω is supposed to be unrelated to the fundamental frequency $\omega_0 = 2\pi f_0 = 2\pi/\tau$.

The power spectrum is

$$g(\omega) = \lim_{N\to\infty} \left| N^{-1} \sum_{l=1}^{N} x(l\tau) \exp(-il\omega\tau) \right|^2 \quad . \tag{3}$$

Assume the control parameter a is chosen such that the unperturbed mapping has a stable solution with period $2^k\tau$. Then for sufficiently small Δ linear response theory[3] yields sharp lines in the spectrum at frequencies

$$\omega = \nu\Omega + 2\pi n/2^k\tau \tag{4}$$

with integer ν and n. Within linear response theory it is easily seen[3] that the spectrum behaves as

$$g(\omega) \sim (1 - 2\chi\cos\hat{\omega}_\nu + \chi^2)^{-1} \tag{5}$$

with $\hat{\omega}_\nu = \nu\Omega + 2\pi\hat{n}/2^k\tau$ and $\hat{n}$ such that $-\pi < \hat{\omega}_\nu < \pi$ and ω is given by (4). The quantity

$$\chi = \prod_{l=1}^{2^k} f_a'(x(l\tau)) \tag{6}$$

is the Lyapunov number and decreases monotonously from $\chi = 1$ at $a = a_k$ where period 2^k_τ sets in in the unperturbed system to $\chi = -1$ just below $a = a_{k+1}$ where period 2^{k+1}_τ appears first. Obviously $g(\omega)$, Eq. (5), has a resonance structure for values $a \approx a_k$ and frequencies $\omega \approx 2\pi n/2^k\tau$. If the frequency of the perturbation or one of its harmonics is near resonance, the signals resulting from it are enhanced.

This picture is supported by numerical calculations. I have chosen $h(\Omega t) = 1$ for $0 < \Omega t < \pi$ and $h(\Omega t) = -1$ for $\pi < \Omega t < 2\pi$. This yields odd harmonics only. The choice $\Omega\tau/2\pi = \frac{3}{80}$ produces doublets with spacing $\delta\omega/2\pi = 1/20\tau$ near $a = a_4$ [Fig. 1(d)] and the third harmonic doublets with spacing $\delta\omega/2\pi = 1/40\tau$ near $a = a_3$ [Fig. 1(b)] reproducing qualitatively the observations made in the experiment. Other choices such as $\Omega\tau/2\pi = \frac{17}{80}, \frac{23}{80}$, or $\frac{37}{80}$ give similar results. The computations shown in Fig. 1 were done with $\Delta = 0.001$. This choice not only matches the order of magnitude of the observed signals but also destroys higher bifurcations. As an example Fig. 1(e) is computed for $a = 0.892$ for which period 16 is stable for the unperturbed map. The signal corresponding to period 16 is clearly absent and other values of a also never show this signal. Furthermore, this value yields already chaotic behavior[4] with sensitive dependence on initial conditions contrary to the other values used in Figs.

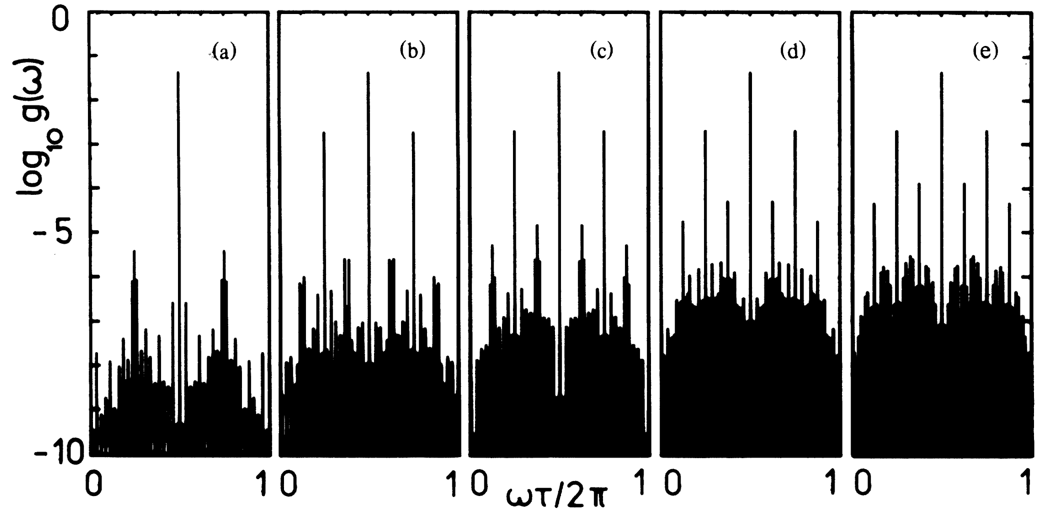

FIG. 1. Power spectrum $\log_{10} g(\omega)$ computed from Eq. (1) for $a = 0.8625$ (a), $a = 0.885$ (b), $a = 0.887$ (c), $a = 0.889$ (d), and $a = 0.892$ (e).

1(a)–1(d). The remaining Figs. 1(a) and 1(c) show the spectra just above the onset of periods 4 and 8, respectively.

The above-mentioned shift in the onset of chaotic behavior as well as the fact that the doublets appear already prior to the bifurcations may explain the deviations of δ from Feigenbaum's value found in the experiment.

The above results suggest performing experiments on period-doubling systems with controlled periodic perturbations applied. Such a study has actually been performed by Gollub and Benson,[5] again on Rayleigh-Benard convection. Their system shows only one period doubling, even without external perturbation, and then the appearance of a second incommensurate frequency as the Rayleigh number is increased. Because of this more complicated behavior of the unperturbed system no analysis using the above model has been undertaken.

I am indebted to M. Giglio for stimulating discussions. Correspondence and discussions with T. Geisel, H. Thomas, J. Heldstab, and M. Lücke are also acknowledged.

[1]M. Giglio, S. Musazzi, and U. Perini, Phys. Rev. Lett. 47, 243 (1981).

[2]M. Feigenbaum, J. Stat. Phys. 19, 25 (1978); 21, 669 (1979).

[3]H. Horner (unpublished); J. Heldstab, H. Thomas, T. Geisel, and G. Radons, Z. Phys. B (in press).

[4]A similar observation has been made by M. Lücke and Y. Saito, Phys. Lett. 91A, 205 (1982).

[5]J. P. Gollub and S. V. Benson, Phys. Rev. Lett. 41, 948 (1978).