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# Fractal nature of turbulence as manifested in turbulent diffusion

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Scaling concepts and fractal statistics are used to assess the effects of intermittency on the important process of diffusion in fully developed turbulence. Sizable corrections to Richardson's "$\frac{4}{3}$ law" and related laws are found. Reexamination of existent data shows agreement with theory.

The Kolmogorov theory[1] of fully developed turbulence from 1941, which enjoys ingenious simplicity, could have been a precise description of turbulence in the inertial range if not for the intermittent nature of turbulence at smaller scales.[2-6] As energy is cascaded from large to small scales the turbulent activity gets concentrated in smaller and smaller fractions of space: the active region becomes a fractal.[3,7]

The most commonly quoted manifestation of the intermittent nature of turbulence is the long tail in the correlation function of the viscous dissipation, $\epsilon(\vec{r})$.[8] Experimentally one finds[5,9]

$$\langle \epsilon(\vec{r})\epsilon(\vec{r}+\vec{l})\rangle = \bar{\epsilon}^2(l_0/l)^{\mu} \ , \tag{1}$$

where $l$ is in the inertial range, $l_d << l << l_0$, and $l_0$, $l_d$ are the stirring and dissipation length scales, respectively. $\bar{\epsilon}$ is the mean energy input per unit mass per unit time. Experimentally one finds[8,9] $0.25 \lesssim \mu \lesssim 0.50$ and theoretically one argues[3,7] that $\mu = d - D$, where $d$ is the spatial dimension, and $D$ the fractal dimension of the active region. In the case of fractally homogeneous turbulence we have recently estimated theoretically $2.50 \lesssim D \lesssim 2.75$ in agreement with experiment.[10]

Clearly, intermittency will also give rise to observable properties of turbulent transport processes. In this Communication we discuss the effects of intermittency on the important process of turbulent diffusion.[11] Concentrating on the relative diffusion of test particles, we find the intermittency corrections to Richardson's "$\frac{4}{3}$ law"[12] and to other laws which stem from Kolmogorov's similarity theory.[5,13] A reexamination of the available experimental data shows agreement with our analysis. It is thus possible to suggest turbulent diffusion as an interesting probe of the fractal nature of turbulence.

Consider then the relative motion of two particles immersed in a turbulent medium without affecting its properties. Denoting the separation between the particles by $\vec{R} \equiv \vec{r}_1 - \vec{r}_2$, the three quantities of major interest would be $d\langle R^2\rangle/dt$, $\langle (d\vec{R}/dt)^2\rangle$, and $\langle R^2(t)\rangle$, where angle brackets denote averaging over many realizations of the experiment. All theories based on dimensional analysis in the inertial range would predict[1,5,13]

$$\frac{dR^2}{dt} \sim \bar{\epsilon}^{1/3} R^{4/3} \ ; \ \left\langle \left(\frac{d\vec{R}}{dt}\right)^2 \right\rangle \sim \bar{\epsilon}^{2/3} R^{2/3} \ ; \quad R^2(t) \sim \bar{\epsilon} t^3 \ , \tag{2}$$

where here and below $R \equiv \langle R^2\rangle^{1/2}$. The corrections due to intermittency are dimensionless. We shall write

$$\frac{dR^2}{dt} \sim \bar{\epsilon}^{1/3} R^{4/3}\left(\frac{R}{l_0}\right)^{\alpha} \ ; \left\langle \left(\frac{d\vec{R}}{dt}\right)^2 \right\rangle \sim \bar{\epsilon}^{2/3} R^{2/3}\left(\frac{R}{l_0}\right)^{\beta} \ ; \quad R^2(t) \sim \bar{\epsilon} t^3 \left(\frac{t}{t_0}\right)^{\gamma} \ , \tag{3}$$

where $t_0 \equiv (l_0^2/\bar{\epsilon})^{1/3}$.

The sign of $\alpha$, $\beta$, and $\gamma$ can be determined by physical considerations alone. If turbulence were space filling, the laws [Eq. (2)] would hold. Clearly if deviation from this behavior exists, it would become more pronounced at smaller length scales, where turbulence becomes very spotty. If the test particles are caught in an inactive region, their relative diffusion would become molecular and thus negligible.[11,12] Therefore, the smaller the separation between particles, the more susceptible they are to intermittency and the reduction in their relative diffusion is thus greater. Since $R << l_0$, $\alpha$ and $\beta$ must then be positive. The exponent $\gamma$ is completely determined by $\alpha$, and by integrating $dR^2/dt$ we find $\gamma = 9\alpha/(2-3\alpha)$. Evidently $\alpha$ must be smaller than $\frac{2}{3}$.

Theoretical estimates for the numerical value of the exponents are obtained by rewriting the above quantities in terms of velocity correlations functions. In the main body of the discussion we shall assume that the turbulence is homogeneous over the fractal[3,7,10] ("fractally homogeneous turbulence" or "absolute curdling"). We shall present, however, the results pertaining to lognormal statistics[5] as well. Using $\vec{R}(t)=\vec{R}(0)+\int_0^t \vec{V}(\tau)d\tau$, where $\vec{V}$ is the relative velocity, we find

$$\frac{dR^2}{dt}=2\int_0^t \langle \vec{V}(t)\cdot\vec{V}(\tau)\rangle d\tau \ ;$$

$$\left\langle \left(\frac{d\vec{R}}{dt}\right)^2 \right\rangle = \langle \vec{V}(t)\cdot\vec{V}(t)\rangle \ . \tag{4}$$

The second of these quantities is simpler to estimate, being a one-time correlation function. In fact, this correlation function is precisely the square of the velocity difference across a distance $R$ at time $t$. For fractally homogeneous turbulence it is simply

$$\langle [\vec{v}(\vec{r})-\vec{v}(\vec{r}+\vec{R}(t))]^2\rangle \sim V_R^2(R/l_0)^{\mu} \ , \tag{5}$$

where $V_R$ is the velocity difference across distance $R$ in an *active region*. The reason for Eq. (5) is the $\vec{v}(\vec{r})$ and $\vec{v}(\vec{r}+\vec{R})$ are correlated only if they belong to the same active region. The weight of such an occurrence on a fractal whose dimension is $D$ is[3] $(R/l_0)^{d-D}=(R/l_0)^{\mu}$. The velocity difference across a length $R$ can be found by equating $\bar{\epsilon}$ to the rate of transfer on length scales $R$, which *in an active region* is[4] $V_R^3/R$. Thus $\bar{\epsilon}\sim(R/l_0)^{\mu}V_R^3/R$ and $V_R\sim\bar{\epsilon}^{1/3} \times R^{1/3}(R/l_0)^{-\mu/3}$. Consequently

$$\left\langle \left(\frac{d\vec{R}}{dt}\right)^2 \right\rangle \sim \bar{\epsilon}^{2/3}R^{2/3}\left(\frac{R}{l_0}\right)^{\mu/3} \ . \tag{6}$$

We comment that since $0.25<\mu<0.5$ the correction to dimensional analysis is quite sizable here.

The quantity $dR^2/dt$ is slightly more difficult to obtain. We rewrite the integral of the time correlation function as

$$\int_0^t d\tau \langle \vec{V}(t)\cdot\vec{V}(\tau)\rangle d\tau = \langle \vec{V}(t)\cdot\vec{V}(t)\rangle \int_0^t d\tau g\left(\frac{t-\tau}{t_R}\right) \ , \tag{7}$$

where $t_R$ is the correlation time between velocity differences across a scale $R$ in an active region,

$$t_R\sim R/V_R\sim\bar{\epsilon}^{-1/3}R^{2/3}(R/l_0)^{\mu/3} \ , \tag{8}$$

and $t=0$ is the time origin for the inertial subrange-dominated phase of relative diffusion. In writing Eq. (7) we have assumed that the dominant contribution to the integral comes from $\tau\sim t$, and thus $R(\tau)\sim R(t)$. A change of variables leads to

$$\langle \vec{V}(t)\cdot\vec{V}(t)\rangle t_R\int_0^{t/t_R} ds\, g(s) \sim \begin{cases} \langle \vec{V}(t)\cdot\vec{V}(t)\rangle t_R \ , & t\gg t_R \\ \langle \vec{V}(t)\cdot\vec{V}(t)\rangle t \ , & t\ll t_R \ . \end{cases} \tag{9}$$

Using Eqs. (4), (7), and (9) we then find

$$\frac{d\langle R^2\rangle}{dt}\sim\begin{cases} \bar{\epsilon}^{1/3}R^{4/3}\left(\frac{R}{l_0}\right)^{\mu/6} , & t\ll t_R \\ \bar{\epsilon}^{1/3}R^{4/3}\left(\frac{R}{l_0}\right)^{2\mu/3} , & t\gg t_R \ . \end{cases} \tag{10}$$

These results can be integrated in a straightforward fashion to yield

$$R^2(t)\sim\begin{cases} \bar{\epsilon}t^3\left(\frac{t}{t_0}\right)^{3\mu/(4-\mu)} , & t\ll t_R \\ \bar{\epsilon}t^3\left(\frac{t}{t_0}\right)^{3\mu/(1-\mu)} , & t\gg t_R \ . \end{cases} \tag{11}$$

One should stress that the two regimes of $t$ compared to $t_R$ *are not* short- and long-time regimes. In fact when intermittency does not exist $t$ scales like $t_R$. With intermittency included, $t_R=Ct(t/t_0)^{\delta}$ where $\delta$ is shown to be positive by using Eq. (8), and $C$ is a dimensionless constant. As will be shown below, there are experiments in which $C$ can be estimated. When we have no such possibility we shall assume $C\sim O(1)$ and thus for $t\ll t_0$, $\delta>0$ leads to $t\gg t_R$.

It is interesting to compare these results to experiments. A set of reasonably accurate data for turbulent diffusion is complied in Ref. 14. Here puffs of smoke resulting from explosions were followed, and $R^2$ has been measured as a function of time. Gifford showed[14] a $t^3$ law was consistent with the data at intermediate time ($10\lesssim t\lesssim 20$ sec). We have reanalyzed the data and found that a least-squares fit resulted in a $t^{3+\gamma}$ law, where $0.45>\gamma>0.15$. (The lower estimate obtains if all 35 data points for $t>10$ sec are taken, whereas the higher is found if the 14 data points with $t>14$ sec are taken.) One can easily see that in these experiments $t<t_R$. In fact, leaving out intermittency for this estimate,

$$t_R\sim\bar{\epsilon}^{-1/3}R^{2/3}\sim\bar{\epsilon}^{-1/3}R_0^{2/3}[R^2(t)/R^2(0)]^{1/3} \ ,$$

where $R(0)$ is the average size of the puff at $t=0$. Gifford estimates[13] $\bar{\epsilon}^{-1/3}R_0^{2/3}\sim 10$ sec. Here typically $[R^2(t)/R^2(0)]^{1/3}\sim 5$. Since $10<t<20$ the first regime in Eq. (12) is realized and therefore our theory would predict $\gamma=3\mu/(4-\mu)$. Accordingly $\mu$ is found to be $0.2<\mu<0.5$, which is in the right vicinity.

It is difficult to resist the temptation of reexamining the classical figure of Richardson from Ref. 12. Richardson has plotted the diffusivity $dR^2/dt$ over five orders of magnitude of $R$. In Fig. 1 we replot the data, excluding the lowest point which pertains to molecular diffusivity. The line with a slope of $\frac{4}{3}$, in addition to a line of least-squares fit, is shown. The

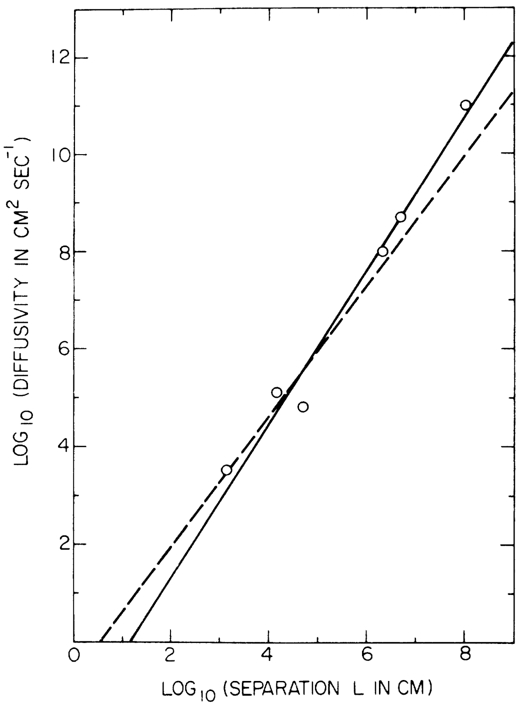


FIG. 1. Turbulent diffusivity in the atmosphere as a function of the length scale. The dashed line is the original line suggested by Richardson (Ref. 12) with a slope of $\frac{4}{3}$. The continuous line is a least-squares fit with a slope of 1.57. The difference is attributed to an intermittency exponent $\mu \sim 0.36$.

latter yields a slope of 1.57. If one excludes the highest point one finds a slope of 1.48. Assuming $C$ in Eq. (13) to be $O(1)$ we use the second of Eqs. (11) to compare with these results (i.e., the regime $t \gg t_R$). Thus $0.15 \lesssim 2\mu/3 \lesssim 0.24$, or $0.22 \lesssim \mu \lesssim 0.36$ which again falls in the correct range.

Additional support for the above approach can be obtained from experiments on two-dimensional (2D) turbulent diffusion. In 2D we expect $\mu = 0$.[9,15] Indeed, a least-squares fit to the results on the 2D turbulent diffusion of constant level balloons in the stratosphere agrees very well with theoretical predictions based on dimensional analysis alone.[16] In addition we have convinced ourselves that also the curves shown in Ref. 5 which pertain to two dimensions call for no modification.

TABLE I. Intermittency corrections. The exponents $\alpha$, $\beta$, $\gamma$ are defined in Eq. (3).

| | | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|---|
| Absolute curdling | $t \ll t_R$ | $\mu/6$ | $\mu/3$ | $\frac{3\mu}{4-\mu}$ |
| | $t \gg t_R$ | $2\mu/3$ | $\mu/3$ | $\frac{3\mu}{1-\mu}$ |
| Lognormal | $t \ll t_R$ | $\mu/18$ | $\mu/9$ | $\frac{3\mu}{12-\mu}$ |
| | $t \gg t_R$ | $\mu/9$ | $\mu/9$ | $\frac{3\mu}{6-\mu}$ |

Although it seems that theories relying on absolute curdling agree better with experiments then theories of intermittency based on lognormal statistics,[10,17] we have studied for completeness the effects on turbulent diffusion within the lognormality assumption as well. The corrections to dimensional analysis are much smaller, and are summarized, together with the absolute curdling results, in Table I.

We think that the most important conclusion of the above analysis does not lie in the comparison of theory to previously performed experiments. In fact it seems that careful modern experiments should be done to test the various predictions shown in Table I. Currently most of the attempts at investigating the intermittency exponent $\mu$ concentrate on measuring higher-order moments of the velocity field.[18] This approach is plagued by inaccuracies at the tails of the distribution functions. In contrast, turbulent diffusion and other transport processes seem to offer a good method of investigation of the fractal nature of fully developed turbulence.

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