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Adiabatic compression of rotating plasmas

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The adiabatic compression of plasmas with mass flow is treated via a variational principle based on a few constants of motion. It is then possible to adapt Grad's alternating-dimensions numerical method to this case.

The behavior of confined plasma systems with macroscopic flows is of considerable interest, mainly because of the generation of large flows of Mach number close to unity during heating by neutral beam injection. Such flows cause an appreciable change in the equilibrium state as well as in its stability properties. Moreover, if the rotating plasma is compressed in order to further increase its temperature, conservation of angular momentum and other quantities will cause the flow to speed up, perhaps to supersonic levels, with the possibility of creating shock waves and even losing confinement.

In this article we present a theory and a numerical scheme that will enable a thorough investigation of an adiabatically compressed, rotating plasma. The work generalizes ideas of Grad[1,2] concerning static equilibria and similarly considers plasmas governed by the ideal magnetohydrodynamics (MHD) equations. But while Grad introduced a new type of differential equation, the so-called "queer differential equation," to handle the problem, we treat it via a new variational principle. We envision a plasma in a toroidal device with nested magnetic surfaces, which may also be surrounded by a vacuum region. The plasma boundary is assumed to be a flux surface. Imagine now a slow compression of the plasma (say, by increasing the magnetic flux in the vacuum region) on a time scale much slower than typical (Alfven) wave transit time across the major dimensions of the device. The waves then cause fast equilibration, and on the compression time scale the plasma may be seen as creeping from one equilibrium state to the next. The purpose of this work is to describe how these different equilibrium states may be determined *without* following the full dynamics.

In general, an axisymmetric ideal MHD equilibrium state with flow is determined[3,4] if one supplies five arbitrary functions of ψ, the poloidal magnetic flux, which we denote by $\Phi(\psi)$, $\Omega(\psi)$, $H(\psi)$, $I(\psi)$, and $S(\psi)$. Let the magnetic field be $\vec{B}=\vec{\nabla}\psi\times\vec{\nabla}\theta+B_\theta\hat{\theta}$ (θ is the ignorable toroidal angle), and denote by $\vec{u}$, p, ρ, and s the plasma velocity, pressure, density, and specific entropy, respectively. We have (in cylindrical coordinates)

$$\vec{u}=\rho^{-1}\Phi\vec{B}+r\,\Omega\hat{\theta}\ , \tag{1}$$

$$B_\theta=\frac{1}{1-\Phi^2/\rho}\left[\frac{I}{r}+r\Phi\Omega\right]\ , \tag{2}$$

$$\frac{\Phi^2}{2\rho^2}\left[\frac{1}{r^2}|\nabla\psi|^2+B_\theta^2\right]-\tfrac{1}{2}r^2\Omega_2+\frac{\gamma}{\gamma-1}S\rho^{\gamma-1}=H(\psi)\ . \tag{3}$$

Equation (1) implies that the flow must be parallel to $\vec{B}$, with magnitude derermined by Φ, up to a rigid rotation of each individual flux surface with frequency $\Omega(\psi)$. The function $I(\psi)$ determines the familiar $1/r$ dependence of the toroidal magnetic field, now modified by the presence of the poloidal flow, and $H(\psi)$ is the analog of the Bernoulli function in fluid dynamics. The equation of state $p=S(\psi)\rho^\gamma$ was used, with γ the adiabatic constant usually taken to be $\frac{5}{3}$. $S\equiv\exp(s)$ must be a flux surface quantity if there is any poloidal flow ($\Phi\neq 0$) because of the energy equation $\vec{u}\cdot\nabla s=0$. The function $\psi(\vec{x})$ is determined from the partial differential equation

$$\vec{\nabla}\cdot\left[\left[1-\frac{\Phi^2}{\rho}\right]\frac{1}{r^2}\nabla\psi\right]+\vec{u}\cdot\vec{B}\dot{\Phi}+r\rho u_\theta\dot{\Omega}+\frac{1}{r}B_\theta\dot{I}+\rho\dot{H}-\frac{1}{\gamma-1}\rho^\gamma\dot{S}=0\ . \tag{4}$$

Here the dot denotes $d/d\psi$, and one substitutes relations (1)–(3) into (4). In particular, Eq. (3) serves to define $\rho=\rho(r,\psi,\nabla\psi)$, and Eq. (4) is elliptic for slow poloidal flow,[5]

$$\Phi^2/\rho<\gamma p/(\gamma p+\vec{B}^2)\ .$$

During the adiabatic compression, the functions Φ, Ω, H, and I change and need to be determined. This is what distinguishes the present problem from the problem of finding, or computing, an equilibrium state. We notice that the magnetic flux is frozen into the plasma, hence constant ψ surfaces always consist of the same fluid particles and retain the same value of ψ. Also the entropy, or $S(\psi)$, is unchanged, by adiabaticity. In addition there are four conservation laws for the dynamically evolving plasma within each moving flux tube, which will serve to determine the four missing functions. The conservation laws are

$$\int_\psi \rho = M(\psi), \quad \int_\psi r\rho u_\theta = L(\psi) \ ,$$
$$\int_\psi B_\theta / r = F(\psi), \quad \int_\psi \vec{u}\cdot\vec{B} = C(\psi) \ ,$$

where the volume integrals are taken within each ψ tube, and the time independence of M, L, F, and C expresses the conservation of mass, angular momentum, toroidal magnetic flux, and circulation, respectively. Following Ref. 1, we introduce a surface average $\langle f\rangle \equiv \oint f\, dS/|\nabla V|$, where dS is an area element on the magnetic surface, and $V(\psi)$ is the volume contained within each surface. Notice that $\oint dS/|\nabla V| = 1$ and $(d/dV)\int_\psi f = \langle f\rangle$. The conservation-law constraints can be written as

$$\langle \rho\rangle = m(\psi)\psi' \ , \tag{5}$$

$$\langle r\rho u_\theta\rangle = l(\psi)\psi', \quad \langle B_\theta/r\rangle = f(\psi)\psi' \ , \tag{6}$$

$$\langle \vec{u}\cdot\vec{B}\rangle = c(\psi)\psi', \quad S = S(\psi) \ , \tag{7}$$

where m,l,f,c are the ψ derivatives of the capital-lettered quantities, also unchanged during the motion, and the prime denotes d/dV. One can also average Eq. (4), obtaining

$$[\langle (1-\Phi^2/\rho) r^{-2} |\nabla V|^2\rangle \psi']' + c\Phi' + l\Omega' + mH' + fI' - \langle \rho^\gamma\rangle \dot{S}/(\gamma-1) = 0 \ . \tag{8}$$

Grad observed[1] that the geometry of flux surfaces,the function $V(\vec{x})$, generally converges much faster than the profile $\psi(V)$ during standard numerical iterations and therefore suggested iterating between a geometry calculation and a cheap profile calculation, the so-called "alternating-dimensions method" (ADM). This technique was applied successfully to many static equilibrium configurations.[2] In our case an extension of ADM will consist of the following steps: (i) Assume magnetic surface geometry $V(\vec{x})$ in the plasma region. (ii) Use the constraints (5)–(7) and relations (1)–(3) to determine Φ, Ω, I, and H as functions of V, ψ, and ψ'. (iii) Solve the ordinary differential equation (ODE) (8) for $\psi(V)$. (iv) Translate the present information into $\Phi(\vec{x})$, $\Omega(\vec{x})$, $I(\vec{x})$, $H(\vec{x})$, and $S(\vec{x})$ and solve Eq. (4) in both the plasma and vacuum regions. (v) Extract a new geomtry $V(\vec{x})$ and go to step (ii). Continue until convergence is achieved. Step (ii) will be rather cumbersome and we will later show how to carry it out efficiently in a more special, although realistic, case. But first we make some remarks.

We first notice from Eq. (1) that u_θ cannot in general vanish. That is, there is in general no equilibrium with purely poloidal flow. Such a flow implies $\Phi I/r + r\rho\Omega = 0$, making ρ too simple to describe nontrivial cases. Another simple case is a purely toroidal flow ($\Phi = 0$). However, during adiabatic compression the $\int_\psi \vec{u}\cdot\vec{B}$ constraint will give rise to poloidal flow. Indeed, assuming $\Phi = 0$, Eqs. (5)–(7) imply $\langle \rho r^2\rangle\langle 1/r^2\rangle/\langle\rho\rangle = fl/(cm)$. This again cannot be expected to be satisfied in general. An exception is the case of no toroidal field, $B_\theta = 0$ as in a multipole configuration. Then $c = f = 0$, $\Phi = I = 0$, and the system is not overdetermined.

As a final remark we point out that it may also be possible to solve the compression problem numerically by a relaxation method, as was suggested in the static case.[6] One may solve the time-dependent ideal MHD equations, adding a damping term to the momentum equation:

$$\rho\frac{\partial\vec{u}}{\partial t} + \rho\vec{u}\cdot\nabla\vec{u} + \vec{\nabla}p - (\vec{\nabla}\times\vec{B})\times\vec{B} = -\alpha\vec{u}_\perp \ , \tag{9}$$

where $\vec{u}_\perp$ is the component of $\vec{u}$ along $\nabla\psi$, and α is some positive coefficient. It can easily be seen that all the previous conservation laws still hold but the total energy is decaying at a rate of $\int \alpha|\vec{u}_\perp|^2 d\vec{x}$. Relaxation will be achieved only when the flow is within flux surfaces. A disadvantage of this method is the small time step necessary for numerical stability in regions where the density ρ is small, i.e., when the Alfven speed $|\vec{B}|/\sqrt{\rho}$ is large.

We now treat a slightly nonideal, but more realistic, plasma. It is well known that due to very high heat conductivity along the magnetic field, each flux surface tends to have constant temperature $T = T(\psi)$ where $p = T\rho$. This condition is more appropriate than $S = S(\psi)$. Moreover, the poloidal flow damps out on a similar time scale as a result of the magnetic pumping effect in the torus.[7] Assuming compression to be much slower than the parallel thermalization time we may consider Φ to vanish. The equilibrium state is determined from

$$\mathrm{div}\left(\frac{1}{r^2}\nabla\psi\right) + r^2 p\,\Omega_0\dot{\Omega}_0 + p\dot{H}_0 + \frac{1}{r^2}I\dot{I} = 0 \ , \tag{10}$$

where Eq. (3) is replaced by

$$\ln p = H_0 + \tfrac{1}{2}r^2\Omega_0^2 \ . \tag{11}$$

Notice that the use of p rather than ρ in (10) and (11) has eliminated an explicit use of T and the equilibrium state depends on only three functions of ψ: $\Omega_0 = \Omega/\sqrt{T}$, $H_0 = H/T + \ln T$, and I. In the adiabatic compression problem constraints (5) and (6) are kept but (7) is ignored. Instead, we notice that since there is no heat flow *across* flux surfaces, the total entropy is conserved inside each ψ surface.

Thus we impose the constraint

$$\langle \rho s \rangle = \sigma(\psi)\psi' \tag{12}$$

with $\sigma(\psi)$ unchanged during compression. Equation (12) determines the value of $T(\psi)$. The actual evaluation of the arbitrary functions can be carried out by using a variant of a variational principle for ideal plasmas due to Woltjer.[3] The equilibrium state is obtained by minimizing the total energy E,

$$E = \int \left[\frac{1}{2r^2}|\nabla\psi|^2 + \frac{1}{2}B_\theta^2 + \frac{1}{2}\rho u_\theta^2 + \frac{p(\rho,s)}{\gamma-1} \right] d\vec{x} \ , \tag{13}$$

subject to constraints (5), (6), and (12), as well as an appropriate constraint for the vacuum region, e.g., total toroidal flux. Equations (1), (2), and (11) follow after variations with respect to u_θ, B_θ, and ρ, and $T = T(\psi)$ is found after a variation with respect to s. Ω, I, H, and T are in fact the Lagrange multipliers. It is now possible to express these functions in terms of geometrical factors, conserved quantities, and ψ'. From the relation $\rho^\gamma e^s = T\rho$ we get

$$\rho \ln T = (\gamma - 1)\rho \ln\rho + \rho s \ .$$

A surface average yields an expression for T. It is convenient to define

$$\rho \equiv m\psi' R, \quad \langle R \rangle = 1 \ . \tag{14}$$

R may be viewed as determining the spatial dependence of ρ, and is independent of $\psi(V)$. Equation (6) now yields

$$\Omega = \frac{l/m}{\langle r^2 R \rangle}, \quad I = \frac{f\psi'}{\langle 1/r^2 \rangle} \ . \tag{15}$$

Notice that from (11) we get that $\ln R$ is linear in r^2 on each flux surface, thus

$$R = e^{ar^2}/\langle e^{ar^2} \rangle, \quad a(V) = \Omega^2/(2T) \ . \tag{16}$$

Substituting these results into the energy functional (13) and writing $d\vec{x} = dV\, dS/|\nabla V|$, we get

$$E = \int \left[\frac{1}{2}K\psi'^2 + \frac{1}{2}\frac{\nu^2}{\langle 1/r^2 \rangle}\psi'^2 + \frac{1}{2}\lambda^2 \frac{A}{A_a}\psi' + \frac{\mu\psi'^\gamma}{\gamma-1}\frac{\exp[(\gamma-1)aA_a/A]}{A^{\gamma-1}} \right] dV \ , \tag{17}$$

where $K(V) \equiv \langle |\nabla V|^2/r^2 \rangle$ is the inductance coefficient,

$$\lambda(\psi) \equiv l/\sqrt{m}, \quad \mu(\psi) \equiv m^\gamma e^{\sigma/m}, \quad \nu(\psi) \equiv f \tag{18}$$

are given functions, and

$$A(a) \equiv \langle e^{ar^2} \rangle, \quad A_a(a) \equiv \langle r^2 e^{ar^2} \rangle \ . \tag{19}$$

For a given geometry $V(\vec{x})$, $K(V)$ is known and E is a functional to be minimized with respect to $\psi(V)$ (subject to the two boundary values) and with respect to $a(V)$, $0 \leq a < \infty$. Once these functions are found, Ω_0, H_0, and I are known on each flux surface. Indeed,

$$\frac{1}{2}\Omega_0^2 = a, \quad H_0 = (\gamma-1)aA_a/A + \ln(\mu\psi'^\gamma/A^\gamma) \ . \tag{20}$$

The energy minimization may be carried out by guessing $a(V)$, say $a = 0$ as in the static case, then minimizing (17) with respect to ψ by solving the nonlinear, second-order ODE, Euler equation. One then finds a new approximation for a by minimizing E algebraically, separately on each flux surface, and so on until the procedure converges. This step replaces the profile determination step in the ADM, after which one solves (10) for $\psi(\vec{x})$. To conclude, the adiabatic compression problem is solved by finding a sequence of equilibrium states, with different external or boundary conditions prescribed by the compression method, in such a way that the dynamical conservation laws of an evolving plasma are obeyed. A successful numerical simulation of a similar problem will be reported elsewhere.[8]

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