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Stationary convection in a cylindrical plasma

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It is shown that viscosity and thermal conductivity leads to large-scale steady convection in a cylindrical current-carrying plasma, under the influence of a magnetic field satisfying $(B_\theta/B_z) >> 1$. This state is the analog in a plasma of stationary convection in ordinary hydrodynamics.

The stability of a current-carrying cylindrical plasma limited by fixed conducting boundaries in the presence of a longitudinal magnetic field has been extensively studied within the framework of ideal magnetohydrodynamics (MHD). For a shearless magnetic field, it has been shown that the growth rate, as a function of parallel wave number $k_\parallel$, has, in the incompressible case, two maxima located symmetrically around $k_\parallel=0$ and vanishes at $k_\parallel=0$.[1,2] It turns out, however, that there is no linear solution of the MHD equations if the plasma is incompressible.[3]

In the following, it will be shown that a new situation arises when nonideal effects such as viscosity and thermal conductivity are taken into account. Even when the corresponding coefficients are small, the nonideal effects play an important role for perturbations with $k_\parallel=0$. This is due to the fact that viscosity has the effect of removing the singularity at $k_\parallel=0$ occurring in the ideal case. In particular, there is a mode with $K_\parallel=0$ and a finite adiabaticity coefficient which, for a critical value of the pressure gradient, characterizes the onset of large-scale steady convection in the plasma.

The basic equations describing the system are

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}\right)=\vec{j}\times\vec{B}-\vec{\nabla}p-\mu_\perp\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) \ , \tag{1a}$$

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0 \ , \tag{1b}$$

$$\frac{\partial p}{\partial t}+\vec{v}\cdot\vec{\nabla}p-\tfrac{2}{3}\kappa\nabla^2p-S_0=-\gamma p\vec{\nabla}\cdot\vec{v} \ , \tag{1c}$$

$$\vec{E}+\vec{v}\times\vec{B}=0 \ , \tag{1d}$$

$$\frac{\partial\vec{B}}{\partial t}=-\vec{\nabla}\times\vec{E} \ , \tag{1e}$$

$$\vec{\nabla}\cdot\vec{B}=0 \ , \tag{1f}$$

$$\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{j} \ . \tag{1g}$$

In these equations only the prependicular part of the viscosity tensor has been considered. The other terms will be shown to be unimportant.

The equilibrium is characterized by

$$\vec{B}^{(0)}=B_l\left(\frac{r}{a}\right)^2\hat{e}_\theta+B_0\hat{e}_z \ , \qquad p^{(0)}=p_0-\frac{B_l^2}{4\pi}\left(\frac{r}{a}\right)^2 \ , \tag{2}$$

where $\vec{B}^{(0)}$ is the equilibrium magnetic field, $p^{(0)}$ the equilibrium pressure, a the radius of the cylinder, and B_0, B_l, and p_0 constants.

In Eq. (1c) κ is the heat conductivity and S_0 is a constant heat source which mantains the equilibrium pressure profile, i.e.,

$$S_0=\tfrac{2}{3}\kappa\frac{B_l^2}{\pi a^2} \ . \tag{3}$$

The rotational transform is constant and, therefore, the magnetic field is shearless:

$$q=\frac{2\pi rB_z^{(0)}}{LB_\theta^{(0)}}=\frac{2\pi aB_0}{LB_l} \ , \tag{4}$$

where L is the length of the cylinder.

Assuming a nearly constant density, $\rho\simeq\rho_0$, and linearizing Eqs. (1) for perturbations of the form $f^{(1)}(r,\theta,z)=f^{(1)}(r)\exp(im\theta+ika+\omega t)$, the following equation for the displacement vector $\vec{\xi}$ is obtained:

$$[\hat{\omega}^2+\hat{\mu}_\perp\hat{\omega}\beta^2a^2+(m-nq)^2]\vec{\xi}=-\vec{\nabla}p^{(1)}-2i(m-nq)(\xi_\theta\hat{e}_r-\xi_r\hat{e}_\theta) \ . \tag{5}$$

In Eq. (5),

$$\omega\vec{\xi}=\vec{v}^{(1)} \ , \quad \hat{\omega}=\left(\frac{4\pi\rho_0a^2}{B_l^2}\right)^{1/2}\omega \ ,$$
$$\hat{\mu}_\perp=\left(\frac{4\pi}{a^2\rho_0B_l^2}\right)^{1/2}\mu_\perp \ , \quad n=-\frac{kL}{2\pi} \ , \tag{6}$$
$$\hat{p}^{(1)}=\frac{4\pi}{B_l^2}\left(p^{(1)}+\frac{(\vec{B}\cdot\vec{B})^{(1)}}{8\pi}\right) \ ,$$

and the plasma has been assumed incompressible, i.e., $\vec{\nabla}\cdot\vec{v}=0$, which, in order to satisfy the whole set of Eqs. (1), implies $\gamma=\infty$ in Eq. (1c).

Writing the components of Eq. (5) in terms of $\hat{p}^{(1)}$ only and using the incompressibility condition $\vec{\nabla}\cdot\vec{\xi}=\vec{0}$ yields the following equation for $\hat{p}^{(1)}$:

$$\nabla^2 p^{(1)}+k^2\sigma^2\hat{p}^{(1)}=0 \quad , \tag{7}$$

where

$$\sigma=\frac{2(m-nq)}{\hat{\omega}^2+\hat{\mu}_\perp\hat{\omega}\beta^2a^2(m-nq)^2} \quad . \tag{8}$$

In Eq. (8) β is a constant defined by

$$\vec{\nabla}\times\vec{\xi}=\beta\vec{\xi} \tag{9}$$

which, by taking the curl of Eq. (5), is found to be

$$\beta=k\sigma \quad . \tag{10}$$

The solution of Eq. (7) regular at $r=0$ is

$$\hat{p}^{(1)}=\alpha J_m[k(\sigma^2-1)^{1/2}r] \quad . \tag{11}$$

Defining

$$\Lambda=[\hat{\omega}^2+\hat{\mu}_\perp\hat{\omega}k^2a^2\sigma^2+(m-nq)^2](\sigma^2-1) \quad , \tag{12}$$

the components of $\vec{\xi}$ are given by

$$\xi_r=\frac{a^2}{\Lambda}\left(\frac{\partial\hat{p}^{(1)}}{\partial r}+\frac{m}{r}\sigma\hat{p}^{(1)}\right) \quad ,$$
$$\xi_\theta=\frac{ia^2}{\Lambda}\left(\frac{m}{r}\hat{p}^{(1)}+\sigma\frac{\partial\hat{p}^{(1)}}{\partial r}\right) \quad , \tag{13}$$
$$\xi_z=\frac{ia^2k}{\Lambda}(1-\sigma^2)\hat{p}^{(1)} \quad .$$

The assumption of perfectly conducting walls and perpendicular viscosity implies the following boundary conditions:

$$\xi_r(r=a)=0 \quad , \quad \xi_\theta(r=a)=0 \quad . \tag{14}$$

From the first two of Eqs. (13) it follows that the boundary conditions (14) can be satisfied only for $\sigma=1$. Therefore, assuming $\sigma\simeq1$, and using Eq. (11) and the relation between the Bessel functions and their first derivatives, it follows that the boundary conditions are satisfied for σ values fulfilling

$$J_{m-1}[k(\sigma^2-1)^{1/2}a]=0 \quad . \tag{15}$$

Thus, setting the argument of J_{m-1} equal to the value where the function takes its first zero yields

$$k^2(\sigma^2-1)a^2=Z_{m-1}^2 \quad , \tag{16}$$

where Z_{m-1} is the first zero of J_{m-1}.

From Eq. (16) it follows that

$$\sigma^2=1+\frac{Z_{m-1}^2}{k^2a^2} \quad , \tag{17}$$

which, for large k^2a^2 values, i.e., $k^2a^2 >> Z_{m-1}$, gives $\sigma\simeq1$.

Therefore, there is a solution to the problem which satisfies $\vec{\nabla}\times\vec{\xi}\simeq\beta\vec{\xi}$ provided that $|k|a >> Z_{m-1}$.

On the other hand, from Eq. (8) and the condition $\sigma\simeq1$, it follows that

$$\hat{\omega}\simeq-\tfrac{1}{2}\hat{\mu}_\perp k^2a^2+[\tfrac{1}{4}\hat{\mu}_\perp^2k^2a^2+2(m-nq)-(m-nq)^2]^{1/2} \quad . \tag{18}$$

In the dispersion relation (18) the sign of the square root has been chosen in such a way that $\hat{\omega}$ be real and positive. Otherwise, from Eq. (8) it follows that $\hat{\omega}$ is complex and Eqs. (14) are not satisfied.

In contrast to the ideal chase,[3] close to $k_\parallel=m-nq\simeq0$, $\hat{\omega}$ and Λ behave like

$$\hat{\omega}\simeq\frac{1|m-nq|}{\hat{\mu}_\perp k^2a^2} \quad ,$$
$$\Lambda\simeq2|m-nq|\left(\frac{Z_{m-1}}{ka}\right)^2 \quad , \tag{19}$$

so that

$$\frac{\hat{\omega}}{\Lambda}=\frac{1}{\hat{\mu}_\perp Z_{m-1}^2} \tag{20}$$

is now finite at $k_\parallel=0$. In other words, since this ratio is finite, it follows that a finite pressure perturbtaion leads to a finite velocity $\vec{v}^{(1)}=\omega\vec{\xi}$.

In the case of a marginally stable state $\omega=0$, Eq. (1e) implies $\vec{\nabla}\times\vec{E}=\vec{0}$, so that $\vec{E}=-\vec{\nabla}\phi$. Therefore, the linearized equation (1d) takes the form

$$\vec{v}^{(1)}\times\vec{B}^{(0)}=\vec{\nabla}\phi^{(1)} \quad . \tag{21}$$

From the components of this equation it follows that for marginal states occurring at $m=nq$, $\vec{\nabla}\cdot\vec{v}=0$. On the other hand, Eq. (18) shows that for $m=nq$ there is a marginal state with $\omega=0$. Therefore, according to the aforementioned argument, such state is incompressible. It will now be proved that this mode satisfies the whole set of Eq. (1), including Eq. (1d) with $\kappa\neq0$ and arbitrary finite γ.

To this end, assuming $\omega=0$ at $m=nq$, Eq. (1d) reduces to (remember that $\vec{\nabla}\cdot\vec{v}=0$)

$$v_r\frac{dp^{(0)}}{dr}=\tfrac{2}{3}\kappa\nabla^2p^{(1)} \quad . \tag{22}$$

Using Eqs. (20) and (6), it follows that the last equation is satisfied provided that

$$\mu_\perp\kappa=-\frac{3}{k^4a}\left.\frac{dp^{(0)}}{dr}\right|_{r=a} \quad . \tag{23}$$

Thus, when this relation between the pressure gradient and the physical constant $\mu_\perp$ and κ is attained, a marginal solution of the entire problem exists for arbitrary γ.

This state is actually the analog in a plasma of the well-known stationary convection in ordinary hydrodynamics.[4] In fact, it it possible to define the plasma analog of the Rayleigh number and express Eq. (23) in the form

$$\mathcal{R}_{\rm crit}=\left(\frac{1}{r_c}\left|\frac{dp^{(0)}}{dr}\right|_{r-a/2}a^4\right)\Big/\mu_\perp\kappa=\frac{k^6a^6}{6m^2}\ . \qquad (24)$$

In complete analogy with the demonstrations given in Ref. 4 (Chap. II, 11 and Appendix I) and in Ref. 5, one shows that the condition $\mathcal{R} \geqslant \mathcal{R}_{\rm crit}$ characterizes the onset of large-scale steady convection in the plasma.[5]

From Eq. (18) it follows that for the unstable modes $ng \simeq m$, so that the condition $|k|a >> Z_{m-1}$ implies [see Eqs. (4) and (6)] $(B_\theta/B_z) >> 1$.

The flux pattern can be studied in the usual way. The result for $m=1$ is illustrated in Fig. 1. The center line of each tube conforming the flux surfaces

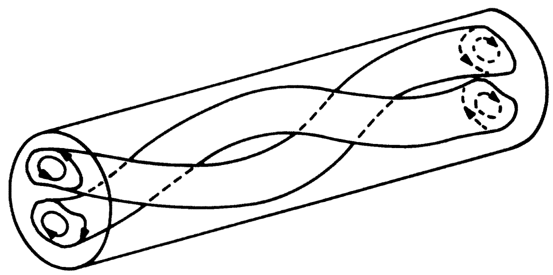

FIG. 1. Flux surfaces for convective $m=1$ modes.

can be shown to satisfy

$$v_\theta^{(1)}=(B_\theta^{(0)}/B_z^{(0)})v_z^{(1)}\ , \qquad (25)$$

so that $v_\theta^{(1)} >> v_z^{(1)}$, which justifies the neglect of parallel viscosity.

A detailed account of this paper, as well as of the effect of resistivity, are planned to be given elsewhere.

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