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Phase dynamics for the wavy vortex state of the Taylor instability

H. Brand and M. C. Cross
Bell Laboratories, Murray Hill, New Jersey 07974
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We present dynamic equations for the slow macroscopic variables of the wavy vortex state. The static solutions explain in a qualitative way recent experimental results of Ahlers *et al.* on the variation of the vortex diameter throughout the cell. In addition, we predict the existence of a pair of propagating or overdamped normal modes (depending on the wave vector of the disturbance) formed by the slow variables.

The Couette-Taylor system, consisting of a fluid contained between two concentric cylinders with the inner one rotating, is particularly convenient for the experimental investigation of the sequence of hydrodynamic instabilities leading to turbulent behavior (for a recent review **see Ref.** 1). Above a critical rotation rate the uniform (Couette) flow is unstable to the Taylor state of azimuthal vortices. The first time-dependent state, in which periodic displacements in the vortices propagate around the cylinders, tends to occur for only slightly larger rotation rates. In this paper we study the static and dynamic behavior of slow perturbations of this "wavy vortex flow," with the Taylor vortex flow a special case. This work was motivated to a large degree by the recent observation by **Ahlers** *et al.*[2] that the wavelength of the vortices in the wavy state is not constant over the cell, but varies over a rather long length scale (of order ten rolls). This is in sharp contrast to the situation in Rayleigh-Benard convection, where the roll wavelength in similar configurations is essentially constant over the bulk of the cell. In addition to accounting for this observation, we predict a new mode for variations of the vortex diameter that is propagating at long wavelengths, and becomes overdamped, and finally diffusive as the wavelength of the perturbation is decreased. In the course of this work we explore the analogies with the hydrodynamic description of slow perturbations in equilibrium systems. We also present for the first time the full two-dimensional amplitude equation describing the behavior of the Taylor vortex state near onset in the small gap limit.

Our method of approach is to construct using general symmetry arguments coupled equations for the slowly varying "phase" variables characterizing the wavy state. Two phases are necessary: the first ψ giving the position of the vortices along the axis; the second Φ giving the azimuthal position of the waves. These phases would be identified as the variables characterizing the spontaneously broken continuous symmetries in an equilibrium system. This description of the wavy flow should be contrasted with the "phase diffusion" equation for the single slow phase variable in Rayleigh-Benard convection.[3–5] The various coefficients in the equations are estimated in the limit of a small gap between the cylinders using the proximity of the wavy state to the onset of the Taylor vortex state. This generalizes the procedure of Davey *et al.*[6] to a spatially nonuniform wavy flow state.

The main result of the present Communication is the set of dynamic equations for the two phases Φ and ψ:

$$\dot{\psi} = D_1\psi'' + C_1(q_y)\Phi' \ , \tag{1}$$

$$\dot{\Phi} = D_2\Phi'' + C_2(q_y)\psi' \ , \tag{2}$$

where the dot denotes a time derivative and the prime a spatial derivative along the axis (x direction). These equations are given by the restriction that they reduce to a simple phase diffusion equation in the Taylor vortex state, and invariance under the symmetry $x \rightarrow -x$, $\psi \rightarrow -\psi$. In Eqs. (1) and (2) we have ignored azimuthal (y direction) inhomogeneities in the phases. The parameter q_y is the azimuthal wave number of the wavy vortex state: Although all parameters depend on q_y, in general, we have emphasized the q_y dependence of the cross-coupling coefficients C_1, C_2, since the symmetry $y \rightarrow -y$, $q_y \rightarrow -q_y$, $\psi \rightarrow \psi$, $\Phi \rightarrow -\Phi$, and the assumption of analyticity imply $C_1, C_2 \propto q_y$ for small q_y. Note that ψ' gives the change in the vortex wave number, and $\dot{\Phi}$ the change in the frequency of the wavy motion.

To study the static solutions of Eqs. (1) and (2) it is useful to add a phenomenological source term $(\delta\Omega)\delta(x)$ to Eq. (2) corresponding to a local tendency to increase the wavy mode frequency by $\delta\Omega$ at $x=0$. The solutions are then

$$\delta k = k - k_\infty = \delta k_0 \exp(-|x|/l) \ , \tag{3}$$

$$\delta\Phi = \Phi - \Phi_\infty = (D_1/C_1)\delta k \ , \tag{4}$$

with k the local wave vector and k_∞ the value at large distances from the source, and δk_0 proportional to the source strength $\delta\Omega$. The "penetration length" l

is given by

$$l^2 = D_1 D_2 / C_1 C_2 \quad (5)$$

and becomes long (justifying the use of the phase equations) for C_1, C_2 small.

Equations (1) and (2) may now be applied to the observations of Ahlers *et al.* Here a local tendency towards a different wave propagation speed due to end perturbations may be expected to drive a distortion in $\Phi(x)$ and hence, through Eq. (1), a local perturbation of the wave vector. Although the details of the resulting source terms in Eqs. (1) and (2) are not clear, the value of the penetration length l, and the relationship between $\delta\Phi$ and δk are independent of them; and Eqs. (3) and (4) should apply near each end. The long penetration length for the wave number perturbation observed by Ahlers *et al.* may therefore be ascribed directly to the coupling to the azimuthal phase.

The time evolution of slow disturbances of the wave number can also be studied from the phase equations. Introducing a perturbation proportional to $\exp[i(Kx-\omega t)]$ we find a propagating mode for $K \to 0$ with frequency $\omega = \pm\sqrt{C_1 C_2} K$ and $0(K^2)$ damping. For larger K we get

$$\omega = -\tfrac{1}{2} i (D_1 + D_2) K^2 \pm \tfrac{1}{2} K [4 C_1 C_2 - K^2 (D_1 - D_2)^2]^{1/2} \quad (6)$$

for $4C_1C_2 > K^2(D_1 - D_2)^2$, and

$$\omega = -i \{ \tfrac{1}{2}(D_1 + D_2) \pm \tfrac{1}{2} K [K^2 (D_1 - D_2)^2 - 4 C_1 C_2]^{1/2} \} \quad (7)$$

for $4C_1C_2 < K^2(D_1 - D_2)^2$. Thus we find by increasing the wave vector, first, a propagating mode, then an overdamped mode with a velocity dependent on the damping coefficients, and, finally, a pure diffusive mode.[7] Note that we assume C_1C_2 to be positive: This result follows from the analysis presented later in the small gap limit near the onset of the wavy mode and is required for stability. C_1 or C_2 passing through zero well above threshold would indicate the onset of an additional instability. It is interesting to point out that for a system near thermodynamic equilibrium the structure of Eqs. (1) and (2), including relationships between the coefficients, would be pinned down by the behavior of ψ, Φ under spatial inversion *and* time reversal.[8] For the present nonequilibrium system the situation is slightly different. To derive Eqs. (1) and (2) we have only used the symmetry $x \to -x$ and $\psi \to -\psi$, $\Phi \to \Phi$. It is *a posteriori* that we can say, taking into account the time-reversal symmetry of ψ and Φ, that C_1 and C_2 are reversible, whereas D_1 and D_2 are dissipative. This transformation has not, however, entered our derivation of the basic equations in any way.

The coefficients in Eqs. (1) and (2) may be roughly estimated by considering the small gap limit, using the following two observations:

(i) Although the first instability of Couette flow is to the stationary Taylor vortex state, at slightly higher Taylor numbers Couette flow is also unstable to vortices with azimuthally propagating waves. The wavy vortex state may be understood by considering the rather delicate competition between the $q_y = 0$ state and a state with nonzero q_y.[6]

(ii) Since the wavy vortex state first occurs close to the first instability in the small gap limit, its onset may be approximately treated using a lowest order "amplitude equation" found by expanding about the critical Taylor number T_c for the *Taylor vortex* state ($T_c = 3390$).

The amplitude equation may be derived using the general procedure of Newell.[9] This leads to the equation at lowest order in $\epsilon = (T - T_c)/T_c$:

$$\partial_\tau A = \tau_0^{-1} [A + (\xi_x^2 \partial_X^2 + \xi_y^2 \partial_Y^2) A - g |A|^2 A] + i\sqrt{T_c} s_1 \partial_X \partial_Y A \quad , \quad (8)$$

where $A(X, Y, \tau)$ is the complex amplitude function in terms of which the deviations of the fluid velocities from Couette flow take the form (e.g., for the radial velocity w)

$$w = \epsilon^{1/2} [A e^{ik_0 x} + \text{c.c.}] w_0(z) + 0(\epsilon) \quad , \quad (9)$$

with $w_0(z)$ a known function. The variables $X = \epsilon^{1/2} x$, $Y = \epsilon^{1/2}(y - s_0 \sqrt{T_c} \tau)$ with $y = (\frac{1}{2}\delta)^{-1/2}\theta$ and $\tau = \epsilon t$ are slow length and time scales with x the axial coordinate scaled by the gap d, θ the azimuthal angle, and τ the time scaled with d^2/ν, with ν the viscous diffusivity. The parameter δ is the ratio of the gap to the average radius of the cylinders. In the small gap limit $\delta \to 0$ the azimuthal variation becomes "slow" in the y coordinate, even though periodic in θ. Y is the (scaled) azimuthal coordinate moving with the waves. The angular propagation speed of the waves at onset, expressed as a fraction of the rotation rate Ω, is s_0, and the coefficient s_1 gives the change in this speed when the axial wave number is changed. In deriving Eq. (8) we have made use of the explicit form of the linear operator in the small gap limit.[6] The amplitude equation [Eq. (8)] takes the familiar form except for the additional last term: It is this term that leads to the phase coupling we seek.

The wavy vortex state is given[6] by the superposition

$$A = |B| e^{i\psi} + i |C| e^{i\psi} (e^{i(QY + \Phi)} + \text{c.c.}) \quad , \quad (10)$$

where $|C| \to 0$ at ϵ_w, the value of ϵ at the onset of the wavy state, and we neglect harmonics in this limit. This equation precisely defines the phases ψ, Φ introduced earlier. Equations (1) and (2) for the slow dynamics about the wavy vortex state are found after eliminating the fast variables $|B|$, $|C|$, with the identities $D_1 = D_2 = \tau_0^{-1} \xi_x^2$, $C_2(q) = \sqrt{T_c} s_1 q$, and

$C_1(q) = C_2(q) r^2$, with $r^2 = |C|^2/(|C|^2 + \frac{1}{2}|B|^2)$. From the calculations of Davey *et al.*[6] we find

$$|C|^2/|B|^2 = \xi_y^2 q_y^2 (\epsilon - \epsilon_w)/\epsilon\epsilon_w \quad .$$

Note that $C_1(q)$ goes to zero at $\epsilon = \epsilon_w$ as would be expected, but that the other parameters are independent of ϵ in the limit we consider (i.e., $\epsilon - \epsilon_w$ small).

We estimate the values of the parameters in Eq. (8) ($\tau_0^{-1} \sim 13$, $\xi_x^2 \simeq 0.15$, $\xi_y^2 \simeq 0.14$, and $s_0 \simeq 0.53$) from Chandrasekhar,[10] and the tabulated results of Davey *et al.*[6] and Krueger *et al.*[11] The important coefficient s_1 is not given by these authors: Instead we estimate $s_1 \simeq 0.014$ directly from measurements by Ahlers *et al.*[1] at $T/T_c = 1.44$ for the $m = 3$ state with $\delta = 0.1$. These values lead to the prediction for the healing length

$$l = 2.4\, q_y^{-1} r^{-1} \quad , \tag{11}$$

which diverges for small q_y as well as for small modulation ratio $r \simeq |C|/|B|$.

Substituting the values appropriate to the experiments of Ahlers *et al.* into these equations we estimate the value $l \sim 6$. From the experimental data we estimate $l \sim 10$ (although there is considerable asymmetry between the top and bottom of the cell that is not understood). This discrepancy may be due to the approximations made in evaluating the parameters, namely, the small gap limit and the proximity to onset of the wavy flow, which may well not apply for these experiments. For a better test of the quantitative predictions of the amplitude equation experiments nearer the onset of the wavy vortex state would be highly desirable.

To summarize, we have discussed the phase dynamics of the Taylor wavy mode and have found a unique structure for its dynamics, which has not been found for any hydrodynamic instability before. This unique structure is mainly due to the fact that one has two scalar quantities characterizing the long-wavelength, low-frequency behavior of this system, one even, one odd, under spatial inversion along the cylinder axis. This situation is to be contrasted, for example, with the onset of convection or with the single model laser where one has only one phase variable giving rise to a purely dissipative slow dynamics. The predictions concerning the propagating and overdamped character of the normal modes in the wavy mode are certainly a new challenge for future experiments.

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