Probability density of the Lorenz model

M. Dörfle and R. Graham
Fachbereich Physik, Universität Essen, West Germany
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The three-dimensional flow of the Lorenz model on its strange attractor is approximated by a two-dimensional flow with a branch curve with the use of the approximation of the Lorenz attractor by invariant two-dimensional manifolds obtained in earlier work. The Poincaré map on the branch curve and the associated invariant measure are determined. The probability density generated by the flow on the invariant manifolds in the steady state is related to and computed from the invariant measure on the branch curve. In a second part of this paper it is shown how the same probability density arises in the Lorenz model subject to stochastic forcing as a self-consistent approximation for the case of very small but finite noise intensity. The distribution transverse to the attractor is determined in the same approximation.

I. INTRODUCTION

Within the last decade it has become a generally accepted idea that turbulence may be associated with the appearance of strange attractors in dissipative dynamical systems. According to this idea, the phase-space trajectory of the system, after its evolution into a turbulent steady state, is captured on an attracting invariant point set of vanishing volume in phase space with an infinitely folded highly complex topological and geometrical structure.

On such a strange attractor the dynamics of the system is ergodic under rather general conditions and may even be mixing, and a statistical description becomes useful. In the steady state, the system must then be described by a time-independent probability measure, which is left invariant by the flow of the system in phase space. This invariant measure is concentrated on the strange attractor in much the same way as the microcanonical ensembles of closed, ergodic, Hamiltonian systems are concentrated on energy hypersurfaces in phase space.

Unfortunately, the explicit construction of invariant probability measures in dissipative systems meets with great difficulties because neither energy conservation nor Liouville's theorem hold for such systems, and one is left without the most important tools for the explicit construction of invariant measures in conservative ergodic systems. In this situation it is useful to study simple model systems.

The model of Bénard convection with three variables x,y,z, satisfying

$$\begin{aligned} \dot{x} &= -\sigma(x-y) , \\ \dot{y} &= -y+(r-z)x , \\ \dot{z} &= -bz+xy , \end{aligned} \tag{1.1}$$

was first studied by Lorenz[1] in this context.

In the following we chose the standard parameter values

$$b=\tfrac{8}{3}, \quad \sigma=10, \quad r=28 . \tag{1.2}$$

The Lorenz model has played an important role in exemplifying many features of continuous dynamical systems with strange attractors.

In a preceding paper,[2] henceforth quoted as I, the first steps towards the explicit construction of a time-independent probability distribution for the Lorenz model have been reported. As already indicated, for egodic systems two difficulties must be overcome in such a construction, which, although interrelated, may be dealt with separately. The first difficulty, stemming from the lack of energy conservation, is to find the support of the measure. It has been dealt with in I where the strange attractor was approximated analytically by three pieces of two-dimensional invariant manifolds, each associated with one of the three fixed points

$$\begin{aligned} P_0 &= (0,0,0) , \\ P_\pm &= [\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1] , \end{aligned} \tag{1.3}$$

of the Lorenz model.

The second difficulty is the construction of the probability density once its support is known. This problem is associated with the lack of conservation of phase-space volume and has not been solved in I, where only certain local results could be obtained. In the present paper we want to present a solution of this second problem and construct the time-independent probability density of the Lorenz model, making use of the fact that its support can be approximated by the two-dimensional invariant manifolds determined in I.

In Secs. VI and VII of this paper we show how the approximate description employed in its first part may be obtained as a weak noise limit of the stochastically forced Lorenz system

$$\begin{aligned}\dot{x}&=-\sigma(x-y)+\sqrt{\epsilon}\,\xi(t)\;,\\ \dot{y}&=-y+(r-z)x+\sqrt{\epsilon\alpha}\,\eta(t)\;,\\ \dot{z}&=-bz+xy+\sqrt{\epsilon\beta}\,\zeta(t)\;.\end{aligned}\tag{1.4}$$

Here ξ,η,ζ are Gaussian white-noise forces with

$$\begin{aligned}&\langle\xi(t)\rangle=\langle\eta(t)\rangle=\langle\zeta(t)\rangle=0\;,\\ &\langle\xi(t)\xi(0)\rangle=\langle\eta(t)\eta(0)\rangle\\ &\qquad=\langle\zeta(t)\zeta(0)\rangle=\delta(t)\;,\\ &\langle\xi(t)\eta(0)\rangle=\langle\xi(t)\zeta(0)\rangle\\ &\qquad=\langle\eta(t)\zeta(0)\rangle=0\;.\end{aligned}\tag{1.5}$$

The parameters α,β are constants of order 1. The weak noise limit is $\epsilon \ll 1$.

The remainder of the paper is organized as follows: In Sec. II we make use of a construction due to Williams[3] and describe the approximation of the Lorenz attractor by a two-dimensional branched manifold M consisting of sufficiently large pieces of the two unstable invariant manifolds M_+,M_- of the two fixed points P_+,P_-, which are glued together along a branch curve C. The local law of conservation of probability on M is formulated in Sec. III and solved in Sec. IV. In Sec. V we determine the boundary condition satisifed by the time-independent probability density on the branch curve C. It involves the Poincaré map on the branch curve C and its invariant measure, which are both determined numerically for the parameter values (1.2). The final expression for the time-independent probability density is obtained and evaluated numerically for the parameter values (1.2). We also present a comparison with the result of a direct numerical simulation of Eq. (1.2). In Sec. VI we consider the Fokker-Planck equation associated with Eq. (1.4) and simplify it under the assumptions that $\epsilon \ll 1$ and that the probability density is concentrated in the vicinity of a two-dimensional manifold $\tilde{M}$, over which it is smoothly distributed. In Sec. VII we show that the solution of that equation, under certain consistency conditions, implying small but finite noise intensity, may be separated into two parts. The first part does not involve the noise in an explicit way and leads back to the description presented in Secs. II–VI, in particular $\tilde{M}=M$. The second part depends on the noise explicitly and determines a narrow Gaussian distribution of the probability density transverse to the attractor.

II. APPROXIMATION OF THE LORENZ ATTRACTOR BY A BRANCHED MANIFOLD

The Lorenz attractor locally is the Cartesian product of a two-dimensional surface and a Cantor set. As was shown by Williams[3] a very useful topological simplification of the attractor results, if it is approximated by a two-dimensional manifold, which consists of two branches glued together along a branch curve. That such an approximation holds to high accuracy for the parameter values (1.2) is already obvious from the numerical work in Refs. 1 and 4. In I the Lorenz attractor was approximated analytically by pieces of the two unstable two-dimensional invariant manifolds M_+,M_- of the fixed points P_+,P_- (1.3) and the two-dimensional invariant manifold M_0 associated with P_0 containing the one-dimensional unstable manifold of P_0 and the z axis.

The three manifolds $M_\pm,M_0$ were constructed in I in a vicinity of $P_\pm,P_0$ in the local form,

$$\begin{aligned}&\text{for } M_+,\\ &\quad x=f_+(y,z)\;,\\ &\text{for } M_-,\\ &\quad x=f_-(y,z)\;,\\ &\text{for } M_0,\\ &\quad x=f_0(y,z)\;,\end{aligned}\tag{2.1}$$

where each of the functions $f_\pm,f_0$ was determined as power series in $(y-y_\pm),(z-z_\pm)$ or y,z, respectively. Such power series have been shown to converge in a sufficiently close neighborhood of the respective fixed point.[5] In addition, using Eqs. (1.1) the manifolds M_+,M_-,M_0 at least in principle may be extended uniquely to a region outside the domain of convergence of the power series.[5]

For use in Secs. III–VI we now approximate the Lorenz attractor by a branched two-dimensional manifold M, which we represent locally by

$$x=f(y,z)\;.\tag{2.2}$$

M is chosen to consist of two sufficiently large pieces of M_+ and M_- which are formally joined with each other along a branch curve C. The choice of the pieces of M_+,M_- and the branch curve C is shown in Fig. 1.

Thus the manifold $x=f(y,z)$ is defined to consist of the following branches

$$f(y,z)=\begin{cases}f_+(y,z)\\ \text{for } y>y_-;y<y_-,z<z_0\\ f_-(y,z)\\ \text{for } y<y_+;y>y_+,z<z_0\end{cases}\;.\tag{2.3}$$

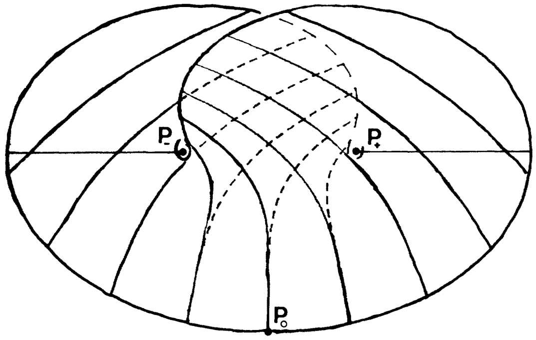

FIG. 1. The branched invariant manifold M obtained by formally joining M_+,M_- along the branch curve C (full horizontal lines). The boundary of M is the one-dimensional unstable manifold of P_0 before the second intersection with C from below.

The branch curve C is defined to consist of the two parts

$$x = \begin{cases} f_-(y,z_0) & \text{for } y < y_- \\ f_+(y,z_0) & \text{for } y > y_+ \ , \end{cases} \tag{2.4}$$

i.e., C is the intersection of M_- (M_+) with the plane

$$z = z_0 \equiv r - 1 \tag{2.5}$$

for $y < y_-$ ($y > y_+$).

The joining of M_+,M_- along the branch curve C is defined as follows: For $y < y_-$ consider the two curves obtained by the intersection of the plane (2.5) with the manifolds M_-,M_+

$$x = f_-(y,z_0) \ , \tag{2.6}$$

$$x = f_+(y,z_0) \ , \tag{2.7}$$

defining C and C_+, respectively. We join M_-,M_+ by identifying points on C_+ with points on C by their values of y. A corresponding construction is made for $y > y_+$ where we obtain the intersections

$$x = f_+(y,z_0) \ , \tag{2.8}$$

$$x = f_-(y,z_0) \ , \tag{2.9}$$

defining C and C_-, respectively, of M_+,M_- with the plane (2.5), and where we identify points on C_- with those points on C which have equal values of y.

As a consequence of this construction, a smooth flow on the branched manifold M which, for $y < y_-$ arrives at the plane $z = z_0$ from below on the branch M_-, continues smoothly across the plane $z = z_0$ and remains on M_-. If, on the other hand, it arrives on M_+, our definition of joining implies that the flow jumps discontinuously from M_+ onto M_- as soon as it traverses from below the plane $z = z_0$ for $y < y_-$. Corresponding statements, with M_+ and M_- interchanged, hold for $y > y_+$.

In the following, we make use of the fact that M constructed in this way gives a satisfactory approximation of the Lorenz attractor for the parameter values of Eq. (1.2) in the sense that each point of the Lorenz attractor is close (within a distance $\lesssim 10^{-2}$) to a point of M. In Secs. VII and VIII of this paper we show how this approximate description also arises in a natural and self-consistent way if one considers the noisy Lorenz model (1.4) for very small but finite noise intensity. The immense advantage of this approximation is that one avoids the necessity to consider the Cantor set substructure of the strange attractor explicitly.

For parameter values different from (1.2) (in particular, larger r) this approximation may not be as satisfactory. However, the Cantor set structure of the attractor necessarily involves arbitrarily small length scales in the distances of the respective sheets in phase space. Thus one would still expect that the gross features of the attractor may be approximated by invariant two-dimensional manifolds, possibly folded into several sheets if these are resolved on the length scale of interest in phase space. All the remaining substructure of the Cantor set may then be neglected after introducing a suitable branch curve C, along which the two-dimensional manifolds are connected. In this way it should also be possible to improve the construction of M described above if this is desirable, however, at the cost of calculating larger pieces of M_+,M_-, containing also backfolded parts of these manifolds, requiring more numerical work. For example, in Ref. 6 a large piece of the folded stable two-dimensional manifold of P_0 has been calculated.

III. CONSERVATION OF PROBABILITY ON THE BRANCHED MANIFOLD M

In the steady state, the trajectory of the system lies on the strange attractor in phase space. In our approximation, it is then sufficient to consider the flow (1.1) on the branched manifold M defined in Sec. II. We obtain, locally on M

$$\begin{aligned} \dot{x} &= f_y \dot{y} + f_z \dot{z} \ , \\ \dot{y} &= -y + (r - z) f = g(y,z) \ , \\ \dot{z} &= -bz + yf = h(y,z) \ , \end{aligned} \tag{3.1}$$

where $f_y = \partial f / \partial y$, etc., and f is given by Eq. (2.3). In Eq. (3.1) and in the following, we label a point on M by its (y,z) coordinates.

Moving forward in time starting at an arbitrary point on the curve C (2.4), the flow (3.1) first moves upwards in z, curves around P_+ or P_-, then moves downwards in z between P_+,P_-, curves again around one of the points P_+ or P_-, and then again intersects from below the plane $z=z_0$ for $y<y_-$ or $y>y_+$. This intersection defines a new point on C by the joining rule laid down in Sec. II. From the new point on C the flow starts again. It is clear that this flow on M is a semiflow, which can be followed forward in time uniquely, but cannot be followed backward in time through the branch curve C.

We now consider a two-dimensional probability measure

$$d\sigma=\rho(y,z,t)dA(y,z)$$

on M with density $\rho(y,z)$ and $dA(y,z)=dy\,dz$. The permanent trapping of phase points on M implies that probability is conserved on M under the semiflow (3.1). This conservation law is expressed by the continuity equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial y}g\rho+\frac{\partial}{\partial z}h\rho=0 \ . \tag{3.2}$$

Equation (3.2) must be accompanied by a boundary condition on C, which can be derived from probability conservation, and the fact that ρ must be finite along C. In the usual manner we infer from both conditions that the normal component of the probability current density

$$\vec{j}=\begin{bmatrix} j_y \\ j_z \end{bmatrix}=\begin{bmatrix} g \\ h \end{bmatrix}\rho \tag{3.3}$$

is continuous across C. The drift velocity $\binom{g}{h}$ has a small discontinuity across C, which is caused by the discontinuity of the joining condition along C. Our basic approximation consists in neglecting this discontinuity. The quality of the this approximation depends on how close M_+ and M_- are to each other along C. For the choice of C given in Sec. II and the parameter values (1.2) the distance of M_+ and M_- along C is of the order 10^{-2}, which defines the quality of our approximation in the numerical examples given in Sec. V. If the drift velocity is taken as continuous across C, then probability conservation implies that the probability density must also be continuous. Using the notation

$$\rho=\begin{cases} \rho_+(y,z,t) & \text{on } M_+ \\ \rho_-(y,z,t) & \text{on } M_- \ , \end{cases} \tag{3.4}$$

the continuity condition along C may be expressed as

$$\begin{aligned} \rho_+(y,z_0^-,t)+\rho_-(y,z_0^-,t)&=\rho_+(y,z_0^+,t) \ , \\ \rho_+(y,z_0^-,t)+\rho_-(y,z_0^-,t)&=\rho_-(y,z_0^+,t) \ , \end{aligned} \tag{3.5}$$

for $y>y_+$ and $y<y_-$, respectively, where

$$z_0^{\pm}=r-1\pm\epsilon \quad (\epsilon>0,\epsilon\to 0) \ . \tag{3.6}$$

Equation (3.5) serves as a boundary condition for Eq. (3.2).

IV. SOLUTION OF THE CONTINUITY EQUATION ON M

Equation (3.2) must be solved with the boundary condition (3.5) and some given initial condition

$$\rho(y,z,0)=\rho_0(y,z) =\begin{cases} \rho_{0+}(y,z) & \text{on } M_+ \\ \rho_{0-}(y,z) & \text{on } M_- \ . \end{cases} \tag{4.1}$$

The procedure to solve Eq. (3.2) may be subdivided in two parts:

(i) Finding $\rho(y,z,t)$ along the branch curve C,
(ii) computing $\rho(y,z,t)$ on M.

Both parts require some numerical work, which will only be carried out for the case of the steady state in Sec. V. The second part is comparatively easy.

Once $\rho(y,z,t)$ is known along C, we obtain, by integrating Eq. (3.2) alongs its characteristics satisfying Eq. (3.1),

$$\rho(y,z,t)=\exp\left[-\int_0^{\tau(y,z)}\lambda_{||}(y(\tau),z(\tau))d\tau\right]\rho(\bar{y}(y,z),z_0^+,t-\tau(y,z)) \ . \tag{4.2}$$

Here,

$$\begin{aligned} \lambda_{||}(y,z)&=\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z} \\ &=-1+(r-z)f_y-b+yf_z \end{aligned} \tag{4.3}$$

is the local rate of divergence of trajectories on M, $(y(\ ),z(\ ))$ is the trajectory of the semiflow (3.1), passing through (y,z) and, a time interval $\tau(y,z)$ earlier, through $(\bar{y}(y,z),z_0)$ on C. The necessary integrations along the characteristics can only be performed numerically.

In order to solve the first part of the problem, i.e., to determine $\rho(y,z,t)$ along C, one has to evaluate the continuity condition (3.5). For this purpose we need to consider the return map $y\to y'$ of the branch curve C into itself, which is induced by the semiflow (3.1). It is obtained by integrating Eqs. (3.1) for one round trip, starting from any point on C. A numerical construction of this Poincaré map for the parameter values (1.2) is given in Fig. 2. It consists of two parts

$$y' = \begin{cases} F_+(y) & y > y_+ \\ F_-(y) & y < y_- \end{cases} \tag{4.4}$$

which arise from the two parts of the branch curve C. In addition to the return map, the return time T from C to C is of importance. Considered as a function of the end point on C labeled by y, $T(y)$ also has two branches, which distinguish whether the trajectory arrives at (y,z_0^-) on M_+ or on M_-.

By definition

$$T=\tau(y,z_0^-) = \begin{cases} T_+(y) & (y,z_0^-) \text{ on } M_+ \\ T_-(y) & (y,z_0^-) \text{ on } M_- \,. \end{cases} \tag{4.5}$$

Now Eq. (3.5) may be restated in the form

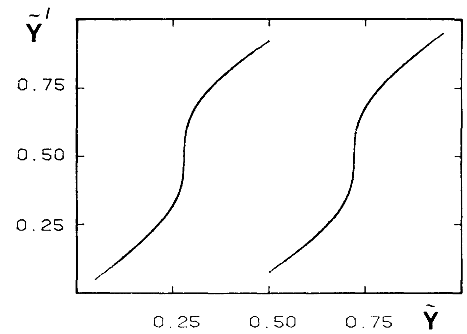

FIG. 2. The return map $\tilde{y} \to \tilde{y}'$ of the branch curve C of Fig. 1 into itself. The curve C is parametrized by a parameter $\tilde{y}$ which is linearly related to y and chosen to run from 0 to 1/2 on the left part of C in Fig. 1, starting in P_-, and from 1/2 to 1 on the right part of C in Fig. 1.

$$\rho_+(F_+^{-1}(y),z_0^+,t-T_+(y))\exp\left[-\int_0^{T_+(y)} \lambda_{||}(y(\tau),z(\tau))d\tau\right] + \rho_-(F_-^{-1}(y),z_0^+,t-T_-(y))\exp\left[-\int_0^{T_-(y)} \lambda_{||}(y(\tau),z(\tau))d\tau\right] = \rho_+(y,z_0^+,t), \quad y > y_+ \tag{4.6}$$

and corresponding for $y < y_-$. In each term of this equation ρ_+ and ρ_- may be replaced by ρ, since there is no ambiguity concerning the branch of M on which each expression is defined. It is convenient to introduce the time-dependent surface element

$$dA(y,z,t) = \exp\left[\int_{\tilde{t}}^{t} \lambda_{||}(y(\tau),z(\tau)d\tau)\right] dA(\tilde{y},\tilde{z},\tilde{t}) \,, \tag{4.7}$$

where the integration is carried out over a trajectory of (3.1) which passes through $(\tilde{y},\tilde{z})$ at time $\tilde{t}$ and through (y,z) at time t $[t-\tilde{t} \le \tau(y,z)]$.

Probability conservation is then expressed by

$$\rho(y,z,t)dA(y,z,t) = \rho(\tilde{y},\tilde{z},\tilde{t})dA(\tilde{y},\tilde{z},\tilde{t}) \,. \tag{4.8}$$

Equation (4.6) now assumes the form

$$\rho(F_+^{-1}(y),z_0^+,t-T_+(y))dA(F_+^{-1}(y),z_0^+,t-T_+(y)) + \rho(F_-^{-1}(y),z_0^+,t-T_-(y))dA(F_-^{-1}(y),z_0^+,t-T_-(y)) = \rho(y,z_0^+,t)dA(y,z_0^+,t) \quad (y > y_+) \,. \tag{4.9}$$

The same equation is also obtained for $y < y_-$.

Defining

$$\mu(y,t) = \rho(y,z_0^+,t)\frac{dA(y,z_0^+,t)}{dy\, d\lambda} \,, \tag{4.10}$$

where $d\lambda$ is a differential independent of y,t we obtain from Eq. (4.9)

$$\frac{\mu(F_+^{-1}(y),t-T_+(y))}{|F'_+(F_+^{-1}(y))|} + \frac{\mu(F_-^{-1}(y),t-T_-(y))}{|F'_-(F_-^{-1}(y))|} = \mu(y,t) \,. \tag{4.11}$$

Equation (4.11) shows that the one-dimensional time-dependent measure $\mu\, dy$ on the branch curve C is conserved. This conserved measure is related to the two-dimensional conserved probability measure $d\sigma = \rho\, dA$ by

$$d\sigma(y,z_0^+) = \rho(y,z_0^+,t)dA(y,z_0^+,t) = \mu(y,t)dy\, d\lambda \,. \tag{4.12}$$

In order to construct the explicit connection between ρ, and μ we still have to construct the relation between dA and $dy\,d\lambda$. To this end we use Eq. (3.1) and write

$$dz = h(y,z)c\,d\lambda\ , \tag{4.13}$$

where c is a constant of normalization and $c\,d\lambda$ is chosen as the invariant increment of the time parameter along the trajectories. Then we obtain

$$dA(y,z_0^+,t) = dy\,|h(y,z_0^+)|\,c\,d\lambda\ . \tag{4.14}$$

We therefore obtain from Eq. (4.10)

$$\rho(y,z_0^+,t) = \frac{1}{c}\frac{\mu(y,t)}{|h(y,z_0^+)|}\ . \tag{4.15}$$

The continuous-delay equation (4.11) has to be solved numerically for $\mu(y,t)$. Equation (4.15) then yields the time-dependent probability density ρ along the branch curve C.

In the present paper we only want to construct the time-independent probability density by explicit computation. This will be done in the following section.

V. PROBABILITY DENSITY IN THE STEADY STATE

In the steady state $\rho(y,z,t) = \rho_\infty(y,z)$ is independent of t and Eq. (3.2) reduces to

$$\frac{\partial}{\partial y}g\rho + \frac{\partial}{\partial z}h\rho = 0\ . \tag{5.1}$$

The solution (4.2) becomes time independent and reduces to

$$\rho_\infty(y,z) = \exp\left[-\int_0^{\tau(y,z)}\lambda_{||}(y(\tau),z(\tau))d\tau\right] \times\rho_\infty(\bar{y}(y,z),z_0^+)\ . \tag{5.2}$$

The probability density on the branch curve ρ_∞ is also time independent and related to a time-independent measure $\mu_\infty(y)$ on the branch curve by Eq. (4.15). The measure $\mu_\infty(y)$ satisfies the time-independent version of Eq. (4.11)

$$\frac{\mu_\infty(F_+^{-1}(y))}{|F'_+(F_+^{-1}(y))|} + \frac{\mu_\infty(F_-^{-1}(y))}{|F'_-(F_-^{-1}(y))|} = \mu_\infty(y)\ . \tag{5.3}$$

Remarkably, the return time $T_\pm(y)$, which appeared in the time-dependent equation (4.11), has dropped out from Eq. (5.3). Equation (5.3) is satisfied by the invariant measures of the return map (4.3).

The invariant measure of the return map of Fig. 2 was determined numerically. Quick convergence to the invariant measure μ_∞ shown in Fig. 3 was obtained by iterating a number of different initial distributions under the map. The invariant measure of the return map was then used in Eq. (4.15) together with the invariant manifolds f_+, f_- determined in I in order to obtain the boundary condition $\rho_\infty(y,z_0^+)$ for Eq. (5.1) along the curve C. This boundary condition is shown in Fig. 4. It must have the property to reproduce itself under the flow (3.1) after one round trip [by the time-independent version of Eq. (4.6)], a property which we checked and found to be satisfied within the accuracy of our approximation. The boundary condition was then used in Eq. (5.2) in order to construct the two-dimensional density $\rho_\infty(y,z)$. Our result is shown in Figs. 5(a) and 5(b).

In I we have also generated the two-dimensional density ρ_∞ histogrammatically by numerical integration of Eqs. (1.1) for long times. The data obtained in this way are in very good agreement with the two-dimensional density of Fig. 5, as can be seen from the comparison in Fig. 6.

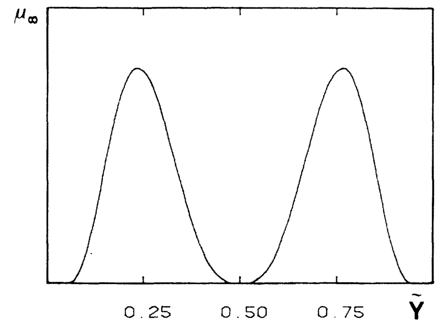

FIG. 3. The invariant measure μ_∞ (arbitrary units) of the return map of Fig. 2, obtained by iterating an arbitrary initial distribution under the map until convergence is obtained (20–30 iterates are needed in practice).

VI. STOCHASTICALLY FORCED LORENZ MODEL

In the approximation of the Lorenz attractor by a branched manifold, the Cantor set substructure of

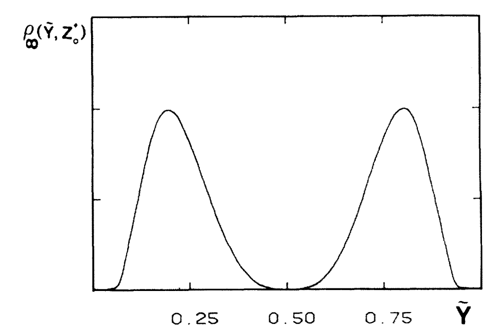

FIG. 4. The invariant probability density $\rho_\infty(\bar{y},z_0^+)$ along the branch curve obtained from μ_∞ in Fig. 3 by Eq. (4.17).

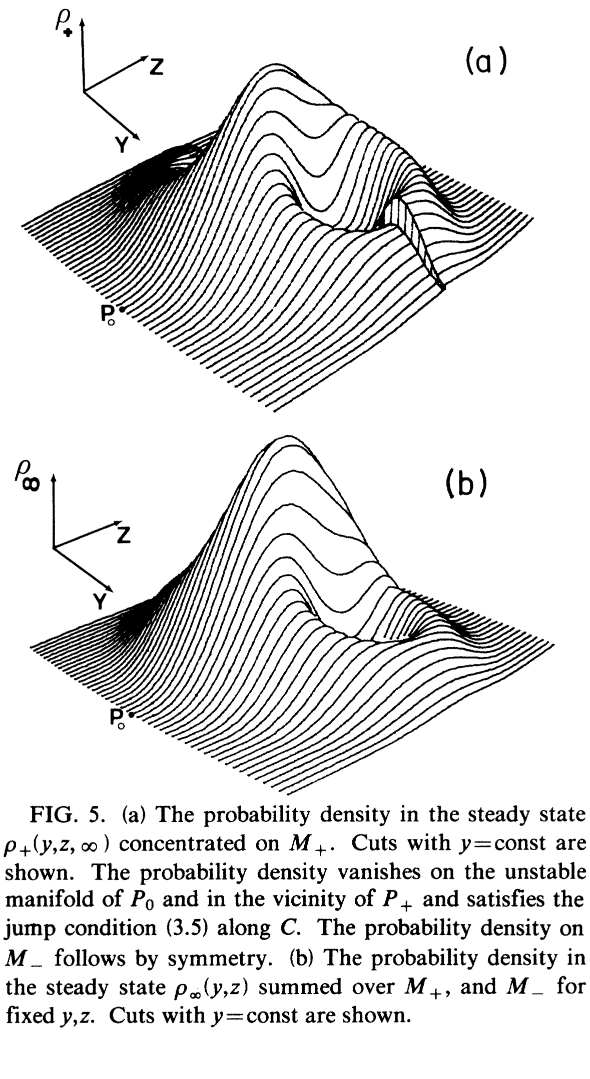

FIG. 5. (a) The probability density in the steady state $\rho_+(y,z,\infty)$ concentrated on M_+. Cuts with $y=\text{const}$ are shown. The probability density vanishes on the unstable manifold of P_0 and in the vicinity of P_+ and satisfies the jump condition (3.5) along C. The probability density on M_- follows by symmetry. (b) The probability density in the steady state $\rho_\infty(y,z)$ summed over M_+, and M_- for fixed y,z. Cuts with $y=\text{const}$ are shown.

the attractor is neglected. Intuitively, it is clear that this corresponds to some sort of course graining in phase space. One way in which such a course graining arises naturally, is in the presence of noise, like in Eqs. (1.4). Therefore, we want to consider here and in the following section, how the description given in Secs. II—IV arises naturally from Eqs. (1.4) in the limit of weak but finite noise $0<\epsilon\ll 1$. We start analyzing Eqs. (1.4) by assuming that the pro-

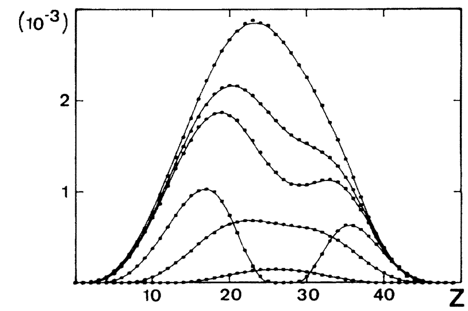

FIG. 6. Comparison of $\rho_\infty(y,z)$ of Fig. 5(b) obtained from Eqs. (5.2), (5.3) (full lines) with the histogrammatically determined probability density (dots), as a function of z (for $y=0,3,4,10,16,22$ from top to bottom for $Z=20$). Both probability densities are normalized to 1. No fit parameters are involved in the comparison.

cess is sharply concentrated around a two-dimensional manifold $\tilde{M}$, which we represent locally by

$$x=\tilde{f}(y,z,\epsilon)\ . \tag{6.1}$$

We also assume that the process has a smooth probability density on the manifold $\tilde{M}$. Later, the consistency of these assumptions with the stochastic dynamics described by Eqs. (1.4) has to be checked. It will then turn out that $\tilde{M}$ cannot be chosen in an arbitrary way, and must possess properties which, under certain conditions on ϵ, forces the choice $\tilde{M}=M$

$$\tilde{f}(y,z,\epsilon)=f(y,z)\ , \tag{6.2}$$

where M

$$x=f(y,z) \tag{6.3}$$

is the branched manifold defined in Sec. II.

In order to consider Eqs. (1.4) in a close vicinity of $\tilde{M}$ (6.1) we introduce the scaled variable

$$\mu=\frac{1}{\sqrt{\epsilon}}[x-\tilde{f}(y,z,\epsilon)] \tag{6.4}$$

and consider the three-dimensional probability density $P(\mu,y,z,t;\epsilon)$ which depends on ϵ.

It satisfies the Fokker-Planck equation

$$\begin{aligned}\frac{\partial P}{\partial t}=&\frac{1}{\sqrt{\epsilon}}\frac{\partial}{\partial\mu}[\sigma(\tilde{f}-y)+\tilde{f}_y((r-z)\tilde{f}-y)+\tilde{f}_z(y\tilde{f}-bz)]P+\frac{\partial}{\partial\mu}[\mu(\sigma+\tilde{f}_y(r-z)+\tilde{f}_zy)]P\\&+\frac{\partial}{\partial y}[y-(r-z)\tilde{f}]P+\frac{\partial}{\partial z}(bz-y\tilde{f})P+\frac{1}{2}\frac{\partial^2}{\partial\mu^2}(1+\alpha\tilde{f}_y^{\,2}+\beta\tilde{f}_z^{\,2})P\\&+\sqrt{\epsilon}\left[\frac{\partial}{\partial\mu}\left[-\frac{\alpha}{2}\tilde{f}_{yy}P-\frac{\beta}{2}\tilde{f}_{22}P\right]+\frac{\partial}{\partial y}[-(r-z)\mu P]+\frac{\partial}{\partial z}(-y\mu P)\right.\\&\left.-\frac{\partial}{\partial\mu}\left[\frac{\partial}{\partial y}\alpha\tilde{f}_yP+\frac{\partial}{\partial z}\beta\tilde{f}_zP\right]\right]+\frac{\epsilon}{2}\left[\alpha\frac{\partial^2P}{\partial y^2}+\beta\frac{\partial^2P}{\partial z^2}\right]\ .\end{aligned}\tag{6.5}$$

The boundary conditions are that P and its derivatives vanish at ∞. We now consider the case of small ϵ, with the goal of keeping only terms up to order ϵ^0. This makes it necessary to choose $\tilde{f}(y,z,\epsilon)$ in such a way that a term of order $\epsilon^{-1/2}$ does not appear. This term is eliminated from Eq. (6.5) by requiring that $\tilde{f}$ satisfies, to lowest order in ϵ,

$$\sigma(\tilde{f}(y,z,0)-y)+[(r-z)\tilde{f}(y,z,0)-y]\tilde{f}_y(y,z,0) +[y\tilde{f}(y,z,0)-bz]\tilde{f}_z(y,z,0)=0\;. \quad (6.6)$$

This equation is locally satisfied by any two-dimensional invariant manifold $x=\tilde{f}(y,z,0)$ of the Lorenz model. Thus, *locally*, the manifold $\tilde{M}$ $[x=\tilde{f}(y,z,\epsilon)]$ must be approximated everywhere by a two-dimensional invariant manifold of (1.1). However, *globally*, the topological structure of $x=\tilde{f}(y,z,\epsilon)$ need not be similar to the topological structure of a two-dimensional invariant manifold of (1.1), even with a function $\tilde{f}$ depending very weakly on ϵ. Indeed, in order to eliminate the $\epsilon^{-1/2}$ term in (6.5) to order ϵ^0, it is already sufficient to require

$$\sigma(\tilde{f}(y,z,\epsilon)-y)+[(r-z)\tilde{f}(y,z,\epsilon)-y]\tilde{f}_y(y,z,\epsilon) +[y\tilde{f}(y,z,\epsilon)-bz]\tilde{f}_z(y,z,\epsilon)=O(\epsilon^{(1/2)+\lambda}) \quad (6.7)$$

with any $\lambda>0$. Thus the requirement on $\tilde{M}$ merely is that the (Euclidean) distance between $\tilde{M}$ and an invariant two-dimensional manifold of (1.1) must be of order $\epsilon^{(1/2)+\lambda}$. We note that this condition can be satisfied by the choice $\tilde{M}=M$, if ϵ is sufficiently large to satisfy

$$|f_+(y,z_0)-f_-(y,z_0)|=O(\epsilon^{(1/2)+\lambda}) \quad (6.8)$$

for all points on curve C (2.4). In the following, we assume that ϵ is sufficiently large so that Eq. (6.8) is satisfied, at least for all points in the support of the invariant measure μ_∞ on C which was determined in Sec. V. For the parameter values as given in Eq. (1.1) this requires $\epsilon\cong 4\times 10^{-4}$. It then remains to be shown that the choice of $\tilde{M}=M$ is not only possible, but indeed necessary, and to determine the probability density P in the vicinity of $\tilde{M}$.

In the following the weak ϵ dependence of $\tilde{f}$ is neglected, when the order of magnitude of terms in Eq. (6.5) is considered. Then, the Fokker-Planck equation to order ϵ^0 reads

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial\mu}[\mu(\sigma+(r-z)\tilde{f}_y+y\tilde{f}_z)]P +\frac{\partial}{\partial y}[y-(r-z)\tilde{f}]P+\frac{\partial}{\partial z}(bz-y\tilde{f})P +\frac{1}{2}\frac{\partial^2}{\partial\mu^2}(1+\alpha\tilde{f}_y^2+\beta\tilde{f}_z^2)P\;. \quad (6.9)$$

In the following section we consider the solution of Eq. (6.9) and further consistency conditions which have to be satisfied by $\tilde{M}$.

VII. PROBABILITY DENSITY OF THE STOCHASTICALLY FORCED MODEL

The solution of Eq. (6.9) may be approached in two steps. In the first step we integrate over μ and consider the reduced equation satisfied by the two-dimensional probability density

$$\rho(y,z,t)=\int_{-\infty}^{\infty}d\mu\,P(\mu,y,z,t)\;. \quad (7.1)$$

According to our assumptions, $\rho(y,z,t)$ may be considered as a probability density of $\tilde{M}$. It satisfies

$$\frac{\partial\rho}{\partial t}=\frac{\partial}{\partial y}[y-(r-z)\tilde{f}]\rho +\frac{\partial}{\partial z}(bz-y\tilde{f})\rho\;. \quad (7.2)$$

In a second step, we enter Eq. (6.9) with the hypothesis

$$P(\mu,y,z,t)=\tilde{P}(\mu\,|\,y,z;t)\rho(y,z,t) \quad (7.3)$$

and fix the normalization of $\tilde{P}$ by

$$\int_{-\infty}^{\infty}d\mu\,\tilde{P}(\mu\,|\,y,z;t)=1\;. \quad (7.4)$$

Using Eq. (7.2) we obtain

$$\frac{\partial\tilde{P}}{\partial t}=\frac{\partial}{\partial\mu}\mu(\sigma+(r-z)\tilde{f}_y+y\tilde{f}_z)\tilde{P} +\frac{1}{2}\frac{\partial^2}{\partial\mu^2}(1+\alpha\tilde{f}_y^2+\beta\tilde{f}_z^2)\tilde{P} +[y-(r-z)\tilde{f}]\frac{\partial\tilde{P}}{\partial y}+(bz-y\tilde{f})\frac{\partial\tilde{P}}{\partial z}\;. \quad (7.5)$$

Equation (7.5) is solved exactly by the normalized Gaussian

$$\tilde{P}(y\,|\,y,z,t)=\frac{1}{\sqrt{2\pi\varphi(y,z,t)}}\times\exp\left[-\frac{\mu^2}{2\varphi(y,z,t)}\right] \quad (7.6)$$

with the local width $\sqrt{\varphi(y,z,t)}>0$ satisfying the linear equation

$$\dot{\varphi}+2[\sigma+(r-z)\tilde{f}_y+y\tilde{f}_z]\varphi-[y-(r-z)\tilde{f}]\frac{\partial\varphi}{\partial y}$$
$$-(bz-y\tilde{f})\frac{\partial\varphi}{\partial z}=1+\alpha\tilde{f}_y^2+\beta\tilde{f}_z^2 . \quad (7.7)$$

We note that the inhomogeneity in Eq. (7.7) is entirely due to the noise in Eq. (1.4). In fact, this is the only place where the noise enters at all.

Not for all choices of $\tilde{f}(y,z,\epsilon)$ do Eqs. (7.2) and (7.5) possess solutions which are consistent with our initial assumptions. We now show that in order to satisfy all consistency conditions, $\tilde{M}$ must be a branched two-dimensional manifold in the $O(\epsilon^{(1/2)+\lambda})$ vicinity of the attractor of Eq. (1.1). Then, provided Eq. (6.8) is satisfied, the simplest branched manifold with these properties is $\tilde{M}=M$. There exist more complicated choices of $\tilde{M}$ (containing more branches) which approximate the attractor even better. However, if Eq. (6.8) is satisfied, these more complicated choices will not lead to distinguishable results for P if inserted in Eq. (7.3). On the other hand, if ϵ is so small that Eq. (6.8) is not satisfied, one is forced to make a better choice of $\tilde{M}$. Thus, as ϵ decreases below the order of magnitude given by Eq. (6.8), more and more of the complex substructure of the attractor has to reappear in the branched manifold $\tilde{M}$.

It has already been shown that $\tilde{M}$ must locally be approximated to order $O(\epsilon^{(1/2)+\lambda})$ by a two-dimensional invariant manifold of (1.1). From the condition $\varphi>0$ it follows that $\tilde{M}$ must be locally attracting. This condition must be satisfied by an acceptable solution of Eq. (7.7) (cf. below). In order that (7.2) has a nontrivial solution for $t\to\infty$, $\tilde{M}$ must be close to the invariant point set of (1.1) approached for $t\to\infty$. By the same condition we must require that all trajectories of the flow

$$\dot{y}=-y+(r-z)\tilde{f} , \quad (7.8)$$

$$\dot{z}=-bz+y\tilde{f} , \quad (7.9)$$

starting out on $\tilde{M}$ remain on $\tilde{M}$ for all times. On the other hand, $\tilde{M}$ is two dimensional and cannot contain a fixed point. This necessarily means that the flow (7.9) must be a semiflow, and $\tilde{M}$ must be a branched two-dimensional manifold. Therefore, we must identify $\tilde{M}$ by M in order to achieve self-consistency for the ansatz (6.1). Equation (7.2) is then identical to Eq. (3.2) and solved by the invariant probability density ρ_∞ determined in Sec. V.

We finally turn to the solution of Eq. (7.7). The characteristics of Eq. (7.7) satisfy Eq. (7.9). Integrating along the characteristic $[y(\ ),z(\ )]$ passing through (y,z) at time t and through $(\tilde{y},\tilde{z})$ at time $t=0$ we obtain

$$\varphi(y,z,t)=\int_0^t d\tau\, e^{2\int_\tau^t\lambda_\perp(\tau')d\tau'}\times[1+\alpha f_y^2(\tau)+\beta f_z^2(\tau)]+e^{2\int_0^t\lambda_\perp(\tau')d\tau'}\varphi_0(\tilde{y},\tilde{z}) , \quad (7.10)$$

where we assumed as initial condition

$$\varphi(y,z,0)=\varphi_0(y,z) \quad (7.11)$$

and introduced

$$\lambda_\perp=-\sigma-(r-z)f_y-yf_z . \quad (7.12)$$

Noting that $\lambda_\perp$ (7.12) and $\lambda_{||}$ (4.3) satisfy

$$\lambda_\perp(y,z)+\lambda_{||}(y,z)=\frac{\partial\dot{x}}{\partial x}+\frac{\partial\dot{y}}{\partial y}+\frac{\partial\dot{z}}{\partial z} , \quad (7.13)$$

where $\dot{x},\dot{y},\dot{z}$ are given by Eq. (1.1), we recognize $\lambda_\perp(y,z)$ as the local rate of attraction (if $\lambda_\perp<0$) transverse to the manifold M. Clearly

$$\lim_{t\to\infty} e^{\int_0^t\lambda_\perp(\tau)d\tau}\to 0 \quad (7.14)$$

must be satisfied, otherwise the steady-state distribution $P(u,y,z,\infty)$ does not exist independent of the initial distribution. In this case we find

$$\varphi(y,z,\infty)=\int_{-\infty}^t d\tau\, e^{2\int_\tau^t\lambda_\perp(\tau')d\tau'}\times[1+\alpha f_y^2(\tau)+\beta f_z^2(\tau)] . \quad (7.15)$$

Equation (7.15) may be simplified under the assumption that $[1+\alpha f_y^2(\tau)+\beta f_z^2(\tau)]$ varies slowly with τ compared to the fast decay of the exponential $\exp[2\int_\tau^t\lambda_\perp(\tau')d\tau']$ with increasing $|\tau-t|$ for fixed t. The integral over τ in Eq. (7.15) is then dominated by its contribution for $\tau=t$ and we obtain

$$\varphi(y,z,\infty)\approx-\frac{[1+\alpha f_y^2(y,z)+\beta f_z^2(y,z)]}{2\lambda_\perp(y,z)} . \quad (7.16)$$

The width of the Gaussian (7.6) expressed in the original x variable is $\sqrt{\epsilon\varphi(y,z,\infty)}$. It must be small compared to distances of order 1, otherwise the course graining introduced by the noise would be too strong. This condition is equivalent to

$$\left[\frac{\epsilon}{|\lambda_\perp(y,z)|}\right]^{1/2}\ll 1 . \quad (7.17)$$

If $\lambda_{||}(y,z)\geq 0$ everywhere on M, then by Eq. (7.13)

$$\lambda_1(y,z) \le \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$$

$$= -(\sigma + b + 1) \tag{7.18}$$

and Eq. (7.17) is satisfied if

$$\sqrt{\epsilon} \ll \sqrt{\sigma + b + 1} \; . \tag{7.19}$$

On the other hand, as has already been discussed above, the noise must be sufficiently strong to smooth out the small-scale structure of the attractor of Eq. (1.1). This small scale can be inferred from the homogeneous part of the solution (7.10) as of the order of $e^{-|\lambda_1|T}$ where T is a typical round trip time, and of order 1 for the parameter values (1.2). Thus we obtain the condition on ϵ

$$e^{-(\sigma+b+1)T} \ll \left[\frac{\epsilon}{\sigma+b+1} \right]^{1/2} \ll 1 \; . \tag{7.20}$$

The lower bound on $\sqrt{\epsilon}$ in (7.20) is just a reformulation of Eq. (6.6).

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