Fully quantized many-particle theory of a free-electron laser

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A fully quantized many-particle theory of the standard free-electron laser in the small-signal, cold-beam regime is presented. The approach is based on an evaluation of the time-evolution operator in the interaction picture to first order in the quantum-mechanical recoil. For algebraic convenience we use the moving (Bambini-Renieri) frame, in which resonance occurs for zero electron momentum. Though we neglect space-charge effects, genuine many-particle contributions still show up, because the radiation emitted by one electron can be amplified by another electron. Our main results are gross features of the amplification, such as gain and spread, are virtually without many-particle effects. These effects are mainly important in the case of spontaneous emission. For a sufficiently high current, the buildup of the laser field from vacuum is enhanced by amplified spontaneous emission. Incoherence of the spontaneous radiation from several electrons induces deviations from Poisson statistics even if gain is neglected. For a dilute electron beam, spontaneous radiation is slightly antibunched for negative gain. Squeezing is obtained for positive gain independent of the number of electrons. However, owing to some idealizations used in the model, it is uncertain whether this applies to a physically realizable situation.

I. INTRODUCTION

Apart from its rapidly increasing practical significance, the free-electron laser[1–3] (FEL) provides a fascinating example of the interrelations between classical and quantum physics. Though all properties of the FEL which are relevant to its practical performance can be understood in classical terms, quantum mechanics provides an alternative, seemingly very different, but depending on one's taste and viewpoint, even simpler approach to its basic concepts. Moreover, it has recently been shown[4,5] that spontaneous radiation from a FEL exhibits nonclassical effects such as photon antibunching and squeezing which are obviously outside of the scope of a classical approach. The objective of the present paper is to provide a fully quantized many-particle treatment of the free-electron *amplifier* in the small-signal cold-beam regime. Though sloppy, but in agreement with common usage, we will refer to it as a free-electron *laser.* We shall assume that the electron beam is sufficiently dilute to make space-charge effects negligible. The emphasis will then be on a clean separation of one- and many-particle contributions to the basic quantities such as gain and spread as well as to the quantum-statistical properties of the radiation. Even with the neglect of Coulomb effects, many-particle contributions occur, because the radiation from one electron can be amplified by another, an effect which is obviously absent in a one-particle description. Many previous classical approaches and all quantum-mechanical ones consider the case of just one electron interacting with the electromagnetic field. The tacit assumption is then that relevant quantities like the gain are just the single-particle quantities multiplied by the number of electrons. We shall also investigate how far this commonly adopted procedure is justified.

The formalism we shall employ is a generalization of an earlier approach[5] which had been developed for the one-electron case. It relies on an evaluation of the time-evolution operator in the interaction picture which is expanded to first order in the quantum-mechanical recoil. Although our superficial expansion parameter will be $\hbar\omega/mc^2$, this is nevertheless not an expansion in terms of overall powers of $\hbar$, since other quantities that occur in the expansion coefficients, such as the number N of initial photons, are implicitly proportional to $\hbar^{-1}$. For algebraic expediency we formulate our approach in the Bambini-Renieri frame in which the laser and the wiggler frequency coincide. In this frame the electron motion can be described nonrelativistically, which is the reason for its simplicity. We shall also restrict ourselves to the standard Stanford-type FEL with a circularly polarized wiggler,[1,2] since this gives rise to the simplest Hamiltonian. Both restrictions are not necessary, and we expect qualitatively similar results to hold for a much wider class of external field configurations. This will be eventually considered in a separate publication.

The outline of this paper is as follows: In Sec. II we derive the time-evolution operator in the interaction picture and introduce its linear-recoil approximation. Although the present paper is supposed to be self-contained, we shall occasionally refer to Ref. 5 for some calculational details as well as more extensive arguments. In Sec. III we apply the formalism to calculate various quantities of interest. Gain and spread incorporate both one- and two-particle contributions. One-particle contributions are, e.g., the usual terms in the gain, which refer to spontaneous and stimulated emission. They arise from the present approach identical in shape to a one-particle theory, but multiplied by the number N_e of electrons. There are no additional two-particle contributions to the stimulated gain. Spontaneous emission, however, is modified by a two-particle term, which is proportional to $N_e(N_e-1)$. This term describes amplified spontaneous emission: Initial spontaneous emission, which is proportional to N_e, is then amplified by one of the other electrons which introduces a further factor of N_e-1.

In the one-particle case the photons emitted by spontaneous emission satisfy Poisson statistics apart from small corrections which are related to the presence of gain. These can have either sign, thus inducing a slight amount of photon bunching or antibunching. In the many-particle case there is an additional two-particle term which increases the spread in photon number under all circumstances. Because this term is dominant it conceals the slight gain-related effects on the photon statistics. This term expresses the fact that spontaneous emission from several electrons is incoherent.

Whereas photon antibunching can only be observed, if at all, in the one-particle case, the small amount of squeezing surprisingly formally persists in the many-particle case. There are, however, serious doubts as to whether this has practical significance. (This will be discussed in Sec. IV.) We conclude Sec. III by working out the interrelations between the final state of the electrons and the laser photons, which are dictated by energy conservation.

In Sec. IV we recapitulate our results and discuss more extensively the significance of amplified spontaneous emission. Since this effect shifts the line center of spontaneous emission towards the positive gain side, i.e., to larger wavelengths, by an amount which depends on the magnitude of the electron current, it provides in principle a diagnostic means to determine the electron current.

In Appendix A we rederive the Bambini-Renieri Hamiltonian, pointing out that this requires neither the Weizsäcker-Williams approximation nor a quantized wiggler field. Appendixes B and C deal with some technical details which have been omitted in the main body of the paper. In Appendix D we finally discuss the consequences of relaxing some of the idealizations which we made in specifying the initial state of the electrons and the radiation field.

II. THE TIME-EVOLUTION OPERATOR

The many-electron Hamiltonian in the Bambini-Renieri frame[6] where $\omega \cong ck/2$, as rederived in Appendix A, is

$$H=H_0+H_I\ , \tag{1a}$$

$$H_0=\sum_{i=1}^{N_e}\frac{p_i^2}{2m}+\hbar\omega a^\dagger a\ , \tag{1b}$$

$$H_I=i\hbar g\sum_{i=1}^{N_e}\left(a^\dagger e^{-i(kz_i+\omega t)}-ae^{i(kz_i+\omega t)}\right)\ . \tag{1c}$$

Here N_e is the total number of electrons, p_i and z_i are the operators of momentum and position of the ith electron, $[z_i,p_j]=i\hbar\delta_{ij}$, and m is the *renormalized* electron mass. a and $a^\dagger$ denote the creation and annihilation operators of the laser photons. All quantities in Eqs. (1a)–(1c) refer to the moving frame, and we have omitted the primes, which indicate the moving frame in Appendix A. The coupling constant is

$$g=\frac{e^2A_w}{mc}\left[\frac{\pi}{\hbar\omega V}\right]^{1/2} \tag{2}$$

with A_w the amplitude of the vector potential of the wiggler and V the quantization volume. Equation (1c) suggests the introduction of the operators[7]

$$A_i=ae^{ikz_i} \tag{3}$$

with

$$A_i^\dagger A_i=a^\dagger a\ , \tag{4a}$$

$$[A_i,A_j]=[A_i^\dagger,A_j^\dagger]=0\ , \tag{4b}$$

but

$$[A_i,A_j^\dagger]=e^{ik(z_i-z_j)}\ . \tag{4c}$$

The fact that the last commutator is no longer a c number as it was in the one-particle case is the distinctive difference between the many- and the one-electron formalism.

In the interaction picture we then have[5]

$$H_I(t)=e^{iH_0t/\hbar}H_Ie^{-iH_0t/\hbar}$$
$$=i\hbar g\sum_{i=1}^{N_e}\left(e^{-it(\hbar k^2+2kp_i)/2m}A_i^\dagger-\text{H.c.}\right)\ . \tag{5}$$

We note that if we replace the momentum operators

p_i by a c number p_0, we have $[H_I^0(t)=H_I(t)\,|_{p=p_0}]$

$$[H_I^0(t),H_I^0(t')]=2i(\hbar g)^2\sin\beta(t-t')\times\sum_{i,j=1}^{N_e} e^{ik(z_i-z_j)} \tag{6}$$

with

$$\beta=\frac{\hbar k^2}{2m}+\frac{kp_0}{m}\ . \tag{7}$$

The right-hand side (rhs) of Eq. (6) is proportional to the Hermitian positive operator

$$B=N_e^{-1}\sum_{i,j=1}^{N_e} e^{ik(z_i-z_j)}\ . \tag{8}$$

The latter can often be regarded as a c number inasmuch as

$$[[H_I^0(t),H_I^0(t')],H_I^0(t'')]=0\ . \tag{9}$$

The time-evolution operator in the interaction picture is

$$S(T/2,-T/2)=\mathscr{T}\exp\left[-\frac{i}{\hbar}\int_{-T/2}^{T/2}dt\,H_I(t)\right]\ . \tag{10}$$

As in Ref. 5 we now attempt a first-order expansion of $S(T/2,-T/2)$ with respect to the quantities p_i-p_0. Since these are operators this looks like a questionable procedure. However, what we shall do is equivalent to writing $p_i=p_0+\alpha_i(p_i-p_0)$, expanding with respect to the c number α_i and setting $\alpha_i=1$ in the end. Hence we write

$$S(T/2,-T/2)=S_0(T/2,-T/2)+S_1(T/2,-T/2)+\cdots\ , \tag{11a}$$

$$S_0(t_2,t_1)=\mathscr{T}\exp\left[-i/\hbar\int_{t_1}^{t_2}dt\,H_I^0(t)\right]\ , \tag{11b}$$

$$\begin{aligned}S_1(T/2,-T/2)&=\sum_{i=1}^{N_e}(p_i-p_0)\left[\frac{\partial}{\partial pi}S(T/2,-T/2)\right]\Bigg|_{p_i=p_0}\\&=-ig\frac{k}{m}\int_{-T/2}^{T/2}dt\,S_0(T/2,t)t\sum_{i=1}^{N_e}[e^{-i\beta t}(p_i-p_0)A_i^\dagger+e^{i\beta t}A_i(p_i-p_0)]S_0(t,-T/2)\ .\end{aligned} \tag{11c}$$

The first line of Eq. (11c) is, of course, a symbolic notation. The correct order of the operators is exhibited in the second line. Notice also that the latter expression is properly time ordered.

Our goal is now to commute $S_0(t,-T/2)$ in the second line of Eq. (11c) with the square bracket so that the group property

$$S_0(t_1,t_2)S_0(t_2,t_3)=S_0(t_1,t_3) \tag{12}$$

can be exploited. In order to compute the required commutators, a more explicit representation for $S_0(t_2,t_1)$ is required. Since the commutator of $H_1^0(t)$ with itself at different times is essentially a c number [cf. Eq. (9)] the time ordering in Eq. (11b) can be disposed of, viz.,

$$S_0(t_2,t_1)=\exp i\theta(t_2,t_1)\times\exp\left[-\frac{i}{\hbar}\int_{t_1}^{t_2}dt\,H_I^0(t)\right]\ , \tag{13}$$

where[8]

$$\begin{aligned}i\theta(t_2,t_1)&=-\frac{1}{2\hbar^2}\int_{t_1}^{t_2}dt'\int_{t_1}^{t'}dt''[H_I^0(t'),H_I^0(t'')]\\&=iN_eBF(t_2,t_1)\end{aligned} \tag{14}$$

with

$$\begin{aligned}F(t_2,t_1)&=-g^2\int_{t_1}^{t_2}dt'\int_{t_1}^{t'}dt''\sin\beta(t'-t'')\\&=-g^2\{(t_2-t_1)/\beta\\&\quad-[\sin\beta(t_2-t_1)]/\beta^2\}\ .\end{aligned} \tag{15}$$

Here Eq. (6) has been used and the operator B was defined in Eq. (8). We further split the second exponential in Eq. (13) into a product of contributions from the individual electrons

$$\exp\left[\frac{-i}{\hbar}\int_{t_1}^{t_2}dt\,H_I^0(t)\right]=\exp[j^*(t_2,t_1)A_1^\dagger-j(t_2,t_1)A_1]\cdots\exp[j^*(t_2,t_1)A_{N_e}^\dagger-j(t_2,t_1)A_{N_e}]\times\prod_{i<j}\exp[i\,|\,j(t_2,t_1)\,|^2\sin k(z_i-z_j)] \tag{16}$$

with

$$j(t_2,t_1)=g\int_{t_1}^{t_2} dt\, e^{i\beta t}=j(-t_1,-t_2)^* \ . \tag{17a}$$

In particular,

$$j(T/2,-T/2)\equiv j(T)=2g\beta^{-1}\sin(\beta T/2)=j^*(T) \ . \tag{17b}$$

The last product in Eq. (16) is a consequence of the commutator (4c). If the first exponentials in Eq. (16) are split by means of the Baker-Hausdorff formula, viz.,

$$\exp(j^*A_i^\dagger - jA_i)=\exp(j^*A_i^\dagger)\exp(-jA_i)\exp(-1/2\,|j\,|^2) \ ,$$

the computation of the following commutators is straightforward, though tedious:

$$[A_i,S_0(t_2,t_1)]=j^*(t_2,t_1)\sum_{j=1}^{N_e} e^{ik(z_i-z_j)}S_0(t_2,t_1) \ , \tag{18a}$$

$$[A_i^\dagger,S_0(t_2,t_1)]=j(t_2,t_1)\sum_{j=1}^{N_e} e^{-ik(z_i-z_j)}S_0(t_2,t_1) \ , \tag{18b}$$

$$[p_i,S_0(t_2,t_1)]=-\hbar k S_0(t_2,t_1)\left[2F(t_2,t_1)\sum_{j=1}^{N_e}\sin k(z_i-z_j)+|\,j(t_2,t_1)\,|^2\sum_{j=1}^{N_e}\cos k(z_i-z_j) + j^*(t_2,t_1)A_i^\dagger + j(t_2,t_1)A_i\right] . \tag{18c}$$

By means of Eqs. (18a)–(18c) we can now rewrite Eq. (11c) as

$$S_1(T/2,-T/2)= -\frac{igk}{m}S_0(T/2,-T/2)\times\left\{\int_{-T/2}^{T/2} dt\, te^{-i\beta t}\sum_{i=1}^{N_e}\left[p_i-p_0-\hbar k\left[\,|\,j(t,-T/2)\,|^2\sum_{j=1}^{N_e}\cos k(z_i-z_j) + j^*(t,-T/2)A_i^\dagger + j(t,-T/2)A_i + 2F(t,-T/2)\sum_j \sin k(z_i-z_j)\right]\right]\times\left(A_i^\dagger + j(t,-T/2)\sum_{j=1}^{N_e} e^{-ik(z_i-z_j)}\right)+\text{H.c.}\right\} ; \tag{19}$$

$S(T/2,-T/2)=S_0(T/2,-T/2)+S_1(T/2,-T/2)+\cdots$ is now evidently unitary up to first order since $S_0(T/2,-T/2)$ is unitary. It depends on the expansion parameter p_0. Inasmuch as a first-order expansion like $f(x)=f(x_0)+(x-x_0)f'(x_0)+\cdots$ is actually independent of x_0, we expect it not to depend on p_0 up to first order. Since we have been dealing here with an expansion in terms of operators, we will verify this in Appendix B with the result

$$\frac{\partial}{\partial p_0}[S_0(T/2,-T/2)+S_1(T/2,-T/2)]=O\left[\frac{\hbar k}{m}\right]^2 . \tag{20}$$

We can then safely exploit this fact in Sec. III by making a convenient choice for p_0.

III. GAIN, SPREAD, AND PHOTON STATISTICS

The time-evolution operator $S(T/2,-T/2)$ which we derived in Sec. II transforms initial states $|\,\text{in}\rangle$ prior to the interaction mediated by the wiggler into final states $|\,\text{out}\rangle$ after the interaction time T,

$$|\,\text{out}\rangle = S(T/2,-T/2)\,|\,\text{in}\rangle \ . \tag{21}$$

All quantities of interest, such as gain, spread, etc., are then given by the final-state expectation values of the corresponding operators,

$$\langle \text{out} \,|\, O \,|\, \text{out}\rangle = \langle \text{in} \,|\, S^{\dagger}(T/2,-T/2) \,|\, O \,|\, S(T/2,-T/2) \,|\, \text{in}\rangle \ . \tag{22}$$

As an initial state we shall take

$$|\,\text{in}\rangle = |\,\bar{p}(1),\ldots,\bar{p}(N_e),N\rangle \equiv |\,(\bar{p})^{N_e},N\rangle \ , \tag{23}$$

describing N_e electrons with identical momenta $\bar{p}$ and N laser photons. Ignoring Fermi statistics is justified for all reasonable current densities. The restriction of the initial state of the electrons to a pure state seems questionable. However, we demonstrate in Appendix C that a density-matrix description of the electrons leads to identical results. The operators $p_i, A_i, A_i^{\dagger}$ act on this state like

$$p_i \,|\, \text{in}\rangle = \bar{p} \,|\, \text{in}\rangle \ , \tag{24a}$$

$$A_i \,|\, \text{in}\rangle = \sqrt{N} \,|\, \ldots,\bar{p}(i)+\hbar k,\ldots,N-1\rangle \ , \tag{24b}$$

$$A_i^{\dagger} \,|\, \text{in}\rangle = \sqrt{N+1} \,|\, \ldots,\bar{p}(i)-\hbar k,\ldots,N+1\rangle \ . \tag{24c}$$

We fix the so far arbitrary expansion parameter p_0 by

$$p_0 = \bar{p} - \tfrac{1}{2}\hbar k \ , \tag{25}$$

which implies for the detuning parameter

$$\beta = k\bar{p}/m \ , \tag{26}$$

and therefore resonance at $\bar{p}=0$, which is somewhat obscured otherwise. The derivation of the following results is then straightforward. Some of the calculational details are presented in Appendix D.

A. Gain and spread

The change in the number of photons during the interaction time T comes out to be

$$\langle \text{out} \,|\, a^{\dagger}a \,|\, \text{out}\rangle - N = \Delta N = N_e j^2 - \frac{\hbar k^2}{m}[(2N+1)N_e jj' + N_e(N_e-1)j^3 j'] + \tfrac{1}{2}N_e(N_e-1)\frac{\hbar k^2}{m}\Phi + N_e\delta \ , \tag{27}$$

where $j \equiv j(T)$, $j' = \partial j(t)/\partial\beta$, and

$$\delta = ig\hbar k^2 j/m \int_{-T/2}^{T/2} dt\, t e^{-i\beta t}[2\,|\,j(T/2,t)\,|^2 + 2\,|\,j(t,-T/2)\,|^2 - j^2(T/2,t) - j^2(t,-T/2)] \ , \tag{28}$$

$$\Phi = igj \int_{-T/2}^{T/2} dt\, t e^{-i\beta t}\{\,|\,j(T/2,t)\,|^2 + |\,j(t,-T/2)\,|^2 - 2j^2(T/2,t) - 2j^2(t,-T/2) - 2i[F(t,-T/2) - F(-t,-T/2)]\} \ . \tag{29}$$

By inserting the explicit expressions (15) and (17b) for j and F, these functions can be expressed in terms of j and its derivatives

$$\delta = -\frac{\hbar k^2}{m}[6jj'g^2\beta^{-2} + (2\beta)^{-1}j^4] \ , \tag{30}$$

$$\Phi = -4g^2\beta^{-2}j(j'-\beta j'') + 2j^3 j' \ . \tag{31}$$

The final simple result for the gain is then

$$\Delta N = N_e j^2 - \frac{\hbar k^2}{m}(2N+1)N_e jj' - 2g^2\beta^{-2}\frac{\hbar k^2}{m}j(j'-\beta j'')N_e(N_e-1) + N_e\delta \ . \tag{32}$$

Here the first term represents spontaneous emission which is proportional to the number of electrons. The second gives the usual small signal-gain expression which is also proportional to the number of electrons, as it should be. It is not modified by $O(N_e^2)$ contributions. The third term which is absent for $N_e=1$ represents amplified spontaneous emission. It is proportional to $N_e(N_e-1)$ and antisymmetric with respect to $\beta=0$. Recalling[5] that positive gain implies $j'<0$, i.e., $\bar{p}>0$, we see that spontaneous emission is enhanced on the positive ($\bar{p}>0$) and reduced on the negative-gain side ($\bar{p}<0$). The physical origin of this term is evident: Starting from the field vacuum ($N=0$), spontaneous emission is initially proportional to N_e. This initial radiation is then amplified by stimulated emission, provided the gain is positive, or attenuated otherwise. Since gain is again proportional to N_e, this effect should go as N_e^2. Futher support for this argument comes from the fact that the first term on the rhs of Eq. (32) is proportional to T^2 for small T, whereas the third term starts with T^4. We shall discuss the order of magnitude of amplified spontaneous emission in Sec. IV. The last term in Eq. (32) is negligible under all conditions: For $N_e \gg 1$ it is small compared to the third term, and for $N_e = 1$ it is small compared to the first one. Counting powers of $\hbar$ we conclude that all terms in Eq. (32) contribute as classical terms to the energy change $\hbar\omega\Delta N$, with the only exception being the one within the factor $2N+1$.

The spread is calculated along the same lines. We first compute

$$\begin{aligned}\langle \text{out} \,|\, (a^\dagger a)^2 \,|\, \text{out} \rangle = {} & (N+N_e j^2)^2 + (2N+1)N_e j^2 + N_e(N_e-1)j^4 \\ & - \frac{\hbar k^2}{m} N_e [\,(4N^2+2N+1)jj' + 4(2N+1)N_e j^3 j' + (4N+1)(N_e-1)j^3 j' \\ & \qquad + 4(N_e-1)^2 j^5 j' - \tfrac{1}{2}(N_e-1)(4N+1+4(N_e-1)j^2)\Phi\,] \\ & + [4N+1+2(3N_e-2)j^2]N_e\delta \;. \end{aligned} \tag{33}$$

The spread is then

$$\begin{aligned}\Delta(N^2) &= \langle \text{out} \,|\, (a^\dagger a)^2 \,|\, \text{out}\rangle - \langle \text{out} \,|\, a^\dagger a \,|\, \text{out}\rangle^2 \\ &= (2N+1)N_e j^2 + N_e(N_e-1)j^4 \\ &\quad - \frac{\hbar k^2}{m} N_e \{\, jj' + (2N+1)[3(N_e-1)+2]j^3 j' + 2(N_e-1)(N_e-2)j^5 j' \\ &\qquad - \tfrac{1}{2}(N_e-1)[2N+1+2j^2(N_e-2)]\Phi \,\} + [2N+1+4j^2(N_e-1)]N_e\delta \;. \end{aligned} \tag{34}$$

Utilizing Eq. (31) this expression can be rewritten as

$$\begin{aligned}\Delta(N^2) = {} & (2N+1)N_e \left[j^2 - 2\frac{\hbar k^2}{m} N_e j^3 j' - \frac{2\hbar k^2}{m}(N_e-1)g^2\beta^{-2} j(j'-\beta j'') + \delta \right] + N_e(N_e-1)j^4 \\ & - \frac{\hbar k^2}{m} N_e [\, jj' + 4g^2\beta^{-2}(N_e-1)(N_e-2)j^3(j'-\beta j'')\,] + 4N_e(N_e-1)j^2\delta \;. \end{aligned} \tag{35}$$

It is well known that gain and spread are intimately related. In view of Eqs. (34) and (35) as well as (32), we can infer two relationships. First, there is a derivative relation between the dominant term of the spread, i.e., the first term of Eq. (34) or (35), and the gain due to stimulated emission, i.e., the second term of Eq. (32), which can be written as

$$(\Delta N)_{\text{stim}} = -\frac{1}{2}\frac{\hbar k^2}{m}\frac{\partial}{\partial\beta}\Delta(N^2) \;, \tag{36}$$

having in mind that this only refers to the just-mentioned terms. It might well be that Eq. (36) is actually better than that, i.e., taking the derivative of the large parentheses in Eq. (35) instead of just its first term might yield $O(\hbar k/m)^2$ corrections to the gain. (Note that the latter are not necessarily quantum corrections.) Since, however, we calculated the gain only up to the order of $\hbar k/m$, we are unable to check that possibility. Equation (36) is the well-known gain-spread relation.[9] Note that it refers only to stimulated emission, i.e., the derivative of the term $N_e(N_e-1)j^4$ of the spread is not related to any contribution to the gain. Second, the dominant term of the spread equals $(2N+1)$ times the dominant term which specifies spontaneous emission. This relation can be written so as to include amplified spontaneous emission:

$$\Delta(N^2)_{\text{stim}} = (2N+1)(\Delta N)_{\text{spont}} + 2j^2(\Delta N)_{\text{stim}} \;. \tag{37}$$

We finally note that Eqs. (32), (34), and (35) show a clean distinction between one- and two- (or more) particle contributions: The former such as the gain due to stimulated emission are proportional to N_e whereas the latter are proportional to $N_e(N_e-1)$.

B. Photon statistics of spontaneous emission

Here we deal with the case $N=0$, i.e., no photons being initially present. The notion of spontaneous emission is not supposed to imply that this a genuine quantum-mechanical regime. It could be and is referred to as bremsstrahlung as well.

From Eqs. (32) and (36) we have for $N=0$

$$\Delta(N^2)-\Delta N=N_e(N_e-1)j^4-\frac{2\hbar k^2}{m}N_e^2 j^3 j'+\cdots \ , \tag{38}$$

where the ellipsis represents terms of order j^6. If this quantity is greater than, less than, or equal to zero, the emitted photons are bunched, antibunched, or coherent,[10] respectively. For $N_e=1$ the first term is missing, and we can have either case depending on the sign of the gain. For $N_e>1$ the first term is always dominant, so that there is photon bunching under all circumstances.[11]

For $N_e=1$ we had remarked earlier[5] that deviations from Poisson statistics are only due to the presence of gain, as indicated by the second term in Eq. (38) being proportional to the derivative j'. For $N_e>1$ this is no longer true, since even when gain is neglected, the first term in Eq. (38) still violates Poisson statistics. This had to be expected, because spontaneous radiation from one electron is supposed to be coherent, whereas from several electrons it is not. On the other hand, amplified spontaneous emission is coherent, as indicated by the fact that the term proportional to $j(j'-\beta j'')$ in Eqs. (32) and (35) dropped out of Eq. (38).

The photon bunching or antibunching in the one-electron case can easily be understood intuitively (see Fig. 1). Suppose the electron has initially momentum $\bar{p}>0$. Then $j'<0$ and gain would be positive. By emitting a photon the electron changes its momentum to $\bar{p}-\hbar k$. Since $j(\bar{p}-\hbar k)>j(\bar{p})$ the probability of emitting an additional photon has increased as compared to the initial state, hence we have photon bunching. It is also evident that this

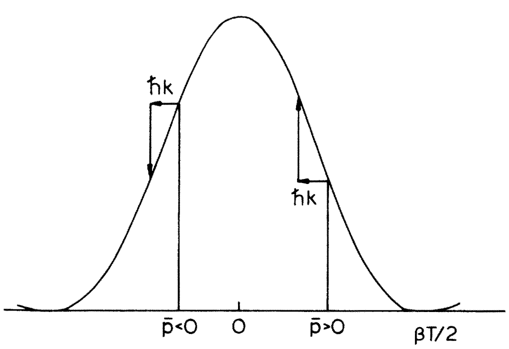

FIG. 1. Spectrum of spontaneous emission vs detuning. The figure shows how by emitting one photon the probability of emitting another photon changes. The magnitude of $\hbar k$ is greatly exaggerated.

effect should be proportional to the probability of emitting one photon times the change in that probability due to the emission of one photon, i.e., proportional to $j^2(\partial/\partial\beta)j^2$, which is exactly the second term in Eq. (38). Obviously, for $\bar{p}<0$ the opposite happens, since $j(\bar{p}-\hbar k)<j(\bar{p})$ for $\bar{p}<0$, and we have antibunching.

For more than one electron this argument is still valid. The much larger incoherence, however, introduced by several electrons radiating, completely masks the effect.

C. Squeezed states in spontaneous emission

It was realized earlier[4] that spontaneous radiation from one electron is in a perfect squeezed state, i.e., in a minimum uncertainty state with either quadrature component squeezed at the expense of the other. Unlike photon antibunching, it is not *a priori* obvious that this effect ought to disappear in the many-electron case. In calculating the squeezing properties we will encounter an essential difference between the one- and the many-electron case.

Owing to conservation of energy and momentum, the annihilation and creation operators of the laser field always appear in the Hamiltonian in company with the operators $\exp(\pm ikz_i)$ which increase or decrease the energy of one of the electrons, as exhibited in Eqs. (3) and (5). Hence taking expectation values of nondiagonal combinations of only the field operators, such as, e.g., $\langle \text{out}|a^2|\text{out}\rangle$, always yields a zero result. The same still holds if the electron coordinates have been eliminated by going over to a reduced density-matrix description. Hence in the one-electron case we defined squeezing in terms of the operators $A=a\exp(ikz)$ instead of just a. The analogous choice for the many-electron case is

$$A=N_e^{-1/2}\sum_{i=1}^{N_e}A_i \ . \tag{39}$$

For $N_e=1$ we had $[A,A^\dagger]=1$ and $A^\dagger A=a^\dagger a$. In general we have instead

$$[A,A^\dagger]=N_e^{-1}\sum_{i,j=1}^{N_e}e^{ik(z_i-z_j)}=B \tag{40}$$

and

$$A^\dagger A=Ba^\dagger a \ . \tag{41}$$

Hence $A^\dagger A$ is no longer the intensity operator of the field. Since, however,

$$\langle \text{in}|B|\text{in}\rangle=1 \ , \tag{42}$$

and we shall see below, also

$$\langle \text{out}|B|\text{out}\rangle=1 \ , \tag{43}$$

for expectation values we still have $\langle [A,A^\dagger]\rangle=1$. Hence we believe that squeezing derived for the operator A would apply to the physically realizable situation where electrons and field are separated after the interaction.

If we introduce the Hermitian quadrature components of A and $A^\dagger$ by

$$A=A_1+iA_2 \;, \tag{44}$$

then we have the uncertainty relation

$$(\langle \Delta A_1^2\rangle\langle \Delta A_2^2\rangle)^{1/2}\ge \tfrac{1}{4}\langle B\rangle \tag{45}$$

for $\Delta A_i=A_i-\langle A_i\rangle$. Since $\langle B\rangle=1$ for all out states that we are considering, we can define a state to be squeezed if

$$\langle \Delta A_i^2\rangle<\tfrac{1}{4} \tag{46}$$

for $i=1$ or $i=2$. Notice that the state does not have to be a minimum uncertainty state for this definition to apply.

In order to evaluate $\langle \Delta A_i^2\rangle$ we need

$$\langle \text{out}\,|\,A^\dagger A\,|\,\text{out}\rangle=N+(2N_e-1)j^2-\frac{\hbar k^2}{m}jj'\{[(2N+1)(2N_e-1)+N(N_e-1)]+4(N_e-1)^2j^2\}$$
$$+\frac{\hbar k^2}{m}(N_e-1)^2\Phi+(3N_e-2)\delta \;, \tag{47}$$

$$\langle \text{out}\,|\,A^2+A^{\dagger 2}\,|\,\text{out}\rangle=2(2N_e-1)j^2-4\frac{\hbar k^2}{m}jj'[(2N+1)N_e+2(N_e-1)^2j^2]$$
$$+2\frac{\hbar k^2}{m}(N_e-1)^2\Phi+2(3N_e-2)\delta \;, \tag{48}$$

$$\langle \text{out}\,|\,A+A^\dagger\,|\,\text{out}\rangle=N_e^{1/2}\left[2j-\frac{\hbar k^2}{m}j'[2N+1+2j^2(N_e-1)]+\frac{\hbar k^2}{2mj}(N_e-1)\Phi+j^{-1}\delta\right], \tag{49}$$

$$\langle \text{out}\,|\,A-A^\dagger\,|\,\text{out}\rangle=O\left[\frac{\hbar k}{m}j^3\right]. \tag{50}$$

Along the same lines Eq. (43) can be verified, thus justifying the definition (46). We can then compute

$$\langle \Delta A_{1,2}^2\rangle=\tfrac{1}{4}\langle 2A^\dagger A+1\pm(A^2+A^{\dagger 2})\rangle\mp\tfrac{1}{4}\langle A\pm A^\dagger\rangle^2 \tag{51}$$

with the result

$$\langle \Delta A_1^2\rangle=\tfrac{1}{4}+(N_e-1)j^2+\frac{\hbar k^2}{m}[-N_e+\tfrac{1}{2}-2(N_e-1)(N_e-2)j^2]jj'$$
$$+\frac{\hbar k^2}{2m}(N_e-1)(N_e-2)\Phi+2(N_e-1)\delta \;, \tag{52}$$

$$\langle \Delta A_2^2\rangle=\tfrac{1}{4}+\frac{\hbar k^2}{2m}jj' \;. \tag{53}$$

Owing to the second term in Eq. (52) for $N_e>1$ we always have[12] $\langle \Delta A_1^2\rangle>\frac{1}{4}$. $\langle \Delta A_2^2\rangle$, however, is independent of the number of electrons. Hence we have squeezing in A_2 for an arbitrary number of electrons on the positive-gain side (i.e., $j'<0$, $\bar{p}>0$ corresponding to a laser wavelength above resonance in the lab frame if $\bar{p}$ is fixed). However, only for $N_e=1$ is minimum uncertainty maintained. Whether or not the squeezing, derived here formally, applies to an experimentally realizable situation, will be discussed in Sec. IV.

D. Electrons after the interaction

It can easily be checked that the operator

$$P=\sum_{i=1}^{N_e}p_i+\hbar k a^\dagger a \tag{54}$$

commutes with $H_I(t)$ and consequently is conserved. Maybe surprisingly, we also have

$$[P,S_0(T/2,-T/2)]=[P,S_1(T/2,-T/2)]=0 \;, \tag{55}$$

which is verified utilizing the commutation relations (18a)–(18c). So, in spite of our approximation of letting $p_i = p_0$, even the zeroth-order approximation is consistent with energy-momentum conservation. This is evident for formal reasons, since the commutators (55) do not care about the dependence of S_0 and S_1 on p_i, because $[P,p_i]=0$ anyway.

Hence the average change in the momentum of the electron is immediately related to the gain via

$$\begin{aligned}\langle \text{out} \,|\, p_i \,|\, \text{out} \rangle &= N_e^{-1} \left\langle \text{out} \left| \sum_{i=1}^{N_e} p_i \right| \text{out} \right\rangle \\ &= N_e^{-1} \langle \text{out} \,|\, P - \hbar k a^\dagger a \,|\, \text{out} \rangle \\ &= \bar{p} - \hbar k N_e^{-1} \Delta N \;, \end{aligned} \tag{56}$$

where

$$P \,|\, \text{out,in} \rangle = (N_e \bar{p} + \hbar k N) \,|\, \text{out,in} \rangle \tag{57}$$

has been used.

To estimate the average electron spread after the interaction,

$$\Delta(p_i^2) = \langle \text{out} \,|\, p_i^2 \,|\, \text{out} \rangle - \langle \text{out} \,|\, p_i \,|\, \text{out} \rangle^2 \;, \tag{58}$$

we write

$$\begin{aligned}\langle \text{out} \,|\, p_i^2 \,|\, \text{out} \rangle &= N_e^{-1} \left\langle \text{out} \left| \left[\sum p_i \right]^2 - \sum_{i \neq j} p_i p_j \right| \text{out} \right\rangle \\ &= N_e^{-1} \langle \text{out} \,|\, P^2 - (\hbar k a^\dagger a)^2 - 2\hbar k \sum p_i a^\dagger a \,|\, \text{out} \rangle - (N_e - 1) \langle \text{out} \,|\, p_1 p_2 \,|\, \text{out} \rangle \;. \end{aligned}$$

If we introduce

$$\Delta(p_i p_j) = \langle \text{out} \,|\, p_i p_j \,|\, \text{out} \rangle - \langle \text{out} \,|\, p_i \,|\, \text{out} \rangle \langle \text{out} \,|\, p_i \,|\, \text{out} \rangle \;, \tag{59}$$

then

$$\Delta(p_i^2) = N_e^{-1} (\hbar k)^2 \Delta(N^2) - (N_e - 1) \Delta(p_1 p_2) \tag{60}$$

with the spread $\Delta(N^2)$ given in Eq. (35). Because Eq. (59) is independent of the particular values of $i \neq j$, the values $i=1$, $j=2$ are arbitrarily chosen for this discussion. Hence for $N_e > 1$, the relation between Δp_i^2 and $\Delta(N^2)$ is not as straightforward as that between Δp_i and ΔN, which is given by Eq. (56). If the momentum of one particular electron, p_1 for example, is above average, this must be compensated for by the momenta of the remaining electrons; hence we expect

$$\Delta(p_1 p_2) < 0$$

and

$$\Delta(p_1 p_2) = 0 \left[\frac{1}{N_e} \right] . \tag{61}$$

The electron momentum spread is therefore increased as compared to the case $N_e = 1$. Of course, $\Delta(p_1 p_2)$ can be evaluated explicitly along the lines of the calculations outlined thus far, utilizing the commutation relation (18c).

IV. DISCUSSION OF RESULTS

In this section we summarize and discuss our main results.

(a) There are no many-particle effects in the gain for stimulated emission.

(b) Spontaneous emission is modified by an additional contribution which is proportional to $N_e(N_e - 1)$, i.e., the third term in Eq. (32). This term shifts the spectrum of spontaneous emission towards the positive-gain side, cf. Fig. 2. If a small shift is assumed, the maximum of spontaneous emission now occurs at

$$\beta = \frac{k^2}{20m} g^2 T^2 N_e \;. \tag{62}$$

Rewritten in the lab frame the corresponding shift in wavelength is

$$\Delta\lambda = \pi / (10\gamma^3)(eA_w / mc^2)^2 r_0 \lambda L^2 (N_e / V) \;. \tag{63}$$

Here A_w denotes the peak amplitude of the vector potential of the wiggler field, L the length of the wiggler, λ the laser wavelength, $E = m_0 c^2 \gamma$ the electron energy, and r_0 the classical electron radius. For the data of Ref. 2, assuming an electron beam area of 0.01 cm^2, Eq. (63) yields $\Delta\lambda = 0.004\ \mu\text{m}$. With amplified spontaneous emission taken into account, the line center is at $\lambda + \Delta\lambda$, where

$$\lambda = \frac{\lambda_0}{2\gamma^2} \left[1 + \left[\frac{eA_w}{mc^2} \right]^2 \right] \times \left\{ 1 + \frac{1}{2\gamma^2} \left[1 + \left[\frac{eA_w}{mc^2} \right]^2 \right] \right\} \tag{64}$$

is the line center for $N_e = 1$ and λ_0 the period of the

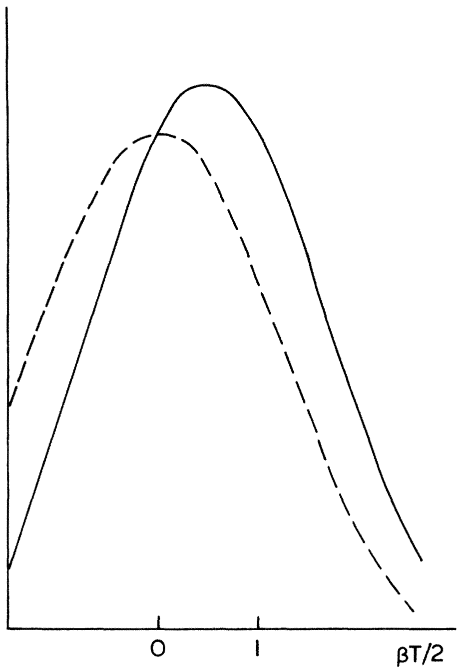

FIG. 2. Spectrum of spontaneous emission for the parameters of Ref. 2 [$J=2.6$ A, $(0.1\ \text{cm})^2$ beam area]; dashed line, usual $(\sin x/x)^2$ spectrum; solid line, corrected for amplified spontaneous emission. The spectrum becomes negative for some values of $\beta T/2$ outside of the figure. This indicates that our first-order expansion is no longer justified. Inclusion of higher orders would restore the spectrum to positive values.

wiggler. Owing to uncertainties in the electron energy and the wiggler field strength, λ is not known with sufficient precision so that $\Delta\lambda$ cannot be inferred from the absolute position of the line center. Since, however, gain still derives from the spontaneous emission line shape for $N_e=1$ [cf. remark (a)], $\Delta\lambda$ should show up as a shift between the center of spontaneous emission and the position of zero gain. This should look like Fig. 9 of the ACO-LURE gain measurement,[13] where an analogous shift of ~ 0.5 MeV exists. (Here the electron energy instead of the wavelength is taken as the variable.) This shift, however, is not explained for by Eq. (62) which yields a result which is smaller by many orders of magnitude owing to the large γ, small L, and the low current of the ACO experiment. The decisive test of whether an observed shift is due to amplified spontaneous emission or something else would be its proportionality to the current density N_e/V.

(c) Spontaneous emission from several electrons is incoherent, amplified spontaneous emission is coherent. The slight quantum effect of photon antibunching, which for $N_e=1$ can lead to sub-Poissonian photon statistics, is for $N_e>1$ hidden behind this larger incoherence.

(d) The squeezing which we have obtained in Eq. (53) is surprising owing to the apparent asymmetry between $\langle \Delta A_1^2 \rangle$ and $\langle \Delta A_2^2 \rangle$ and its independence of N_e. The question arises as to whether this is an artifact of our formalism or reality. We first note that introducing phases into the effect of an operator A_i on the state $|\text{in}\rangle$, i.e., writing

$$A_i\,|\text{in}\rangle=\sqrt{N}e^{i\phi_i}\,|\ldots,\underset{\cdot}{p}(i)+\hbar k,\ldots,N-1\rangle$$

with an arbitrary i-dependent phase in place of Eq. (24b) does not change anything, since in a nonzero expectation value each A_i is matched by a $A_i^\dagger$. Changing, however, the relative phase between the laser and the wiggler field, shifts the squeezing into a different quadrature component. For example, adopting the Hamiltonian (A16) instead of (A13) would interchange the right-hand sides of Eqs. (52) and (53). It is not clear to us how by an actual experiment this phase is fixed and, more important, whether it would have the same value for a sequence of experiments which are performed as identically as possible. Still, it is surprising that all many-particle effects enter only one field-quadrature component (with our choice of the Hamiltonian, $\langle \Delta A_1^2 \rangle$) leaving the other one unaffected. This seems to be an unambiguous consequence of our formalism. It will certainly change if the electrons are described by a nondiagonal density matrix.

(e) The electron-energy spread is increased in comparison to the one-particle case. Gain and electron-energy spread are the same regardless of whether the initial state of the laser field is taken as a photon number eigenstate, a coherent, or a (phase-averaged) generalized coherent state.

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APPENDIX A

The Hamiltonian for the electron in the Bambini-Renieri frame is derived by starting with the classical Hamiltonian for an electron in the laboratory frame

$$H=\{[\vec{p}-e(\vec{A}_w+\vec{A}_L)]^2c^2+m_0^2c^4\}^{1/2}\,, \qquad \text{(A1)}$$

where $\vec{p}$ is the momentum of the electron and m_0 is the rest mass. The vector potentials for the wiggler field A_w and the optical field A_L are given by

$$\vec{A}_w = A_w(\sin k_q z\hat{x} - \cos k_q z\hat{y}) = -iA_w(\hat{e}^* e^{ik_q z} - \hat{e}e^{-ik_q z})/\sqrt{2} \tag{A2}$$

and

$$\vec{A}_L = (A_L \hat{e} e^{i(kz-\omega t)} + A_L^* \hat{e}^* e^{-i(kz-\omega t)})/\sqrt{2} \tag{A3}$$

with $\hat{e} = (\hat{x} + i\hat{y})/\sqrt{2}$. The initial momentum of the electron is chosen to be parallel with the axis of the magnetic field so that

$$\vec{p}\cdot\vec{A}_L = \vec{p}\cdot\vec{A}_w = 0 . \tag{A4}$$

The substitution of the Eqs. (A2) and (A3) into the Hamiltonian (A1) gives, with the aid of condition (A4),

$$H \simeq [p^2c^2 - ie^2A_w(A_L e^{i(k_q+k)z-\omega t} - \text{c.c.})c^2 + m^2c^4]^{1/2} , \tag{A5}$$

where p is the longitudinal component of the momentum and

$$m^2 = m_0^2 + e^2A_w^2/c^4$$

is the renormalized mass. We have also made use of the condition $A_L^2 \ll A_w^2$ to neglect terms proportional to A_L^2.

In order to move to a frame with velocity v along the z direction, the frequencies and wave vectors are transformed to the new coordinate system with the aid of

$$z = (z' + vt')\gamma' , \tag{A6}$$

$$t = \left[t' + \frac{vx'}{c^2} \right] \gamma' , \tag{A7}$$

where primed quantities refer to the new coordinate system and

$$\gamma' = (1 - v^2/c^2)^{-1/2} .$$

With these transformations, the phase of the interaction term in Eq. (A5) becomes

$$(k_q + k)z - \omega t = [k_q + k(1-v)]z' - [\omega - v(k_q + k)]t' .$$

If the velocity of the moving frame is chosen so that the Hamiltonian is time independent,

$$v = c\frac{k}{k + k_q} , \tag{A8}$$

then we are in the Bambini-Renieri frame. The Hamiltonian in this frame is then given by

$$H' = \left[p'^2c^2 + m^2c^4 - i\frac{A_w e^2}{mc^2}(A_L e^{ik'z'} - A_L^* e^{-ik'z'}) \right]^{1/2} , \tag{A9}$$

where

$$k' = k_q + k(1-v) = (2k_q k + k_q^2)^{1/2} \approx (2k_q k)^{1/2} . \tag{A10}$$

The momentum p' in the moving frame is related to the quantities in the laboratory frame through

$$p' = \left[p - \frac{Ev}{c^2} \right]\gamma' = \left[(k + k_q)p - \frac{kE}{c} \right] \Big/ k' \ll mc . \tag{A11}$$

Because p' is nonrelativistic, the Hamiltonian (A9) can be approximated as

$$H' \approx \frac{p'^2}{2m} + mc^2 - \frac{e^2 iA_w}{2mc^2}(A_L e^{ik'z'} - A_L^* e^{-ik'z'}) . \tag{A12}$$

In order to quantize the Hamiltonian, the vector potential (A3) for the optical field is replaced by

$$\vec{A}_L \to c\left[\frac{2\pi\hbar}{\omega V}\right]^{1/2} (\hat{e}ae^{ikz} + \hat{e}^* a^\dagger e^{-ikz}) ,$$

where a ($a^\dagger$) are photon destruction (creation) operators of the laser mode in question, and V is the quantization volume. The net effect of this substitution is to make the replacement

$$A_L \to c\left[\frac{4\pi\hbar}{\omega V}\right]^{1/2} ae^{i\omega t} ,$$

$$A_L^* \to c\left[\frac{4\pi\hbar}{\omega V}\right]^{1/2} a^\dagger e^{-i\omega t} .$$

The quantized version of the Hamiltonian (A12) is then

$$H' = \frac{p'^2}{2m} - ig\hbar(ae^{i(k'z'+\omega' t')} - a^\dagger e^{-i(k'z'+\omega' t')}) , \tag{A13}$$

where p' and z' are now operators, $[p',z'] = -i\hbar$, ω' is the laser frequency in the moving frame

$$\omega' = \frac{ckk_q}{k'} \simeq \frac{ck'}{2} , \tag{A14}$$

and

$$g = \frac{e^2A_w}{mc}\left[\frac{\pi}{\omega V\hbar}\right]^{1/2} . \tag{A15}$$

There are several interesting features associated with this derivation. Unlike previous derivations of the Bambini-Renieri frame, this one does not require the Weizsäcker-Williams approximation or quantization of the wiggler field. Also, the final form of the Hamiltonian depends upon the initial phase of the wiggler field. For instance, if the substitution $z \to z+\pi/2$ is made in Eq. (A2), the wiggler vector potential becomes

$$\vec{A}_w = A_w(\hat{e}^* e^{ik_q z} + \hat{e} e^{-ik_q z})/\sqrt{2} ,$$

which causes the Hamiltonian (A13) to become

$$H' = \frac{p'^2}{2m} + \hbar g (a e^{i(k'z'+\omega't')} + a^\dagger e^{-i(k'z'+\omega't')}) . \quad \text{(A16)}$$

For most calculations this phase-induced change of the Hamiltonian does not affect the results. It does, however, affect the squeezing (see Sec. III C).

The Lorentz transformation, specified by Eqs. (A6) and (A7), depends via Eq. (A8) on the laser frequency ck. Hence it appears as if, once we have chosen the frame, we cannot vary the laser frequency any more. Although this is true in principle, varying the value of p' is essentially equivalent to varying the laser frequency. This is evident from Eq. (A11): The quantity in the parentheses is just the detuning parameter, which can be interpreted either as a detuning in energy for fixed k or a detuning in k for fixed energy.

APPENDIX B

In order to prove Eq. (20) we note that S_0 depends on p_0 only via the parameter β with

$$\frac{\partial \beta}{\partial p_0} = \frac{k}{m} . \quad \text{(B1)}$$

Since we are only interested in terms up to the order of $\hbar k/m$, when differentiating S_1 with respect to p_0, the only contribution comes from the explicit p_0 in the integrand. We then obtain

$$\frac{\partial S_0(T/2,-T/2)}{\partial p_0} = \frac{k}{m} S_0(T/2,-T/2) \left[i N_e B \frac{\partial}{\partial \beta} F(T/2,-T/2) + \frac{\partial j(T)}{\partial \beta} \sum_{i=1}^{N_e} (A_i^\dagger - A_i) \right] , \quad \text{(B2)}$$

$$\frac{\partial S_1(T/2,-T/2)}{\partial p_0} = \frac{k}{m} S_0(T/2,-T/2) \left[-\frac{\partial j(T)}{\partial \beta} \sum_{i=1}^{N_e} (A_i^\dagger - A_i) + i g N_e B \int_{-T/2}^{T/2} dt\, t e^{-i\beta t} [j(t,-T/2) - j(T/2,t)] \right] . \quad \text{(B3)}$$

The use of the explicit representations for F and j, and Eqs. (15), (17a), and (17b), in the above equations gives

$$g \int_{-T/2}^{T/2} dt\, t e^{-i\beta t} [j(t,-T/2) - j(T/2,t)] = -\frac{\partial}{\partial \beta} F(T/2,-T/2) . \quad \text{(B4)}$$

This proves that $S_0 + S_1$ is independent of p_0 at least up the order of $\hbar k/m$, as expressed in Eq. (20).

APPENDIX C

In this appendix we shall discuss the consequences of taking a more general initial state than that adopted in Eq. (23). In a density-matrix description, the final-state expectation value of an operator $\hat{O}$ would be given by

$$\langle \hat{O} \rangle_{\text{out}} = \text{Tr} \rho_{\text{out}} \hat{O} \quad \text{(C1)}$$

instead of Eq. (22) with

$$\rho_{\text{out}} = S(T/2,-T/2) \rho_{\text{in}} S(T/2,-T/2)^\dagger . \quad \text{(C2)}$$

We assume that ρ_{in} factors into an electron and a field part,

$$\rho_{\text{in}} = \rho_e \otimes \rho_f . \quad \text{(C3)}$$

We first note that describing the initial field in terms of a pure coherent state

$$\rho_f = |\alpha\rangle\langle\alpha| \quad \text{(C4)}$$

with $a|\alpha\rangle = \alpha|\alpha\rangle$ instead of a photon number eigenstate $|N\rangle$ introduces only minor changes if N is replaced by $|\alpha|^2$. The reason is that the interaction Hamiltonian (5) does not contain the operators a and $a^\dagger$ separately, but only in the electron-field combination $A_i = a \exp(ikz_i)$. The orthogonality of electron states with different momenta then still guarantees that terms with unequal numbers of a's and $a^\dagger$'s never contribute to any expectation value. The only changes then arise from terms which are quartic in a and $a^\dagger$, e.g.,

$$\langle N\,|\,(a^\dagger a)^2\,|\,N\rangle = N^2 ,$$

whereas

$$\langle \alpha\,|\,(a^\dagger a)^2\,|\,\alpha\rangle = |\,\alpha\,|^2(\,|\,\alpha\,|^2+1) = N(N+1) .$$

It turns out that among the quantities which we have computed in Sec. III only $\langle \text{out}\,|\,(a^\dagger a)^2\,|\,\text{out}\rangle$ and consequently $\Delta(N^2)$ change. For an initial field coherent state $|\,\alpha\,\rangle$ with $|\,\alpha\,|^2=N$ we then have

$$\Delta(N^2)=\Delta(N^2)_N+N-4\frac{\hbar k^2}{m}NN_e jj' , \qquad \text{(C5)}$$

where $\Delta(N^2)_N$ is given by Eq. (35). The quantity, which is easily accessible by experiment, is not $\Delta(N^2)$ but the electron-energy spread $\Delta(p_i^2)$, Eq. (60). It turns out that the additional terms in Eq. (C5) cancel when $\Delta(p_i^2)$ is recalculated for a coherent state. Hence the relevant quantities, the gain and the electron-energy spread, are the same irrespective of the laser field being initially in a photon number or, what is more realistic, in a coherent state. Obviously, this applies even more so to the physically most reasonable description by a generalized coherent state[14]

$$\rho_f=\int_0^{2\pi}\frac{d\phi}{2\pi}\,|\,|\,\alpha\,|\,e^{i\phi}\rangle\langle\,|\,\alpha\,|\,e^{i\phi}\,| \, . \qquad \text{(C6)}$$

From Eq. (38) and its analog for $N\neq 0$, which could be derived from Eqs. (32) and (35), it is then evident that field coherent states are not conserved by the interaction.

Also describing the initial state of the electrons by a density matrix

$$\rho_e=\bigotimes_{i=1}^{N_e}\rho_e^i ,$$

$$\rho_e^i=\sum f_l\,|\,\bar{p}+l\hbar k\,\rangle\langle\,\bar{p}+l\hbar k\,| \qquad \text{(C7)}$$

with $\sum f_l=1$ and $\sum l f_l=0$ does not change anything, again owing to the fact that all operators $\exp(\pm ikz_i)$ for each index i must cancel separately in view of the orthogonality of the electron states. This would no longer hold true, however, in the case of a nondiagonal ansatz for ρ_e instead of (C7).

By adopting the factorization property (C3) we have actually ruled out the possibility of an initial electron-field coherent state.[7] This can be defined in terms of the operator A introduced in Eq. (39) in analogy to the one-particle case

$$A\,|\,|\alpha\rangle=\alpha\,|\,|\alpha\rangle \qquad \text{(C8)}$$

with the solution

$$|\,|\alpha\rangle=C(\alpha)\exp(\alpha A^\dagger B^{-1})\,|\,(\bar{p})^{N_e},0\rangle , \qquad \text{(C9)}$$

where $C(\alpha)$ is a normalization constant and $\bar{p}=N_e^{-1}(P-\hbar kN)$ according to Eq. (57). It has previously been pointed out in Ref. 5 that electron-field coherent states are not sensible initial states since they already involve electron-field correlations which are supposed to evolve only afterwards owing to the interaction mediated by the wiggler. Also, reinjecting the laser output of a first transit does not lead to an electron-field coherent state as an input for the second transit, since in this process phase correlations are destroyed.

APPENDIX D

In this appendix we point out some details of the evaluation of final state expectation values. For an arbitrary operator $O=O(A_i,A_i^\dagger,p_i)$ we have from Eq. (22)

$$\begin{aligned}\langle \text{out}\,|\,O\,|\,\text{out}\rangle &= \langle \text{in}\,|\,S_0^\dagger O S_0\,|\,\text{in}\rangle \\ &\quad +\langle \text{in}\,|\,S_0^\dagger O S_1+S_1^\dagger O S_0\,|\,\text{in}\rangle \\ &\quad +\cdots , \end{aligned}\qquad \text{(D1)}$$

where $S_i=S_i(T/2,-T/2)$. We have on the rhs of Eq. (D1) omitted the contribution proportional to $S_1^\dagger O S_1$, since it is of the same order of magnitude as, e.g., $S_0^\dagger O S_2$, which is beyond the scope of our first-order approximation. The utilization of the commutation relations (18a—18c) and unitarity, i.e., $S_0^\dagger S_0=1$, gives for the first term on the rhs of Eq. (D1):

$$\begin{aligned}\langle \text{in}\,|\,S_0^\dagger O S_0\,|\,\text{in}\rangle &= \langle \text{in}\,|\,O\,|\,\text{in}\rangle \\ &\quad +\langle \text{in}\,|\,S_0^\dagger[O,S_0]\,|\,\text{in}\rangle . \end{aligned}\qquad \text{(D2)}$$

Since the commutators (18a)—(18c) are all proportional to S_0, the S_0 also disappears from the second term on the rhs of Eq. (D2). To evaluate the first-order contribution to Eq. (D1), we note that S_1 as given by Eq. (19) is proportional to S_0, viz., $S_1=S_0\Sigma_1$. Hence we can write

$$\begin{aligned}&\langle \text{in}\,|\,S_0^\dagger O S_1+S_1^\dagger O S_0\,|\,\text{in}\rangle \\ &\quad =\langle \text{in}\,|\,(O+S_0^\dagger[O,S_0])\Sigma_1\,|\,\text{in}\rangle \pm \text{c.c.}\end{aligned}\qquad \text{(D3)}$$

for Hermitian or anti-Hermitian operators, respectively. Exploiting Eqs. (24a)—(24c), which specify the action of the operators A_i, $A_i^\dagger$, and p_i on the states $|\,\text{in}\rangle$, we are finally left with matrix elements of the type

$$\alpha_n=\langle \text{in}\,|\,B^n\,|\,\text{in}\rangle \qquad \text{(D4)}$$

and

$$\beta_n = \left\langle \mathrm{in} \left| \left[\sum_{i=1}^{N_e} e^{-2kiz_i} \right] \left[\sum_{i=1}^{N_e} e^{ikz_i} \right]^2 B^n \right| \mathrm{in} \right\rangle , \tag{D5}$$

where the operator B was defined in Eq. (8). These matrix elements are calculated by recalling

$$e^{ikz_i} | p_1(1),\ldots,p_i(i),\ldots,p_{N_e}(N_e) \rangle = | p_1(1),\ldots,p_i(i)+\hbar k,\ldots,p_{N_e}(N_e) \rangle \tag{D6}$$

as well as the orthogonality of the momentum states (notice that our states are not symmetrized)

$$\langle p_1'(1),\ldots,p_{N_e}'(N_e) | p_1(1),\ldots,p_{N_e}(N_e) \rangle = \delta_{p_1',p_1},\ldots,\delta_{p_{N_e}',p_{N_e}} . \tag{D7}$$

Straightforward manipulations then yield

$$\alpha_n = \frac{(-1)^n (n!)^2 2^{2n}}{N_e^n (2n)!} \frac{\partial^{2n}}{\partial y^{2n}} J_0(y)^{N_e} \Big|_{y=0} \tag{D8}$$

and

$$\beta_n = \frac{(-1)^n n!(n+2)! 2^{2n+2}}{N_e^{n-1}(2n+2)!} \frac{\partial^{2n+2}}{\partial y^{2n+2}} \times [J_0(y)^{N_e-1} J_2(y)]_{y=0} . \tag{D9}$$

We actually only need

$$\begin{aligned} &\alpha_0 = \alpha_1 = 1 , \\ &\alpha_2 = N_e^{-1}(2N_e - 1) , \\ &\alpha_3 = N_e^{-2}(6N_e^2 - 9N_e + 4) , \\ &\beta_0 = N_e , \\ &\beta_1 = 3N_e - 2 . \end{aligned} \tag{D10}$$

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[3]An account of most of the relevant work on the subject can be found in *Novel Sources of Coherent Radiation*, Vol. 5 of Physics of Quantum Electronics, edited by S. F. Jacobs, M. Sargent III, and M. O. Scully (Addison-Wesley, Reading, Mass., 1978); *Free-Electron Generators of Coherent Radiation*, Vol. 7 of Physics of Quantum Electronics, edited by S. F. Jacobs, H. S. Pilloff, M. Sargent III, M. O. Scully, and R. Spitzer (Addison-Wesley, Reading, Mass., 1980); *Free-Electron Generators of Coherent Radiation*, Vols. 8, 9 of Physics of Quantum Electronics, edited by S. F. Jacobs, G. T. Moore, H. S. Pilloff, M. Sargent III, M. O. Scully, and R. Spitzer (Addison-Wesley, Reading, Mass., 1982).

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[8]J. M. Jauch and F. Rohrlich, *The Theory of Electrons and Photons*, 2nd ed. (Springer, New York, 1976), p. 399. Note that the right-hand side of Eq. (12b) of Ref. 5 should have the opposite sign.

[9]J. M. J. Madey, Nuovo Cimento B 50, 64 (1979); N. M. Kroll, in *Free-Electron Generators of Coherent Radiation*, Vol. 8 of Physics of Quantum Electronics, Ref. 3, p. 315; W. Becker, in *Free-Electron Generators of Coherent Radiation*, Vol. 9 of Physics of Quantum Electronics, Ref. 3, p. 985.

[10]In fact, this is only a necessary criterion for coherence.

[11]Apparently, the right-hand side of Eq. (38) can still become negative near the zeros of j. In this region, however, our first-order approximation is insufficient.

[12]Reference 11 applies here as well.

[13]C. Bazin *et al.*, in *Free-Electron Generators of Coherent Radiation*, Vol. 8 of Physics of Quantum Electronics, Ref. 3, p. 89.

[14]R. J. Glauber, in *Quantum Optics and Electronics, Lectures delivered at Les Houches, 1964*, edited by C. de Witt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, New York, 1965), p. 63.