Adiabatic expansion for the single-mode laser

Fritz Haake and Maciej Lewenstein*
Fachbereich Physik, Universitsät Essen—Gesamthochschule, D-4300 Essen 1, Germany
(Received 29 June 1982)

We carry out a systematic adiabatic elimination of the atomic degrees of freedom from the quantum-mechanical master equation for the single-mode laser. We represent the reduced density operator of the field mode by various quasiprobabilities and construct the respective equations of motion. The generators of infinitesimal time translations are obtained as series in powers of two parameters, the smallness of which defines the adiabatic limit. Our "adiabatic" expansion treats that part of the atom-field interaction as a zeroth-order effect which describes the action of the field on the atoms, while the reaction of the atoms is treated perturbatively. The adiabatic equilibrium thus assigned to the atomic variables at all times is a conditional one, contingent on the current state of the field mode. As a result, saturation effects in the atoms are fully accounted for in low orders of our expansion. In second order, especially, we obtain a Fokker-Planck equation for the Wigner function of the field mode which is valid for arbitrary pump strengths below, near, and above threshold. We compare our results with those of previous theories.

I. INTRODUCTION

Nonlinear irreversible processes are, even though ubiquitous and of considerable interest, notoriously difficult to treat. It is quite typical of these difficulties that the literature on adiabatic elimination of fast degrees of freedom for even rather simply systems (such as the single-mode laser and optically bistable devices) contains numerous results which are inconsistent with one another.[1] In some cases, such elimination procedures have even involved nonsystematic *ad hoc* assumptions.

We here describe a scheme for eliminating fast variables which yields a systematic perturbation expansion for the generator of infinitesimal time translation for the slow variables, the expansion parameter being the ratio of the respective time scales. The method has already been employed by one of us in the context of classical Brownian motion[2] and will here be extended to quantum systems, the single-mode laser serving as an example.

Starting from the well-known master equation for the single-mode laser, we derive an equation of motion for the reduced density operator of the field mode. After an initial transient has died out in a time typical of the free atomic relaxation, the field mode appears to undergo a Markov process. It is the generator l of infinitesimal time translations for that process which we calculate through a double perturbation series. The two expansion parameters, which are considered to have the same degree of smallness, are given in terms of the damping constants of the field mode (κ), the atomic polarization ($\gamma_\perp$) and inversion ($\gamma_{||}$), and the atom-field coupling constant (g) as

$$g^2/\gamma^2 \quad \text{and} \quad \kappa/\gamma \,, \tag{1.1}$$

where γ may stand for either $\gamma_\perp$ or $\gamma_{||}$.

It is convenient to represent the field density operator by a quasiprobability density since the generator l then takes the form of a differential operator with respect to a complex field amplitude β. To first order in the perturbation expansion mentioned, l contains only first-order derivatives, and thus describes a deterministic drift of the field amplitude. In second order, diffusion effects manifest themselves by the appearance of second-order derivatives; third- and higher-order derivatives arise only in third and higher orders of the perturbation series.

Even though the atom-field coupling constant g enters one of the parameters (1.1), our expansion is not (a bare) one involving powers of the whole atom-field interaction. For the expansion to be useful it is necessary to treat part of that interaction as a zeroth-order effect. In the generator L of infinitesimal time translations for the combined system of atoms and field, the atom-field interaction can be split into a contribution describing the action of the field on the atoms, and one representing the reaction of the atoms on the field. The first of these must be included in the zeroth-order part L_0 of L together with all terms describing the relaxation and the pumping of the atoms. Such a choice for L_0 implements the qualitative idea that the atoms, starting from whatever initial state is imposed, relax quickly toward a conditional equilibrium contingent on the

initial value of the field, and from then on stay in such conditional equilibrium states as the field changes its value slowly.

The "adiabatic" expansion of l just discussed contains, in any finite order, infinite partial sums of the bare expansion[2] which excludes the whole atom-field interaction from L_0. The drift and diffusion coefficients we obtain up to second order contain the full effect of atomic saturation even for strongly pumped lasers. Only for lasers operated close to threshold do our drift and diffusion coefficients reduce to the simpler ones characteristic of the well-known van der Pol oscillator[3] which can be obtained in fourth order of the bare expansion.[4]

Our results for l, i.e., for the drift and diffusion coefficients, depend on the choice we make for the quasiprobability. We consider a continuous class of quasiprobabilities $\rho_\epsilon(\beta,\beta^*,t)$, labeled by a positive parameter ϵ.[5] The definition involves the diagonal element of the field density operator $\hat{\rho}$ with respect to a coherent state[6] $|\beta\rangle$,

$$Q(\beta,\beta^*,t)=\frac{1}{\pi}\langle\beta|\rho|\beta\rangle\ , \tag{1.2}$$

and reads[7]

$$Q(\beta,\beta^*,t)=\int d^2\alpha\frac{1}{\pi\epsilon}\exp(-|\beta-\alpha|^2/\epsilon)\times\rho_\epsilon(\alpha,\alpha^*,t)\ . \tag{1.3}$$

Expectation values of normally ordered products of the annihilation and creation operators of photons can be calculated with the help of the quasiprobability ρ_ϵ as

$$\langle b^{\dagger n}b^m\rangle=\int d^2\beta\left[\beta^*+(1-\epsilon)\frac{\partial}{\partial\beta}\right]^n\left[\beta+(1-\epsilon)\frac{\partial}{\partial\beta^*}\right]^m\rho_\epsilon(\beta,\beta^*,t)\ . \tag{1.4}$$

Well-known special cases of ρ_ϵ are the so-called Q function (1.2) itself ($\epsilon=0$), Glauber's weight function in a diagonal representation of the density operator with respect to coherent states ($\epsilon=1$),[6(a)] and the Wigner function ($\epsilon=\frac{1}{2}$).[6(c)]

Our results for the diffusion matrix coincide, for the special cases $\epsilon=0$, $\frac{1}{2}$, and 1 with the ones recently given by Lugiato, Casagrande, and Pizzutto.[1] However, these authors obtain a drift coefficient lacking the second-order part. As we shall see in Sec. V, the discrepancy between our results and those of Lugiato *et al.*[1] is, for typical lasers, more interesting from a fundamental point of view than from a practical one.

The generator l_ϵ pertaining to the quasiprobability ρ_ϵ is not, to second order, a Fokker-Planck differential operator in all cases since the diffusion matrix turns out not to be positive for all values of ϵ. We shall see, however, that the diffusion matrix is positive for values of ϵ in some interval around $\epsilon=\frac{1}{2}$.

We shall also show that the adiabatic elimination of the atomic variables leads, in second order, to the same result for l_ϵ if, instead of the exact laser master equation for the full density operator for the atoms and the field, certain approximate versions of that equation are used as a starting point. Especially, the full density operator may be represented by quasiprobabilities with five independent variables referring to the field amplitudes β and β^*, the atomic polarization, and the atomic inversion.[3,8] If the respective equations of motion were taken in the diffuse approximation and if the atomic variables were then eliminated, the resulting l_ϵ would become non-systematic in third order in the parameters (1.1) only. Interestingly enough, our second-order result is reproduced even if the 5×5 diffusion matrix is nonpositive, i.e., even if the diffusion approximation does not admit well-defined solutions for the quasiprobability distribution of the five variables mentioned.

II. LASER MASTER EQUATION

The simplest possible laser model accounts for N two-level atoms interacting with a single mode of the electromagnetic field as well as for suitable heat reservoirs providing the pump and loss mechanisms. The dynamics of the μth atom can be described by means of a triple of operators, two of which, $s_{\mu\pm}$, represent the polarization while the third one $s_{\mu z}$, represents the inversion between the two energy levels. These operators obey the angular momentum commutation relations

$$\begin{aligned}[s_{\mu z},s_{\nu\pm}]&=\pm\delta_{\nu\mu}s_{\mu\pm}\ ,\\ [s_{\mu+},s_{\nu-}]&=2\delta_{\mu\nu}s_{\mu z}\ ,\end{aligned} \tag{2.1}$$

while the annihilation and creation operators of photons have the Bose commutator

$$[b,b^\dagger]=1\ . \tag{2.2}$$

The density operator $\hat{W}$ of the atoms and the field obeys a master equation of the form[3,8]

$$\dot{\hat{W}}(t)=L\hat{W}(t)=(\Lambda_A+\Lambda_F+L_{AF})\hat{W}(t)\ . \tag{2.3}$$

The interaction part L_{AF} of the generator L involves

the commutator with the interaction Hamiltonian

$$L_{AF}\hat{W}=-\frac{i}{\hbar}[H_{AF},\hat{W}]\ ,$$
$$H_{\mathrm{AF}}=i\hbar g(b^{\dagger}S_{-}-bS_{+})\ , \tag{2.4}$$

where the operators

$$S_{\pm}=\sum_{\mu=1}^{N}s_{\mu\pm}\ , \tag{2.5}$$

represent the global polarization of the atoms. The other two terms in L describe the irreversible effect of the heat reservoirs on the atoms and the field mode. They read, if the reservoir providing the field damping is taken to be at zero temperature

$$\Lambda_F W=\kappa([b,b^{\dagger}\hat{W}]+[b\hat{W},b^{\dagger}]+2[b,[\hat{W},b^{\dagger}]]) \tag{2.6}$$

and

$$\Lambda_A\hat{W}=\sum_{\mu}\Lambda_{\mu}\hat{W}\ , \tag{2.7}$$

$$\Lambda_{\mu}\hat{W}=\tfrac{1}{4}\gamma_{||}(1-2\sigma_0)([s_{\mu-},\hat{W}s_{\mu+}]+[s_{\mu-}\hat{W},s_{\mu+}])+\tfrac{1}{4}\gamma_{||}(1+2\sigma_0)([s_{\mu+},\hat{W}s_{\mu-}]+[s_{\mu+}\hat{W},s_{\mu-}])$$
$$+(\gamma_{\perp}-\tfrac{1}{2}\gamma_{||})([s_{\mu z},\hat{W}s_{\mu z}]+[s_{\mu z}\hat{W},s_{\mu z}])\ .$$

Here $2\sigma_0$ is the unsaturated inversion an individual atom would take on if exposed to the heat reservoir but not to the field. Obviously, σ_0 must lie in the region $\sigma_0<\frac{1}{2}$.

If the density operator $\hat{W}$ is represented by a c-number quasiprobability density for the field mode but is left intact as a density operator for the atoms, the generators Λ_F and L_{AF} become differential operators with respect to complex field amplitudes β and β^*. Especially, for quasiprobabilities $\hat{W}_{\epsilon}$ defined as in Eq. (1.2), we have

$$L_{AF}\hat{W}_{\epsilon}=g[\beta^*S_{-}-\beta S_{+},\hat{W}_{\epsilon}]$$
$$+g\frac{\partial}{\partial\beta}[-\epsilon S_{-}\hat{W}_{\epsilon}+(\epsilon-1)\hat{W}_{\epsilon}S_{-}]$$
$$+g\frac{\partial}{\partial\beta^*}[-\epsilon\hat{W}_{\epsilon}S_{+}+(\epsilon-1)S_{+}\hat{W}_{\epsilon}]\ , \tag{2.8}$$

and

$$\Lambda_F=\kappa\left[\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*+2(1-\epsilon)\frac{\partial^2}{\partial\beta\,\partial\beta^*}\right]\ . \tag{2.9}$$

The expressions are quite well known[6,8] for the special cases $\epsilon=0$, $\frac{1}{2}$, and 1. The generalization from $\epsilon=0$ to arbitrary positive values of ϵ is easily obtained by using the definition (1.2) or, equivalently, recently derived convolution identities.[5]

We would like to point out that the commutator term in Eq. (2.8) describes the motion of the atoms in an electric field of fixed complex amplitude β. The remaining terms in (2.8) involve derivatives with respect to β and β^* and thus represent the influence of the atoms on the field mode.

III. FORMAL ELIMINATION OF THE ATOMIC VARIABLES

We may represent the state of the atoms and the field by a quantity $W_{\epsilon}(\beta,\beta^*,t)$ which is a quasiprobability defined as in (1.2) for the field mode but a density operator with respect to the atoms. The field mode alone can then be described by the reduced quasiprobability

$$\rho_{\epsilon}(\beta,\beta^*,t)=\mathrm{tr}_A W_{\epsilon}(\beta,\beta^*,t)\ . \tag{3.1}$$

An equation of motion for $\rho_{\epsilon}(\beta,\beta^*,t)$ can be constructed so as to have the form[2,9]

$$\dot{\rho}_{\epsilon}(\beta,\beta^*,t)=l_{\epsilon}(t)\rho_{\epsilon}(\beta,\beta^*,t) \tag{3.2}$$

with a certain, in general time-dependent generator $l_{\epsilon}(t)$.

In order to construct $l_{\epsilon}(t)$, we must, in order to avoid irrelevant existence problems, require an initial condition such that $\rho_{\epsilon}(\beta,\beta^*,0)$ is nonzero throughout the complex β plane. We then define a time-evolution operator for the field mode as

$$U(t)=\mathrm{tr}_A e^{Lt}\hat{W}_{\epsilon}(\beta,\beta^*,0)\rho_{\epsilon}^{-1}(\beta,\beta^*,0)\ , \tag{3.3}$$

where L is the generator defined in Eqs. (2.3), (2.8), and (2.9). Obviously, $U(t)$ transforms the initial distribution into the one at the current time

$$\rho_{\epsilon}(\beta,\beta^*,t)=U(t)\rho_{\epsilon}(\beta,\beta^*,0)\ , \tag{3.4}$$

provided the initial state of the combined system is given by $\hat{W}(\beta,\beta^*,0)$.

The generator $l_{\epsilon}(t)$ can now be expressed as the product of the time derivative of $U(t)$ with the inverse of $U(t)$ as

$$l_{\epsilon}(t)=\dot{U}(t)U^{-1}(t)\ . \tag{3.5}$$

We shall see that $l_{\epsilon}(t)$ approaches a time-

independent limit after a time of the order of the smaller one of the atomic relaxation times $\gamma_\perp^{-1}$ and $\gamma_{||}^{-1}$. On the much longer time scale characteristic of the motion of the field varibles, $l_\epsilon(t)$ can be replaced by its limit

$$l_\epsilon = \lim_{t\to\infty} \dot{U}(t)U^{-1}(t) . \tag{3.6}$$

IV. PERTURBATION EXPANSION FOR THE GENERATOR l_ϵ

In order to generate an expansion of $l_\epsilon(t)$ in powers of the parameters (1.1) we split the operator L into a zeroth-order part L_0, with

$$L_0\hat{f} = \Lambda_A + g[\beta^* S_- - \beta S_+, \hat{f}] \tag{4.1}$$

and a remainder. Obviously, L_0 describes a relaxation process of the atomic system in the presence of the pump and loss mechanisms as well as of an external electric field with the complex amplitude β. The relaxation takes place with the damping constants $\gamma_\perp$ and $\gamma_{||}$ and tends to lead the atoms into a conditional equilibrium characterized parametrically by the field strength β. That equilibrium is described by a density operator $\hat{A}$, the solution of $L_0\hat{A}=0$ with $\mathrm{tr}_A\hat{A}=1$, which factorizes with respect to all atoms and reads

$$\hat{A} = \prod_{\mu=1}^{N} \hat{A}_\mu ,$$

$$\hat{A}_\mu = \tfrac{1}{2} + \sigma s_{\mu+} + \sigma^* s_{\mu-} + \sigma_z s_{\mu z} \tag{4.2}$$

with

$$\sigma_z = \sigma_0 \left[1 + \frac{4g^2}{\gamma_\perp\gamma_{||}}\beta^*\beta\right]^{-1} ,$$

$$\sigma = (2g/\gamma_\perp)\beta\sigma_z \tag{4.3}$$

$$= (2g\sigma_0/\gamma_\perp)\beta\left[1 + \frac{4g^2}{\gamma_\perp\gamma_{||}}\beta^*\beta\right]^{-1} .$$

The parameters σ and σ_z are, of course, the expectation values of, respectively, the operators $s_{\mu-}$ and $s_{\mu z}$ in the conditional-equilibrium state $\hat{A}$.

The saturation effects manifested in Eqs. (4.3) are important for a laser operated well above its threshold since in that regime the most probable values of the field intensity $\beta^*\beta$ are comparable with or even exceed the so-called saturation photon number

$$n_s = \gamma_\perp\gamma_{||}/4g^2 . \tag{4.4}$$

It is for this reason that we must include in L_0 the parts of L_{AF} proportional to the field amplitude.[2] It is interesting to realize that the saturation effects inherent in Eqs. (4.3) have contributions from all orders in the coupling constant g. The zeroth-order result [(4.2) and (4.3)] contains, in other words, infinite partial sums of a "bare" perturbation expansion which excludes the whole atom-field interaction L_{AF} from L_0.

We shall now expand the right-hand side in Eq. (3.6) in powers of $L-L_0$. To that end we first note the zeroth-order version of the time-evolution operator (3.3) to be the unit operator

$$U^{(0)}(t) = \mathrm{tr}_A e^{L_0 t}\hat{W}_\epsilon(\beta,\beta^*,0)\rho_\epsilon^{-1}(\beta,\beta^*,0)$$

$$= 1 . \tag{4.5}$$

There is thus no zeroth-order contribution to l_ϵ. The first-order part is given by

$$l_\epsilon^{(1)} = \lim_{t\to\infty}\dot{U}_1(t) = \lim_{t\to\infty}\mathrm{tr}_A(L-L_0)e^{L_0 t}\hat{W}_\epsilon(\beta,\beta^*,0)\rho_\epsilon^{-1}(\beta,\beta^*,0) . \tag{4.6}$$

Since L_0 has the unique stationary state (4.2) we immediately find

$$l_\epsilon^{(1)} = -g\left[\frac{\partial}{\partial\beta}\mathrm{tr}_A(S_-\hat{A}) + \frac{\partial}{\partial\beta^*}\mathrm{tr}_A S_+\hat{A}\right] + \Lambda_F$$

$$= \left[\frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\beta^*}\beta^*\right]\left[\kappa - \frac{2N\sigma_0 g^2/\gamma_\perp}{1+(4g^2/\gamma_\perp\gamma_{||})\beta^*\beta}\right] + \frac{\partial^2}{\partial\beta\,\partial\beta^*}2\kappa(1-\epsilon) , \tag{4.7}$$

which obviously is a Fokker-Planck differential operator with respect to all quasiprobabilities with $\epsilon < 1$.

We now proceed to the second-order term

$$l_\epsilon^{(2)} = \lim_{t\to\infty}[\dot{U}^{(2)}(t) - \dot{U}^{(1)}(t)U^{(1)}(t)] . \tag{4.8}$$

It is possible to show that $l_\epsilon^{(2)}$, as well as $l_\epsilon^{(1)}$ and all higher-order contributions, is independent of the initial state $\hat{W}(\beta,\beta^*,0)$. In the interest of brevity we forgo the proof of that independence and choose special initial states which minimize the labor involved in evaluating $l_\epsilon^{(2)}$. These turn out to be

$$\hat{W}_\epsilon(\beta,\beta^*,0)=\hat{A}\rho_\epsilon(\beta,\beta^*,0)\ , \tag{4.9}$$

i.e., states of zeroth-order adiabatic equilibrium of the atoms and arbitrary field distributions. We then immediately have the expression

$$l_\epsilon^{(2)}=\lim_{t\to\infty}\int_0^t dt'\,\mathrm{tr}_A(L-L_0)(e^{L_0t'}-\hat{A}\,\mathrm{tr}_A)\times(L-L_0)\hat{A}\ , \tag{4.10}$$

which is manifestly finite since $\hat{A}\,\mathrm{tr}_A$ is just the time-independent part of the spectral decomposition of $\exp(L_0t)$.

By inserting $L-L_0$ from Eqs. (2.3), (2.8), and (2.9) we find

$$l_\epsilon^{(2)}=-g\lim_{t\to\infty}\frac{\partial}{\partial\beta}\,\mathrm{tr}_A S_-(e^{L_0t}-\hat{A}\,\mathrm{tr}_A)(L-L_0)\hat{A}+\text{c.c.}\ , \tag{4.11}$$

and are thus led to the problem of constructing the time dependence of the quantity

$$f_-(t)=\mathrm{tr}_A S_- e^{L_0t}X\ , \tag{4.12}$$

in which X is an arbitrary operator. We differentiate $f_-(t)$ with respect to t, and find this quantity coupled to similarly defined ones $f_+(t)$ and $f_z(t)$, in which S_- is replaced by S_+ and S_z. The corresponding three equations of motion are easily solved with the result

$$\int_0^\infty dt\,\mathrm{tr}_A S_-(e^{L_0t}-\hat{A}\,\mathrm{tr}_A)X=\frac{1}{\gamma_\perp}\left[1+\frac{4g^2}{\gamma_\perp\gamma_{||}}\beta^*\beta\right]^{-1}\left[\left[1+\frac{2g^2}{\gamma_\perp\gamma_{||}}\right]\mathrm{tr}_A S_-X-\frac{2g^2}{\gamma_\perp\gamma_{||}}\beta^2\mathrm{tr}_A S_+X+\frac{2g}{\gamma_{||}}\beta\,\mathrm{tr}_A S_zX-N\left[1+\frac{2\gamma_\perp}{\gamma_{||}}\right]\sigma\,\mathrm{tr}_A X\right]\ . \tag{4.13}$$

The remaining steps in the construction of $l_\epsilon^{(2)}$ require the evaluation of bilinear expectation values like $\mathrm{tr}_A S_+S_-\hat{A}$, and present no difficulty.

In writing our result for $l_\epsilon=l_\epsilon^{(1)}+l_\epsilon^{(2)}$ we save space and gain physical transparency by using the rescaled field

$$u=\beta/\sqrt{n_s}=\beta 2g/\sqrt{\gamma_\perp\gamma_{||}}\ . \tag{4.14}$$

This is a physically appropriate scaling since it gives to all terms in L_0 the same "order," i.e., coefficient proportional to either $\gamma_\perp$ or $\gamma_{||}$, or $\sqrt{\gamma_\perp\gamma_{||}}$. Each term in the perturbation expansion of l_ϵ in powers of $L-L_0$ then displays its order as the number of factors κ and g^2/γ it contains. Note that, in this sense, the second-order derivative term in Eq. (4.7) now appears as of second order. We may therefore say that to first order in the expansion parameters (1.1) the generator l_ϵ is purely deterministic in character. To second order, l_ϵ takes the form

$$l_\epsilon=\frac{\partial}{\partial u}D_u+\frac{\partial}{\partial u^*}D_u^*+\frac{\partial^2}{\partial u^2}D_{uu}+\frac{\partial^2}{\partial u^{*2}}D_{uu}^*+\frac{\partial^2}{\partial u\,\partial u^*}D_{uu^*}\ , \tag{4.15}$$

with the drift coefficient

$$D_u=u\left\{\kappa-\frac{2\sigma_0Ng^2/\gamma_\perp}{1+u^*u}+\frac{2\sigma_0Ng^2\kappa/\gamma_\perp^2}{(1+u^*u)^3}\left[\left[1+\frac{2\gamma_\perp}{\gamma_{||}}\right]u^*u-1\right]+\frac{4Ng^4\sigma_0^2/\gamma_\perp^3}{(1+uu^*)^4}[N(1-u^*u)-2u^*u]+\frac{Ng^4/\gamma_\perp^2\gamma_{||}}{(1+u^*u)^4}\times[-8\sigma_0^2Nu^*u+4\sigma_0^2(1-u^*u)-(1-2\epsilon)(8\sigma_0)(1+u^*u)+(3+u^*u)(1+u^*u)^2]\right\}\ , \tag{4.16}$$

and the diffusion coefficients

$$D_{uu^*}=\frac{8g^2\kappa}{\gamma_\perp\gamma_{||}}(1-\epsilon)+\frac{2Ng^4/\gamma_\perp^2\gamma_{||}}{(1+u^*u)^3}\left[-4\sigma_0^2\left[1+\frac{\gamma_\perp}{\gamma_{||}}\right]u^*u+(1+u^*u)[(1+u^*u)(2+u^*u)-(1-2\epsilon)(4\sigma_0)]\right]\ ,$$

$$D_{uu}=-\frac{Ng^4}{\gamma_\perp^2\gamma_{||}}\frac{u^2}{(1+u^*u)^3}\left[4\sigma_0^2\left[1+\frac{\gamma_\perp}{\gamma_{||}}\right]+(1+u^*u)[1-(1-2\epsilon)(4\sigma_0)+u^*u]\right]\ . \tag{4.17}$$

It is, of course, not due to fortuitous cancellations that no derivatives of order higher than second appear in Eq. (4.15). By inspecting the structure of L as given by Eqs. (2.3), (2.8), and (2.9) and the scaling (4.14) it becomes obvious that, in nth order with respect to the expansion parameters κ/γ and g^2/γ^2, derivatives up to the nth order can appear in l_ϵ.

The result (4.17) for the diffusion matrix has also been obtained by Lugiato, Casagrande, and Pizzuto[1] (LCP) for the special cases $\epsilon=0$, $\frac{1}{2}$, and 1. The drift coefficient given in Ref. 1 lacks, however, the second-order contribution. We have already argued in a previous paper[5] that neglecting the second-order drift while keeping the second-order diffusion destroys the consistency of the expansion with the quantum-mechanical commutation rule (2.2).

The generator l_ϵ we find is a Fokker-Planck differential operator only if the diffusion matrix is positive. By using polar coordinates in the complex β plane we see that this is the case if the following third-order polynomial in $z=1/(1+u^*u)$:

$$f(z)=\alpha z^3+[4\sigma_0(z\epsilon-1)-\alpha]z^2 +(1-4\sigma_0\epsilon+2\sigma_0)z+\frac{2\kappa\gamma_\perp}{Ng^2}(1-\epsilon) \tag{4.18}$$

[with $\alpha=4\sigma_0^2(1+\gamma_\perp/\gamma_\parallel)$], is positive for $0<z<1$. The positivity of $f(z)$ is guaranteed if either

$$\alpha<2 \text{ and } 0<\epsilon<1\ , \tag{4.19a}$$

or

$$\alpha>2 \text{ and } |\sigma_0(2\epsilon-1)|<\tfrac{1}{4}[\alpha(4-\alpha)]^{1/2} \tag{4.19b}$$

and

$$0<\epsilon<1\ ,$$

or

$$\alpha>2 \text{ and } |\sigma_0(2\epsilon-1)|>\tfrac{1}{4}[\alpha(4-\alpha)]^{1/2} \tag{4.19c}$$

but with $f(z)$ positive at the points where $f'(z)=0$ and with $0<\epsilon<1$. The Wigner function ($\epsilon=\frac{1}{2}$) as well as the quasiprobabilities ρ_ϵ with ϵ sufficiently close to $\frac{1}{2}$ are thus found to obey Fokker-Planck equations.

It is interesting to realize that our second-order results for l_ϵ could also be obtained by rewriting the master equation (2.3) as a c-number equation of motion for a quasiprobability $W_\epsilon(s,s^*,s_z,\beta,\beta^*,t)$ which associates c-number variables, s,s^*,s_z, with the operators S_-,S_+,S_z, respectively,[3,7] and by neglecting, in that equation, all derivatives of higher than second order.[10] This diffusion approximation would lead to erroneous results for l_ϵ in third and higher orders of the perturbation expansion only. The reason for the partial usefulness of the diffusion approximation for the five-variable quasiprobability is easy to understand; if we choose, e.g., a quasiprobability W_ϵ which has as its atomic moments the expectation values of the fully symmetrized products of the corresponding atomic operators, the diffusion approximaton for L reads[11]

$$L=\left[\frac{\partial}{\partial\beta}(\kappa\beta-gs)+\frac{\partial}{\partial s}(\gamma_\perp s-2g\beta s_z)+\text{c.c.}\right]+\frac{\partial}{\partial s_z}[\gamma_\parallel(s_z-N\sigma_0)+g(\beta s^*+\beta^* s)] +\frac{\partial^2}{\partial\beta\,\partial\beta^*}\kappa+\frac{\partial^2}{\partial s\,\partial s^*}N\gamma_\perp+\frac{\partial^2}{\partial {s_z}^2}\gamma_\parallel\left[\frac{N}{4}-\sigma_0 s_z\right]-\left[\frac{\partial^2}{\partial s\,\partial s_z}s+\frac{\partial^2}{\partial s^*\partial s_z}s^*\right]\gamma_\parallel\sigma_0\ . \tag{4.20}$$

Here the variables s, s^*, and s_z are associated with the operators S_-, S_+, and S_z, respectively. The construction of l_ϵ from the c-number L (4.20) proceeds in complete analogy with the one given above, all quantum operators being replaced by their c-number counterparts and the trace operation meaning an integral over the whole complex phase with respect to the "polarization" s and over the whole real axis with respect to the "inversion" s_z. We must now only note that the drift and diffusion coefficients in the c-number L contain no nonlinear terms with respect to the atomic variables. The first and second moments of atomic variables entering (4.7) and (4.11) are thus not influenced at all by the third- and higher-order derivatives not included in (4.20).

V. STATIONARY SOLUTION

We shall here discuss the stationary Wigner function, i.e., the stationary eigenfunctions of the Fokker-Planck differential operator $l_{1/2}$. That distribution function turns out to be independent of the phase of the field amplitude u and can thus be represented as a function of the squared modulus of u,

$$\rho_{1/2}(u,u^*,t\to\infty)=W(z),\quad z=u^*u\ . \tag{5.1}$$

It is convenient to express $W(z)$ in terms of three auxiliary functions $f(z)$, $g(z)$, and $h(z)$. The first of these is a combination of the diffusion coefficients (4.17) and reads

$$f(z)=\frac{2}{n_s\sigma_c}\left[\tfrac{1}{2}\sigma_c+1-z(1+z)^{-1}-4\sigma_0^2\left[1+\frac{\gamma_{\parallel}}{\gamma_{\perp}}\right]z(1+z)^{-3}\right], \tag{5.2}$$

where n_s is the saturation photon number defined in Eq. (4.4) while σ_c denotes the parameter

$$\sigma_c=\kappa\gamma_{\perp}/2Ng^2 . \tag{5.3}$$

According to Eq. (4.7) σ_c is a measure of the "threshold pump strength," i.e., the value of σ_0 required to ensure self-sustained oscillations in the laser; note that the drift coefficient in Eq. (4.7) describes, if saturation is neglected, an amplification process for $\sigma_0>\sigma_c$. The other two auxiliary functions

$$g(z)=2\left[1-\frac{\sigma_0}{\sigma_c}(1+z)^{-1}\right] \tag{5.4}$$

and

$$h(z)=g(z)\frac{\kappa}{\gamma_{\parallel}}\frac{\sigma_0}{\sigma_c}2\left[\left[1+\frac{\gamma_{\parallel}}{2\gamma_{\perp}}\right]z-\frac{\gamma_{\parallel}}{2\gamma_{\perp}}\right](1+z)^{-3}+\frac{1}{8n_s\sigma_c}\left[(3+z)(1+z)^2+4\sigma_0^2(1-z)-8\frac{\gamma_{\parallel}}{\gamma_{\perp}}\sigma_0^2 z\right](1+z)^{-4}$$

represent the first- and second-order drift coefficients, respectively. The stationary Wigner function now reads

$$W(z)=\mathcal{N}\frac{1}{f(z)}\exp\left[-\int_0^z dz'\frac{g(z')+h(z')}{f(z')}\right], \tag{5.5}$$

where $\mathcal{N}$ is a normalization constant defined by $\int_0^{\infty} dz\, W(z)=1$.

In a first application of our result (5.5) we would like to represent the deviation of the threshold value $\sigma_{\rm thr}$ of σ_0 from the parameter σ_c. We define $\sigma_{\rm thr}$ by requiring that the random variable z cease to have its most probable value at zero as σ_0 increases through the threshold, i.e., by the condition[12]

$$\frac{d}{dz}\ln W(z)=0 . \tag{5.6}$$

A series expansion of $\sigma_{\rm thr}$ in powers of our expansion coefficients (1.1) is easily deduced from this condition and reads, to within corrections of second order,

$$\sigma_{\rm thr}=\sigma_c\left\{1+\frac{1}{8n_s\sigma_c}\left[1-4\sigma_c^2\left[1+\frac{2\gamma_{\parallel}}{\gamma_{\perp}}\right]\right]\right\}. \tag{5.7}$$

For typical lasers the first-order term in the expansion (5.7) amounts to a few percent of the zeroth-order contribution σ_c. It is interesting to observe that the nonsystematic result of Ref. 1, which corresponds to neglecting the function $h(z)$ in the stationary distribution (5.5), yields a rather different threshold shift,

TABLE I. Mean intensity (5.9) vs σ_0/σ_c. First column gives the result calculated from Eq. (5.5); second column is obtained with $h(z)=0$; third column contains numbers given in Ref. 1, obtained with a Gaussian fit to Eq. (5.5) with $h(z)=0$.

σ_0/σ_c	(i)	(ii)	(iii)
0.8	2.09×10^{-2}	2.14×10^{-2}	2.10×10^{-2}
0.9	3.18×10^{-2}	3.29×10^{-2}	3.21×10^{-2}
1.00	5.58×10^{-2}	5.84×10^{-2}	5.67×10^{-2}
1.02	6.35×10^{-2}	6.66×10^{-2}	6.46×10^{-2}
1.04	7.27×10^{-2}	7.63×10^{-2}	7.39×10^{-2}
1.06	8.33×10^{-2}	8.75×10^{-2}	8.48×10^{-2}
1.08	9.56×10^{-2}	10.03×10^{-2}	9.73×10^{-2}
1.1	0.110	0.115	0.111
1.2	0.198	0.205	0.202
1.3	0.298	0.304	0.300
1.4	0.398	0.404	0.400

TABLE II. Second-order correlation (5.10) vs σ_0/σ_c. Columns differ from one another as in Table I.

σ_0/σ_c	(i)	(ii)	(iii)
0.8	1.878	1.872	1.876
0.9	1.774	1.763	1.770
1.00	1.578	1.559	1.572
1.02	1.527	1.507	1.520
1.04	1.473	1.453	1.466
1.06	1.418	1.397	1.411
1.08	1.362	1.344	1.356
1.1	1.311	1.293	1.304
1.2	1.126	1.118	1.123
1.3	1.057	1.055	1.056
1.4	1.032	1.031	1.032

TABLE III. Mean intensity and second-order intensity correlation according to the adiabatic expansion [Eqs. (5.9) and (5.4)] and to the fourth-order expansion [Eqs. (5.9) and (5.11)].

	$\langle b^\dagger b\rangle/n_s$		$\langle b^\dagger b^\dagger bb\rangle/\langle b^\dagger b\rangle^2$	
σ_0/σ_c	adiabatic	bare	adiabatic	bare
0.8	2.086×10^{-2}	2.165×10^{-2}	1.878	1.891
0.9	3.177×10^{-2}	3.300×10^{-2}	1.774	1.778
1.0	5.577×10^{-2}	5.665×10^{-2}	1.578	1.572
1.02	6.354×10^{-2}	6.384×10^{-2}	1.527	1.521
1.04	7.267×10^{-2}	7.206×10^{-2}	1.473	1.468
1.06	8.334×10^{-2}	8.133×10^{-2}	1.418	1.415
1.08	9.564×10^{-2}	9.165×10^{-2}	1.363	1.364
1.1	0.1096	0.1029	1.311	1.316
1.2	0.1981	0.1676	1.126	1.144
1.3	0.2975	0.2307	1.057	1.073
1.4	0.3975	0.2857	1.032	1.044
1.5	0.4975	0.3333	1.021	1.030
2.0	0.9975	0.4995	1.005	1.010
2.5	1.4975	0.5999	1.002	1.006

$$[(\sigma_{\rm thr}-\sigma_c)/\sigma_c]_{\rm LCP} = -\frac{1}{2n_s\sigma_c}[1+4\sigma_c^2(1+\gamma_{||}/\gamma_\perp)] . \quad (5.8)$$

Similarly significant differences exist between the mean values calculated from our distribution (5.5) and from the one obtained by setting $h(z)=0$. We illustrate these differences for the mean intensity

$$\langle b^\dagger b\rangle = n_s\int_0^\infty dz\left[z-\frac{1}{2n_s}\right]W(z) \quad (5.9)$$

in Table I and for the second-order correlation

$$\langle b^\dagger b^\dagger bb\rangle/\langle b^\dagger b\rangle^2 = \langle b^\dagger b\rangle^{-2}\int_0^\infty dz(z^2-z/n_s)W(z) \quad (5.10)$$

in Table II. The calculations are based on the parameter values $\sigma_{\rm thr}=0.5\times10^{-2}$, $n_s=10^4$,

$$\gamma_{||}/2\gamma_\perp=1 .$$

As is obvious from Table I, our intensities are smaller than the ones obtained with the approach of Ref. 1, i.e., with $h(z)=0$. This discrepancy can be understood with the help of our above discussion of the threshold shift. While our shift given in Eq. (5.7) is positive, the one obtained with $h(z)=0$, i.e., in Eq. (5.8), is negative. The same argument explains why we obtain intensity fluctuations which are slightly larger than the ones corresponding to $h(z)=0$ (see Table II). The relative differences mentioned amount to a few percent near threshold and decrease as the pump strength σ_0 grows. In view of the small differences between our results and those of Ref. 1 one may conclude that the latter, even though nonsystematic, are useful in practice.

Finally, in Table III we demonstrate that saturation effects become increasingly important as σ_0 grows. Here we compare our results for the intensity $\langle b^\dagger b\rangle$ and the second-order correlation (5.10) with the ones obtained by Risken's approximation.[13] To obtain Risken's stationary distributions, the following replacements are necessary:

$$f(z)=f(0) , \quad g(z)=2[1-(\sigma_0/\sigma_c)(1-z)] , \quad (5.11)$$

and

$$h(z)=0 .$$

The considerable saturation effects displayed in Table III are, of course, the common motivations of all efforts to perform the adiabatic elimination of the atomic variables in a systematic way.

ACKNOWLEDGMENTS

We would like to thank Luigi Lugiato, Robert Graham, and Hajo Leschke for stimulating discussions. The contributions of Martin Wilkens to preliminary discussions and work preceding the investigation presented here have been extremely valuable. Finally, financial support to M. L. by the Deutscher Akademischer Austauschdienst (DAAD) is gratefully acknowledged.

*Permanent address: Institute for Theoretical Physics, Polish Academy of Sciences, 02-668 Warsaw, Poland.

[1]L. A. Lugiato, F. Casagrande, and L. Pizzuto, Phys. Rev. A 26, 3438 (1982). See also references cited herein.

[2]F. Haake, Z. Phys. B 48, 31 (1982).

[3]H. Haken, *Handbuch der Physik* (Springer, Berlin, 1970), Vol. XXV/2c.

[4]T. Arimitsu and T. Tominaga (unpublished).

[5]F. Haake and M. Lewenstein, Z. Phys. B 48, 37 (1982).

[6](a) R. J. Glauber, Phys. Rev. 130, 2529 (1963); 131, 2766 (1963); (b) E. C. G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963); (c) E. P. Wigner, Phys. Rev. 140, 749 (1932).

[7](a) R. Graham, F. Haake, H. Haken, and W. Weidlich, Z. Phys. 213, 21 (1968); (b) K. E. Cahill and R. J. Glauber, Phys. Rev. 177, 1857 (1969); 177, 1882 (1969).

[8]H. Haken, H. Risken, and W. Weidlich, Z. Phys. 206, 355 (1967).

[9]H. Grabert, P. Talkner, and P. Hänggi, Z. Phys. B 26, 389 (1977); H. Grabert, P. Talkner, P. Hänggi, and H. Thomas, *ibid.* 29, 273 (1978).

[10]See, e.g., N. G. Van Kampen, Adv. Chem. Phys. 15, 65 (1969).

[11]M. Gronchi and L. A. Lugiato, Lett. Nuovo Cimento 23, 593 (1978).

[12]This condition contains, of course, an element of arbitrariness since it would yield different values of σ_{thr} if some other quasiprobability ρ_ϵ were used instead of the Wigner function $W(z)$. Since none of the ρ_ϵ is directly measurable it is difficult if not impossible to give an operative definition of the threshold.

[13]H. Risken, Z. Phys. 186, 85 (1965); Fortschr. Phys. 16, 261 (1968).