Higher-order phase transitions in systems far from equilibrium: Multicritical points in two-mode lasers

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The possibility of a system driven far from equilibrium to exhibit multicritical phenomena is discussed. As an example, the laser oscillating in two modes is shown to possess bicritical and tetracritical points depending on the strength of the coupling between the two modes. The phase diagrams indicating the regions and the nature of the four possible phases for the two-mode laser model are presented. The exponents and the scaling characteristics near the bicritical and tetracritical points are also discussed.

I. INTRODUCTION

Recently it has been recognized that a large class of systems in a variety of disciplines such as physics, chemistry, biology, etc., exhibit transitions from certain disordered to ordered states which are strikingly similar to the usual phase transitions in equilibrium physical systems. The remarkable analogy between the phase-transition behavior of these "nonequilibrium" systems and systems in thermodynamic equilibrium has led to considerable activity in a new field of research named synergetics.[1,2] In synergetics, certain suitably defined control or pump parameters play the role of temperature or pressure in a thermodynamic system. As the control parameters are altered from one set of values to another, the system may undergo certain phase transitions and it may be possible to study the critical properties of the system near such transition points.

In addition to myriads of examples in other disciplines,[2,3] in physics itself, phase transitions in several far-from-equilibrium systems have been investigated in detail. Examples are single-mode lasers,[2,4–6] various types of bistable systems (e.g., optical bistability,[7] bistable behavior of Josephson junctions[8]), optical absorption by high-density excitons,[9] nonequilibrium superconductivity,[10] etc. In each case the "macroscopic" state of the system may be characterized in terms of certain appropriately defined order parameters which appear in a generalized potential or a "free-energy" expansion. The latter has a structure similar to that in the usual Landau theory of phase transitions, and one has generally seen either a phase transition of first order or second order.

Recently however in an interesting paper[11] on a single mode laser with saturable absorbers, Walgraef *et al.*, using the model of Lugiato and coworkers,[6] have pointed out that the nature of phase transition may undergo a change from second to first order at a point in the space of control parameters. Such a confluence point at which a line of second-order transition meets one of first order has been termed a tricritical point, in analogy with similar phenomena seen in equillibrium systems of ternary liquid mixtures, mixtures of normal ^{3}He and superfluid ^{4}He, the so-called metamagnets,[12,13] etc. The example treated by Walgraef *et al.* is still characterized by a single-order parameter albeit the corresponding free energy has a higher-order coupling,[5,6] which is absent in the ordinary single-mode laser and which is essential for the occurrence of a tricritical point.

On the other hand, one believes on very general grounds that a system described by two or more coupled-order parameters should exhibit complex critical phenomena of higher order.[14] We demonstrate that such multicritical points can indeed occur in a laser oscillating in two modes,[15,16] such as the Zeeman or the ring laser. We thus provide for the first time an example of a system driven far from equilibrium which exhibits bicritical and tetracritical behavior. The latter, as remarked earlier, is a consequence of the coupling between two order parameters, which may be identified with the amplitudes of the two-mode laser in the present case. The phenomenon is similar to the multicritical behavior observed in the equilibrium phase transitions of certain anisotropic antiferromagnets in the presence of external magnetic fields.[17] The latter have been discussed quite extensively in recent years and may serve to elucidate the important features of multicritical points in nonequilibrium systems too. These features include the exponents and scaling behavior of the free energy which will be discussed at length for the case of a two-mode laser.

II. MULTICRITICAL BEHAVIOR OF THE TWO-MODE LASER

The semiclassical and quantum theories of a laser oscillating in two modes[15,16,18] have been worked out in detail. In order to analyze the phase-transition characteristics of the two-mode laser, we shall recall here some of the basic results.

A. Basic equations for the two-mode operation

The complex amplitudes ϵ_1 and ϵ_2 associated with the two modes of oscillation are known (cf. Ref. 16) to satisfy the Langevin equations

$$\dot{\epsilon}_1=(a_1-|\epsilon_1|^2-\xi|\epsilon_2|^2)\epsilon_1+f_1(t)\,, \qquad (2.1)$$

$$\dot{\epsilon}_2=(a_2-|\epsilon_2|^2-\xi|\epsilon_1|^2)\epsilon_2+f_2(t)\,, \qquad (2.2)$$

where a_1 and a_2 are the dimensionless pump parameters that could be either positive or negative and ξ is a (positive) coupling parameter between the two modes. The structure of a_1, a_2, and ξ depends on the details of the laser which are different, for instance, in ring and Zeeman lasers.[14] The complex quantities $f_i(t)$ in (2.1) and (2.2) are Gaussian, delta-correlated random-noise terms with zero mean, i.e.,

$$\langle f_i^*(t)f_j(t')\rangle=2\delta_{ij}\delta(t-t')\,. \qquad (2.3)$$

The probability distribution of ϵ_1 and ϵ_2 can be calculated either from (2.1) and (2.2) or from the corresponding Fokker-Planck equation. The steady-state distribution turns out to be

$$P_{\rm st}(\epsilon_1,\epsilon_2)\propto\exp(-\Phi)\,, \qquad (2.4)$$

where the generalized potential or the free-energy function Φ has the structure

$$\Phi=-\tfrac{1}{2}a_1|\epsilon_1|^2-\tfrac{1}{2}a_2|\epsilon_2|^2+\tfrac{1}{4}(|\epsilon_1|^4+|\epsilon_2|^4)+\tfrac{1}{2}\xi|\epsilon_1|^2|\epsilon_2|^2\,. \qquad (2.5)$$

Equation (2.5) constitutes a Landau-type expansion of the free energy in terms of two order parameters ϵ_1 and ϵ_2. The phases of the complex quantities ϵ_1 and ϵ_2 do not appear[19] in Φ unless a field is injected externally. From a knowledge of Φ and hence of $P_{\rm st}$, all the steady-state characteristics, e.g., moments like $\langle|\epsilon_2|^{2n}|\epsilon_2|^{2m}\rangle$, which depend on the values of a_1, a_2, and ξ, can be evaluated.

B. Stable solutions of the order-parameter equations

In order to establish the multicritical behavior of the two-mode laser, we examine the extrema of the potential (2.5). The condition

$$\frac{\partial\Phi}{\partial\epsilon_1}=\frac{\partial\Phi}{\partial\epsilon_2}=0 \qquad (2.6)$$

yields the following values of $|\epsilon_1|$ and $|\epsilon_2|$:

(a) if $|\epsilon_1|=0$, then

$$|\epsilon_2|^2=\begin{cases}0 & \text{for } a_2<0,\\ a_2 & \text{for } a_2>0;\end{cases} \qquad (2.7)$$

(b) if $|\epsilon_2|=0$, then

$$|\epsilon_1|^2=\begin{cases}0 & \text{for } a_1<0,\\ a_1 & \text{for } a_1>0;\end{cases} \qquad (2.8)$$

(c) if $|\epsilon|_1\neq 0$, $|\epsilon_2|\neq 0$, then

$$|\epsilon_1|^2=(\xi a_2-a_1)/(\xi^2-1)\,,$$

$$|\epsilon_2|^2=(\xi a_1-a_2)/(\xi^2-1)\,. \qquad (2.9)$$

As $|\epsilon_1|^2$ and $|\epsilon_2|^2$ are positive, Eq. (2.9) leads to the following conditions on the parameters a_1, a_2, and ξ:

$$\text{(i)}\ \xi<1\,,\quad a_2>\xi a_1\,,\quad a_1>\xi a_2\,, \qquad (2.10)$$

and

$$\text{(ii)}\ \xi>1\,,\quad \xi a_1>a_2\,,\quad \xi a_2>a_1\,. \qquad (2.11)$$

The steady-state solution obtained above are "locally" stable, i.e., they represent local minima of the potential if the Hessian matrix

$$H=\begin{vmatrix}\dfrac{\partial^2\Phi}{\partial\epsilon_1\partial\epsilon_1^*} & \dfrac{\partial^2\Phi}{\partial\epsilon_1\partial\epsilon_2^*}\\[2ex] \dfrac{\partial^2\Phi}{\partial\epsilon_2\partial\epsilon_1^*} & \dfrac{\partial^2\Phi}{\partial\epsilon_2\partial\epsilon_2^*}\end{vmatrix} \qquad (2.12)$$

is positive definite. The condition of "global" stability, which must be satisfied in order to have stability rather than metastability, requires, of course, that the solutions correspond to absolute minima of the potential. The form of the potential (2.5) and the requirement of local stability lead to the conditions

$$(1+\tfrac{1}{2}\xi)(|\epsilon_1|^2+|\epsilon_2|^2)>\tfrac{1}{2}(a_1+a_2)\ , \tag{2.13}$$

$$(-\tfrac{1}{2}a_1+|\epsilon_1|^2)(-\tfrac{1}{2}a_2+|\epsilon_2|^2)+\tfrac{1}{2}\xi(|\epsilon_1|^4+|\epsilon_2|^4-\tfrac{1}{2}a_1|\epsilon_1|^2-\tfrac{1}{2}a_2|\epsilon_2|^2)>0\ . \tag{2.14}$$

We examine now cases (a), (b), and (c) [see Eqs. (2.7)–(2.9)] separately. It may be noted that the stability analysis presented here in terms of Hessian is new. Sargent *et al.*[15] have discussed stability using macroscopic equations.

i. Solution: $|\epsilon_1|=|\epsilon_2|=0$ *(Phase I).* Equations (2.13) and (2.14) lead to

$$a_1+a_2<0\ ,\quad a_1a_2>0\ , \tag{2.15}$$

which implies that

$$a_1<0\ ,\quad a_2<0\ . \tag{2.16}$$

We can easily check that under the condition (2.16), the solution $|\epsilon_1|=|\epsilon_2|=0$ corresponds to the only possible minimum of Φ, and hence, are stable.

ii. Solution: $|\epsilon_1|=0,\ |\epsilon_2|^2=a_2,\ a_2>0$ *(Phase II).* The inequalities (2.13) and (2.14) now yield

$$a_2\xi-a_1>0\ ,\quad a_2(1+\xi)-a_1>0\ .$$

Thus the region in the (a_1, a_2) plane in which the solution $|\epsilon_1|=0$, $|\epsilon_2|^2=a_2$ yields the only possible stable phase is given by

$$a_2\xi-a_1>0\ . \tag{2.17}$$

iii. Solution: $|\epsilon_1|^2=a_1,\ |\epsilon_2|=0, a_1>0$ *(Phase III).* Following the argument leading to (2.17), the solution $|\epsilon_1|^2=a_1$, $|\epsilon_2|=0$ is found to be stable provided that

$$a_1\xi-a_2>0\ . \tag{2.18}$$

iv. Solution: $|\epsilon_1|^2=(\xi a_2-a_1)/(\xi^2-1)>0,$ $|\epsilon_2|^2=(\xi a_1-a_2)/(\xi^2-1)>0$ *(Phase IV).* Equations (2.13) and (2.14) result now in the inequality

$$|\epsilon_1|^2|\epsilon_2|^2(1-\xi^2)<0\ . \tag{2.19}$$

Thus phase IV is stable provided that

$$\xi<1\ . \tag{2.20}$$

Of course, one should additionally have the conditions

$$a_1>a_2\xi\ ,\quad a_2>a_1\xi\ , \tag{2.21}$$

since $|\epsilon_1|^2>0$ and $|\epsilon_2|^2>0$.

C. The phase diagrams

The stable phases derived above are represented in Fig. 1 by means of a phase diagram in the space of the pump parameters a_1 and a_2. As argued above, one may observe four distinct phases in different regimes of the parameter space when the coupling constant $\xi<1$. Phase IV corresponds to coherent lasing action in which the intensities of both the modes ($I_i=|\epsilon_i|^2$, $i=1,2$) are nonzero. The phase boundaries are marked by heavy solid lines along which the stable values of the order parameters on both sides of the boundary merge into each other in a continuous fashion. These lines represent therefore lines of second-order phase transition.

On the other hand, for $\xi>1$, the inequality

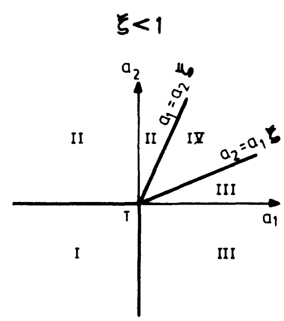

FIG. 1. Phase diagram displaying the four possible phases in the space of the pump parameters a_1 and a_2. For the coupling constant $\xi<1$, four (heavy) lines of second-order phase transition meet at the point T, called the *tetracritical* point.

(2.19) is violated and phase IV becomes unstable. We now have just three different phases I, II, and III indicated in Fig. 2. Applying the condition (2.17) one now finds that phase II extends beyond the $a_2=a_1$ line up to the line defined by $a_2\xi=a_1$. Similarly, phase III extends beyond the $a_2=a_1$ line up to the line $a_1\xi=a_2$ [cf. Eq. (2.18)]. However, on evaluating the respective free energies:

$$\Phi_{\rm II}=-\tfrac{1}{4}a_2^2\ ,\quad \Phi_{\rm III}=-\tfrac{1}{4}a_1^2\ , \qquad (2.22)$$

it becomes evident that in the lower half of the shaded region bounded by the lines $a_1\xi=a_2$ and $a_2\xi=a_1$, phase II is metastable, while in the upper half phase III is metastable. The coexistence curve on which $\Phi_{\rm II}$ and $\Phi_{\rm III}$ match is simply defined by the dashed line $a_1=a_2$. Across this line the set of order parameters undergoes a jump discontinuity and there is a first-order phase transition.

Based on the above considerations, we find that two distinct possibilities[20] exist as far as the shaded region is concerned: (i) Bistable behavior and hysteresis effects: (ii) Simple jump behavior across the line of first-order phase transition $a_1=a_2$. The first feature has been discussed at length by Mandel and co-workers[21] and by Singh.[19] In what follows we shall ignore the question of metastability and address the stable regions of the phase diagram, indicated in Fig. 3.

We are now in a position to compare the phase diagrams for the two-mode laser with those for an equilibrium system having multicritical properties. Consider, for instance, a weakly anisotropic (uniaxial or orthorhombic) antiferromagnet in a magnetic field applied along the easy direction of antiferromagnetic order.[17] Below a certain set of values of the temperature T and the magnetic field H, the system is in the antiferromagnetic (AF) phase characterized by "up"-"down" ordering of the spins (Fig. 4). If one now keeps H fixed but increases T, one goes over into the paramagnetic (PM) phase across the line AB of second-order transition. On the other hand, starting from the AF phase and increasing H while keeping T fixed ($<T_B$), one marches into the so-called spin-flop (SF) phase in which spins switch over from their low-field alignment parallel to the easy axis into a perpendicular alignment transverse to the field. The line BD, which is the boundary between the AF and the SF phase, is a line of first-order phase transition. The SF phase is separated from the PM phase by the second-order line BC. The confluence point B of the two second-order lines AB and BC and the first-order line BD is a multicriti-

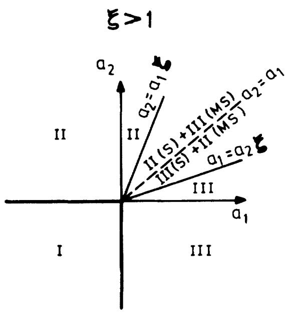

FIG. 2. For $\xi>1$, the fourth phase disappears and there is a first-order transition between phases II and III across the (dashed) line $a_1=a_2$. The regions in which stable (s) phase II (III) coexists with metastable (ms) phase III (II) are also indicated by light hatched lines.

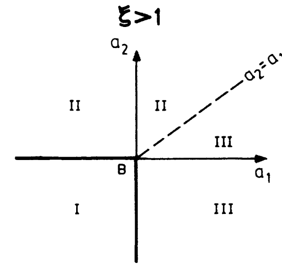

FIG. 3. Stable portions of the phase diagram for $\xi>1$. The point B which is the confluence of two (heavy) lines of second-order transition and one (dashed) line of first-order transition is termed the *bicritical* points.

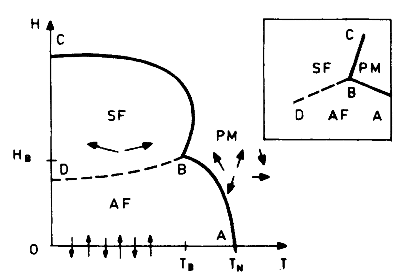

FIG. 4. Phase diagram (schematically) for a weakly anisotropic antiferromagnet in the thermodynamic space of temperature T and applied magnetic field H. The Neel point T_N marks the antiferromagnetic (AF) paramagnetic (PM) transition for $H=0$. The point B (H_B,T_B) at which the AF, PM, and the spin-flop (SF) phases meet is referred to as the bicritical point. The inset on the right shows the three lines of phase transitions in the region asymptotically close to the bicritical point B.

cal point at which the basic characteristics of the phase transition are quantititively different from that near an ordinary critical point. It is evident that the multicritical point B occurs as a result of the competition between *two* distinct sorts of ordering, namely, parallel and perpendicular in the magnetic context. For this reason the point B has been termed the *bicritical point* by Fisher and Nelson.[17] Comparing Fig. 3 and the inset in Fig. 4 we find that a quantatively similar situation exists in the case of a two-mode laser for which phase I, located symmetrically with respect to phases II and III, plays the role of the paramagnetic phase. By analogy, the point B may be referred to as the bicritical point of a two-mode laser.

The case $\xi < 1$ has its counterparts also in antiferromagnetism. Here, a fourth intermediate phase in addition to the flop-phase shows up. One now has four lines of second-order phase transitions meeting at the point T (see Fig. 1), which is another instance of a multicritical point and is known as the tetracritical point (borrowing again the terminology of Fisher and Nelson[17]). The significant feature of the tetracritical point is that, at T, the pump parameters conspire in such a manner as to make the four distinct phases identical.

III. SCALING AND MULTICRITICAL EXPONENTS

We have remarked earlier that a multicritical point (bicritical and tetracritical, in particular) is a special point on a line of critical points at which the basic features of the transition change abruptly. In order to investigate such characteristic changes, it is instructive to examine the scaling behavior of the free energy and the values of the exponents[22] near the multicritical point, a task addressed to briefly in the following.

In the free-energy function

$$\Phi = -\tfrac{1}{2}a_1|\epsilon_1|^2 - \tfrac{1}{2}a_2|\epsilon_2|^2 + \tfrac{1}{4}(|\epsilon_1|^4 + |\epsilon_2|^4) + \tfrac{1}{2}\xi|\epsilon_1|^2|\epsilon_2|^2 , \qquad (3.1)$$

one may note that the pump parameters a_1, a_2, and ξ are the "field" variables (same in the coexisting phases), while the order parameters $|\epsilon_1|$ and $|\epsilon_2|$ are the "density" variables (different in the coexisting phases), in the sense of Griffiths and Wheeler.[23] The scaling properties of Φ can be conveniently discussed in terms of field variables. To this end, we may introduce the "ordering" fields, or the fields conjugate to the order parameters ϵ_1 and ϵ_2 as (the numerical factor 2 arises due to complex nature of the order parameters)

$$E_1 = (2)\frac{\partial\Phi}{\partial\epsilon_1^*} = (-a_1 + |\epsilon_1|^2 + \xi|\epsilon_2|^2)\epsilon_1 , \qquad (3.2)$$

and

$$E_2 = (2)\frac{\partial\Phi}{\partial\epsilon_2^*} = (-a_2 + |\epsilon_2|^2 + \xi|\epsilon_1|^2)\epsilon_2 . \qquad (3.3)$$

In the multicritical region in which $|\epsilon_1|$ and $|\epsilon_2|$ are small, it is possible to invert (3.2) and (3.3) and write the approximate relations

$$\epsilon_1 \approx -\frac{E_1}{a_1}\left[1 + \frac{|E_1|^2}{a_1^3} + \frac{\xi}{a_1}\frac{|E_2|^2}{a_2^2}\right] , \qquad (3.4)$$

$$\epsilon_2 \approx -\frac{E_2}{a_2}\left[1 + \frac{|E_2|^2}{a_2^3} + \frac{\xi}{a_2}\frac{|E_1|^2}{a_1^2}\right] . \qquad (3.5)$$

On substituting (3.4) and (3.5) into (3.1), the free energy can be shown to have the scaling form

$$\Phi(E_1,E_2,a_1,a_2) \approx (a_1)^{2-\alpha} F\left[\frac{E_1}{a_1^{\Delta_1}}, \frac{E_2}{a_1^{\Delta_2}}, \frac{a_2}{a_1^{\phi}}\right] . \qquad (3.6)$$

Imagining the field variable a_1 to play the role of temperature of an equilibrium system, we may identify α as the "heat-capacity" exponent, Δ's the exponents associated with the ordering fields, and ϕ the so-called "crossover " exponent[24] associated with the switching of the behavior from second-order-like to first-order-like at the bicritical point B (Fig. 3). For the two-mode laser model the exponents, of course, have their *classical* values $\alpha = 0$, $\Delta_1 = \Delta_2 = \frac{3}{2}$, and $\phi = 1$. These can be readily deduced by expressing the free energy (3.1) in the structure of (3.6) with the aid of (3.4) and (3.5).

Regarding the other exponents, if one approaches either the tetracritical point T in Fig. 1 or the bicritical point B in Fig. 3 *along* a line of *second*-order transition, the order parameter $|\epsilon_1|$ goes to zero as $(a_1)^{1/2}$. Hence the exponent β_m (m referring to multicriticality) has the value one-half. On the other hand, $\partial^2\Phi/\partial\epsilon_1\partial\epsilon_1^*$ goes to zero as a_1. Thus the "susceptibility" exponent $\gamma_m = 1$. These exponents satisfy the usual scaling relation[24]

$$\gamma_m = 2 - \alpha_m - 2\beta_m , \qquad (3.7)$$

$$\Delta_m = 2 - \alpha_m - \beta_m \ , \tag{3.8}$$

where in the present case of a two-mode laser, $\alpha_m = 0$ and $\Delta_m = \frac{3}{2}$ as mentioned before. We may obtain the additional exponent δ_m from the relation[24]

$$\delta_m = -1 + (2 - \alpha_m)/\beta_m \ . \tag{3.9}$$

In the present case, $\delta_m = 3$.

Turning now to *purely* bicritical behavior (Fig. 3), one may introduce the "subsidiary exponents" which come about if one regards the bicritical point as the terminus of the line of first-order phase transition. The subsidiary exponents specify how the discontinuity in certain density variables (defined below) across the first-order line vanish at the bicritical point B. The corresponding exponents are denoted by the subscript u, adopting the notation of Griffiths.[24] We thus introduce the density variables

$$A_1 = \frac{\partial \Phi}{\partial a_1} \ , \quad A_2 = \frac{\partial \Phi}{\partial a_2} \ , \tag{3.10}$$

out of which A_2 may be viewed to play the role of the magnetization in the corresponding bicritical phenomenon in antiferromagnets discussed earlier. It is clear that the discontinuity in A_2 across the first-order line $a_1 = a_2$ (Fig. 3) is proportional to $|\epsilon_2|^2$. As $|\epsilon_2|$ itself goes to zero at the bicritical point as $(a_2)^{1/2}$, the discontinuity in A_2 vanishes as $(a_2)^{\beta}$, where the exponent $\beta_u = 1$. The corresponding "susceptibility" exponent is $\gamma_u = 0$. These values are of course consistent with the scaling relations[24]

$$\beta_\mu = \phi(1 - \alpha_m) \ , \tag{3.11}$$

$$\gamma_\mu = \phi \alpha_m \ . \tag{3.12}$$

Until now we have discussed only the *static* exponents. It is possible also to evaluate the *dynamic* exponents which determine the relaxation behavior of the two-mode laser near the multicritical point. These exponents are obtained from the eigenvalues of the Hessian matrix in (3.12) which are given by

$$\lambda_{\pm} = \tfrac{1}{2}[-b \pm (b^2 - 4c)^{1/2}] \ , \tag{3.13}$$

where

$$b = \tfrac{1}{2}(a_1 + a_2) - (1 + \tfrac{1}{2}\xi)(|\epsilon_1|^2 + |\epsilon_2|^2) \ , \tag{3.14}$$

and

$$c = (-\tfrac{1}{2}a_1 + |\epsilon_1|^2)(-\tfrac{1}{2}a_2 + |\epsilon_2|^2) + 2\xi[|\epsilon_1|^2(-\tfrac{1}{2}a_1 + |\epsilon_1|^2) + |\epsilon_2|^2(-\tfrac{1}{2}a_2 + |\epsilon_2|^2)] \ . \tag{3.15}$$

We may now study the behavior of the λ's as the tetracritical or the bicritical point is approached along one of the lines of phase transtions. We quote the results.

a. Tetracritical point ($\xi < 1$).

(1) $\lambda_+ = \frac{1}{2}|a_1| \ , \ \lambda_- = 0 \ ,$

$$\text{if } a_2 = 0 \ , \ a_1(<0) \to 0 \ ; \tag{3.16}$$

(2) $\lambda_+ = \frac{1}{2}|a_2| \ , \ \lambda_- = 0 \ ,$

$$\text{if } a_1 = 0 \ , \ a_2(<0) \to 0 \ ; \tag{3.17}$$

(3) $\lambda_+ = \frac{1}{4}a_2 \ , \ \lambda_- = 0 \ ,$

$$\text{if } a_2\xi = a_1 \ ; \tag{3.18}$$

(4) $\lambda_+ = \frac{1}{4}a_1 \ , \ \lambda_- = 0 \ ,$

$$\text{if } a_1\xi = a_2 \ . \tag{3.19}$$

b. Bicritical point ($\xi > 1$). The results (3.16) and (3.17) hold also if one approaches the bicritical point B along one of the two lines of second-order phase transition (cf. Fig. 3). Along the line of first-order phase transition, however, one has

$$\lambda_{\pm} = \tfrac{1}{4}a_1[\xi \pm (\xi^2 + \xi - 1)^{1/2}] \ , \quad a_1 = a_2 \ .$$

Thus the relaxation behavior of our system in the mean-field approximation is rather simple. The exponents presented in this section are valid in the mean-field limit and in practice there might be departures due to field fluctuations (rather than spatial fluctuations) becoming large, see, however, Ref. 25, where the mean-field behavior of a single-mode laser has been seen by careful experimentation.

IV. SUMMARY

In this paper we have analyzed the nature of phase transitions in the nonequilibrium system of a two-mode laser. We might mention here that the laser serves as a prototype model for the understanding of most nonequilibrium phase transitions. The reason is that, for a laser, unlike many other synergetic systems in biology, sociology, etc., the mathematical route one has to follow and the nature of approximations one has to make in deriving the macroscopic order parameter equations from microscopic equations of motion of various atom-field operators, are very clearly laid out. The example of laser is therefore ubiquitous in any discussion of nonequilibrium phase transitions.

To this date, the second-order phase transition and the accompanying critical-point phenomenon in a single-mode laser are very well investigated, both theoretically,[2,4,5] and experimentally.[25] Now, a significant advancement in the last decade in the area of critical phenomena in equilibrium physical systems has been the understanding of higher-order critical points, especially tricritical points. It was therefore interesting to see also an example of a tricritical point[11] in a laser with saturable absorbers. A more complex situation in higher-order critical phenomena arises however when the free energy is a function of *competing* order parameters and possesses certain symmetries, such as those implied by the dependence of the potential only on the modulus of the order parameters [cf. Ref. 19]. In that case, there is the possibility of the occurence of "symmetrical" multicritial points, e.g., the bicritical and tetracritical points. The latter have also been widely studied in equilibrium physical systems, both from classical as well as renormalization group theories.[17] The question therefore naturally arises: Is there a situation in the laser also wherein one may observe multicriticality? The present work provides an answer to this query. We should conclude by remarking that the laser system has the advantage that spatial fluctuations in the order parameters are negligibly small and hence simpler classical theories (as discussed here) would seem adequate for the description of phase transitions. It would be interesting therefore to test the predictions with regard to exponents and scaling behavior as made in this paper.

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