Decay of correlations in certain hyperbolic systems

Giulio Casati and Giorgio Comparin
Istituto di Fisica Università, via Celoria 16, I-20133 Milano, Italy

Italo Guarneri
Istituto di Matematica, Università di Pavia, Pavia, Italy

(Received 25 February 1982)

Some results on the decay of correlation functions in certain unstable systems are presented which appear to confirm previous theoretical estimates.

The behavior of autocorrelation functions of given dynamical variables is of primary importance in the statistical description of dynamical systems; in particular, exponential decay of autocorrelation functions for a class of phase functions sufficiently broad as to include relevant observables is being regarded as a paradigm of stochastic behavior. A satisfactory formalization of this property within the framework of ergodic theory is therefore necessary, and, in fact, it is the object of current mathematical research.[1-3]

For some time it has been commonly believed by physicists, on heuristic grounds, that this property would be displayed by systems with a positive KS entropy and that the decay constant should be proportional to the entropy. However, experimental results showing that even highly stochastic systems may exhibit long-time algebraic decay contradict this belief.[4,5] Only recently rigorous mathematical results on a class of model systems with positive entropy (dispersing billiards, Lorenz gas) have been obtained by Bunimovich and Sinai.[1,3]

Dispersing billiards and the automorphism of the torus commonly known as "Arnold's cat" belong to a class of dynamical systems of hyperbolic nature for which the decay of correlation functions can be analyzed by means of a technique derived by Sinai and others which is based on the construction of the so-called Markov partitions.[6,7] By means of one such partition of the phase space one can construct a symbolic dynamical system that, on one hand, provides a convenient model for studying the statistical properties of the original system, and, on the other, can be "asymptotically" approximated by a Markov chain, i.e., by a system exhibiting exponential decay of correlation functions for a broad class of functions.

However, due to insufficient smoothness of the original system, the character of the approximation by Markov chains may be such that one in this way cannot exclude the case in which the decay of correlation functions is somewhat slowed down. Specifically, Bunimovich and Sinai obtained Markov partitions for the Lorenz gas and dispersing billiards and used them to show that the velocity autocorrelation functions of a particle in a Lorenz gas with a periodic configuration of scatterers and with a uniformly bounded free path is $O(\exp(-\alpha n^{\gamma}))$, $\alpha > 0$, $0 < \gamma \lesssim 1$ as $n \to \infty$, n being the number of collisions. (The displacement of one scatterer is, however, sufficient to turn the type of decay from exponential to an inverse power of time.) An analogous result was found for the dispersing billiard consisting of four circle arcs (Fig. 1). Thus two problems, at least, arise: what the mechanism discriminating exponential from algebraic decay is, and whether it is possible to improve the above estimate into a pure exponential decay ($\gamma = 1$).

In this Communication we present the results of numerical computations of correlation functions in two model systems. One of them is the dispersing billiard: Our results seem to exclude the case in which one can take $\gamma = 1$. The other is a discontinuous map of the torus. We have studied it mainly in order to adjust the numerical technique, but it may have some interest of its own: Here the results seem

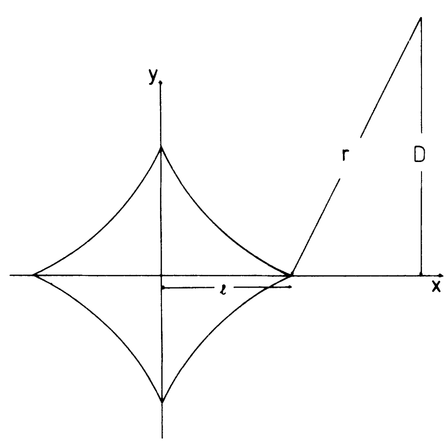

FIG. 1. Exponentially unstable billiard.

to indicate a pure exponential decay of correlation functions.

Consider the mapping of the 2-torus generated (modulo 1) by the matrix

$$A = \begin{vmatrix} 1 & k \\ 1 & 1+k \end{vmatrix}, \quad k > 0 \ . \tag{1}$$

For k an integer this map is a hyperbolic automorphism of the torus: In particular, with $k = 1$ one has Arnold's cat. For this automorphism, Markov partitions were first found by Adler and Weiss[6] and can be constructed as shown in Ref. 7. The corresponding symbolic dynamical system is a true Markov chain; the transition matrix and its eigenvalues can be explicitly calculated and used to conclude that, for sufficiently smooth functions f,

$$\rho(n) = \left| \frac{S(n) - S(\infty)}{S(0) - S(\infty)} \right| \lesssim \exp(-2hn) \ , \tag{2}$$

where $S(n) = \langle f(0) f(n) \rangle$, is the phase-averaged correlation function and h is the KS entropy. We found this decay too fast to be used as a reliability check of our numerical scheme of computation of correlation functions. With $0 < k < 1$ the map has still a hyperbolic character, but it is neither injective nor continuous, and this seems to place it beyond the range of applicability of the Markov partition machinery. However, it may be interesting to study the effect of this discontinuity since, as mentioned above, the slowing down of correlation functions in the Lorenz gas seems to be due precisely to a lack of smoothness.

We computed the phase-averaged correlation function of the characteristic function of the square $\Sigma = [0,d) \times [0,d)$ $(0 < d < 1)$, for different values of k, by integrating 90 000 different trajectories initially started in Σ with a uniform distribution. A typical result is shown in Fig. 2. We fitted the numerical data with the curve $\exp(-\alpha n^{\gamma})$ and found $\gamma = 1$ with a very good accuracy $(\pm 10^{-3})$. We have, therefore, good evidence of a pure exponential decay; the decay constant α is an increasing function of $\ln\lambda(k)$, $\lambda(k)$ being the eigenvalue of A greater in modulus than 1 (Fig. 3). It is, however, apparent that the decay is considerably slowed down with respect to $\exp\{-2\ln[\lambda(k)]n\}$ (which is the correct decay for $k = 1$).

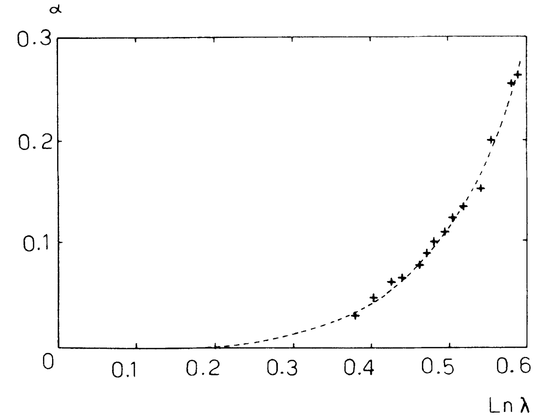

FIG. 3. Dependence of the decay constant α on $\ln\lambda$, where λ is the eigenvalue of A greater than 1.

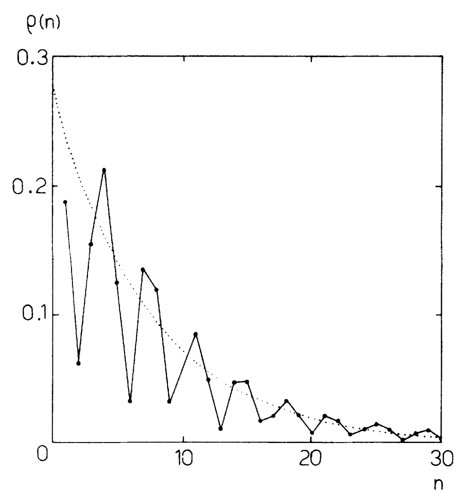

FIG. 2. Absolute value of the correlation function (solid line) vs the number of the iterations n, for the area preserving mapping (1), with $k = 0.28$. The dotted line represents the fitted curve $0.27e^{-0.13n}$.

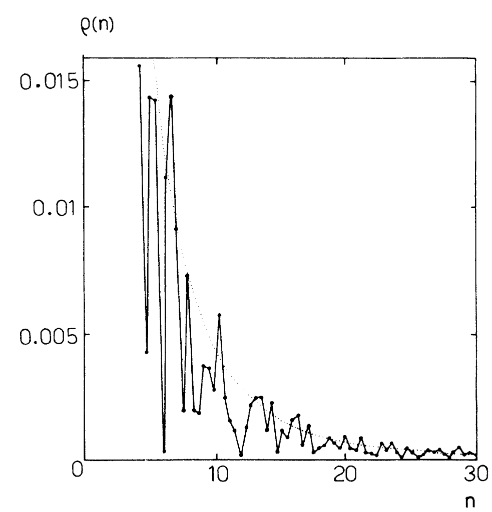

FIG. 4. Absolute value of the correlation function (solid line) for the dispersing billiard of Fig. 1 with $\epsilon = l/D = 0.5$. The values of the correlations are taken at fixed time intervals. The dotted line represents the fitted curve $\exp(-1.4n^{0.42})$.

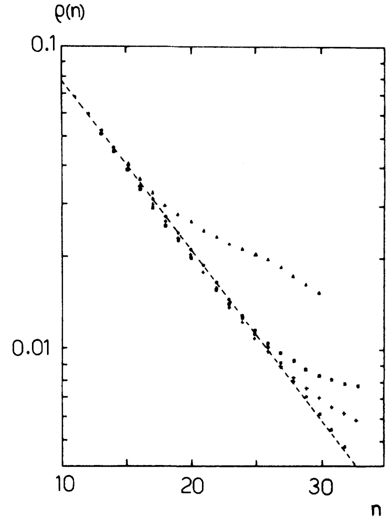

FIG. 5. Correlation function for the case of Fig. 2 after the smoothing, for different numbers N of trajectories: $\Delta, N=2500$; $\square, N=10\,000$; $+, N=40\,000$; $\bullet, N=90\,000$. The dashed line gives the pure exponential decay $0.27e^{-0.13n}$.

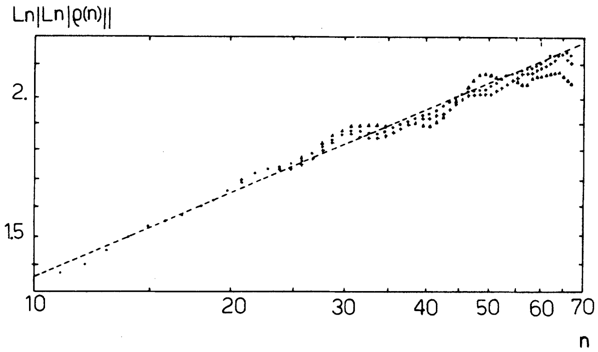

FIG. 6. Correlation function for the case of Fig. 4 after the smoothing, for different numbers N of trajectories: $\circ, N=40\,000$; $+, N=120\,000$; $\cdot, N=250\,000$. The dashed line gives the exponential decay $\exp(-1.4n^{0.42})$.

We considered the dispersing billiard of Fig. 1 characterized by the value of the parameter $\epsilon = l/D = 0.5$ and numerically computed the autocorrelation of the characteristic function f of the region Σ on the energy surface defined by $0 \le x \le l/3$, $0 \le y \le l/2$, $\cos(1.1) \le v_x \le \cos(0.1)$, $\sin(0.1) \le v_y \le \sin(1.1)$. Here we integrated 250 000 different trajectories initially started in Σ with uniform distribution and computed the autocorrelation $\rho(n)$. The result is shown in Fig. 4. A best fit of the numerical data gives $\rho(n) \sim \exp(-1.4n^{0.42})$, thus supporting the view that the results obtained by Bunimovich and Sinai[1-3] are not merely estimates but describe the actual asymptotic behavior of correlations.

Various numerical checks were devised for the accuracy of computer results. Since we perform phase averages and not time averages, we needed to integrate each orbit only for short times, and therefore, we had not to deal with big problems of accuracy for the time evolution of any single orbit. The main restriction comes from the limited number N of orbits one actually integrates, since the error in the corrrelation value decreases as $1/\sqrt{N}$; another source of difficulty is given by the oscillating character of the decay. This last problem was simply handled by introducing a convenient smoothing procedure, thus obtaining a monotone decreasing function. In Figs. 5 and 6 are drawn the smoothed values of the correlation functions for the two models studied and for increasing number N of trajectories.

As it is clearly apparent from the two pictures, the fitting of the data with the dashed lines becomes more and more accurate with increasing N.

ACKNOWLEDGMENTS

We are indebted to Y. Sinai for helpful discussions. This work was supported by Consiglio Nazionale delle Ricerche under Grant No. 80.02384 and by NATO under Grant No. 1645.

[1]L. A. Bunimovich and Ya. G. Sinai, Commun. Math. Phys. 78, 479 (1981).
[2]L. A. Bunimovich and Ya. G. Sinai, Commun. Math. Phys. 78, 247 (1981).
[3]Ya. G. Sinai, in *Works on the Foundation of Statistical Physics*, by N. S. Krylov (Princeton University Press, Princeton, N.J., 1979), Appendix.
[4]B. J. Alder and T. E. Wainwright, Phys. Rev. A 1, 18 (1970).
[5]M. H. Ernst, E. H. Hauge, and J. M. van Leeuwen, Phys. Rev. A 4, 2055 (1971); J. R. Dorfman, in *Proceeding of the 1974 Wageningen Summer School* (North-Holland, Amsterdam, 1979), p. 277; W. W. Wood, *ibid.*, p. 33.
[6]R. Adler and B. Weiss, Mem. Am. Math. Soc. 98, 1 (1970).
[7]Ya. G. Sinai, *Introduction to Ergodic Theory* (Princeton University Press, Princeton, N.J., 1977).