Quantum origin of dephasing and revivals in the coherent-state Jaynes-Cummings model

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The Jaynes-Cummings model of a two-level system interacting with a single-mode coherent-state field is investigated in a transformed representation emphasizing quantum corrections to the semiclassical Rabi problem. We find an intuitive explanation of the collapse and revivals of oscillations in the population inversion.

The Jaynes-Cummings model (JCM) of a single two-state system interacting with a single occupied quantized radiation field mode[1] lies at the heart of modern laser physics and quantum optics.[2] When the exact solutions for this model[1–3] are used to study the effects of field statistics in this nondissipative model, a number of unexpected features are obtained.[4–6] The semiclassical sinusoidal Rabi solution for the two-state inversion W is obtained in a sense when the field is represented by a pure number state. When the field mode is initially prepared in a coherent state, the two-state inversion oscillations decay away (even though there are no losses in the model) and then revive.[4–6] Interest in this model has been greatly stimulated by the recent work of Eberly and colleagues[6] who have elucidated many new characteristics and systematic features of the decay of Rabi oscillations, their revival, and the interference between revivals and resultant apparent irreversibility.

The central problem in studying the *coherent-state* JCM is that the sinusoidal inversion W_n induced by a resonant number state $|n\rangle, W_n = \cos 2\lambda n^{1/2} t$ (where λ is a coupling constant) is replaced by the Poisson-weighted infinite sum

$$W = \exp(-|\alpha|^2) \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \cos 2\lambda n^{1/2} t \ , \quad (1)$$

where $|\alpha|^2 = \bar{n}$ is the mean number of photons in the coherent state $|\alpha\rangle$. This sum cannot be evaluated in closed form and requires careful approximations[6] in order to reproduce all of the features referred to above. We have obtained a new representation of the coherent-state JCM which emphasizes the importance of quantum corrections to semiclassical Rabi oscillations in the JCM and provides a physical interpretation of the collapse and revival[7] in terms of competition from a cascade of quantum-generated eigenfrequencies.

Previous workers have emphasized the Poisson distribution of n's (and indirectly the spread in Rabi frequencies $\lambda\sqrt{n}$) by expanding the coherent state in terms of number states. We choose to emphasize the classical nature of such a field. Instead of using the standard fully quantized single-mode interaction Hamiltonian acting on an initial state $|\alpha\rangle$, we consider the entirely *equivalent* Hamiltonian describing the interaction of the two-state system with a quantized field *and* a classical field of normalized amplitude α, acting on an initially *unoccupied* field-mode state $|0\rangle$.

The coherent state JCM is characterized by two "spin" states $|\pm\rangle$, a *resonant* field state $|\alpha\rangle$ and a Hamiltonian[2,6] in rotating-wave approximation (RWA),

$$H_1 = \tfrac{1}{2}\hbar\omega_0\sigma_3 + \hat{V}_1 \ , \quad (2)$$

where the interaction Hamiltonian $\hat{V}_1 = -\underline{d}\cdot\underline{\hat{E}}$, $\underline{d}$ is the transition dipole operator, $\underline{\hat{E}}$ the electric field operator, and

$$V_1 = \hbar\lambda(\hat{a}^\dagger e^{i\omega t}\sigma_- - \sigma_+\hat{a}e^{-i\omega t}) \ , \quad (3)$$

where $[\hat{a},\hat{a}^\dagger]=1$, $[\sigma_-,\sigma_+] = -\sigma_3$. We transform H_1 using the inverse of the Glauber unitary transformation[8]

$$D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) \ , \quad (4)$$

where $D^{-1}(\alpha)|\alpha\rangle = |0\rangle$, and find our new Hamiltonian $H_2 = D^{-1}(\alpha)H_1D(\alpha) = H_1 + V_2$, where V_2 describes the interaction of the two-state system with a classical field[9] $\underline{E_c}$ of normalized amplitude α,

$$V_2 = -\underline{d}\cdot\underline{E_c} = \hbar\lambda(\alpha^* e^{i\omega t}\sigma_- - \sigma_+\alpha e^{-i\omega t}) \ . \quad (5)$$

In our transformed representation V_2 induces sinusoidal Rabi oscillations between $|+\rangle$ and $|-\rangle$ without changing the number of photons in the field mode (this number is initially zero). The quantized interaction $\hat{V}_1$ *can* change the number of photons in the mode. For example, it can take the two-state system from $|0,+\rangle$, the initial state with no photons present, to $|1,-\rangle$ with one photon present (and return it).

In RWA, the "essential states" coupled by the transformed interaction V are represented in Fig. 1 as a "ladder" of states. The couplings are

$$\langle n+|\hat{V}_1|n+1,-\rangle=\hbar\lambda\sqrt{n+1}\,, \tag{6a}$$

$$\langle n+|V_2|n-\rangle=\hbar\lambda\alpha e^{-i\omega t}\,. \tag{6b}$$

We see now how the distribution of Rabi eigenfrequencies necessary for dephasing arises in our model as a result of the quantum *dynamics*. As successive states $|n\pm\rangle$ are coupled in, quantum Rabi frequencies $\lambda\sqrt{n}$ are generated after a succession of semiclassical oscillations. We expand the wave function in terms of $|n+\rangle$ and $|n-\rangle$,

$$\Psi(t)=\sum_n [C_{n-}(t)e^{-iE_{n-}t/\hbar}|n-\rangle + C_{n+}(t)e^{-iE_{n+}t/\hbar}|n+\rangle]\,, \tag{7}$$

and use the Schrödinger equation and orthonormality to generate equations of motion of the probability amplitudes for the two-level system to be in state $|\pm\rangle$ with k photons present, $C_k\pm(t)$. We choose as our initial condition $C_{0,+}(0)=1$, and find

$$i\dot{C}_{k+}=\lambda\alpha C_{k-}+\lambda\sqrt{k+1}C_{k+1,-} + i\delta_{k0}\delta(t)\,, \tag{8a}$$

$$i\dot{C}_{k-}=\lambda\alpha C_{k+}+\lambda\sqrt{k}\,C_{k-1,+}\,. \tag{8b}$$

We convert these to algebraic equations by Fourier transforms, with

$$C_{k\pm}(t)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-iEt}g_{k\pm}(E)\,dE\,,$$

and find the tridiagonal matrix equation

$$\begin{pmatrix} E & -\lambda\alpha & 0 & 0 & 0 & 0 & \cdots \\ -\lambda\alpha & E & -\lambda\sqrt{1} & 0 & 0 & \vdots & \\ 0 & -\lambda\sqrt{1} & E & -\lambda\alpha & 0 & & \\ \vdots & 0 & -\lambda\alpha & E & -\lambda\sqrt{2} & & \\ & \vdots & 0 & -\lambda\sqrt{2} & E & & \\ & & \vdots & 0 & -\lambda\alpha & & \\ & & & \vdots & 0 & & \end{pmatrix} \begin{pmatrix} g_{0-} \\ g_{0+} \\ g_{1-} \\ g_{1+} \\ g_{2-} \\ g_{2+} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\,. \tag{9}$$

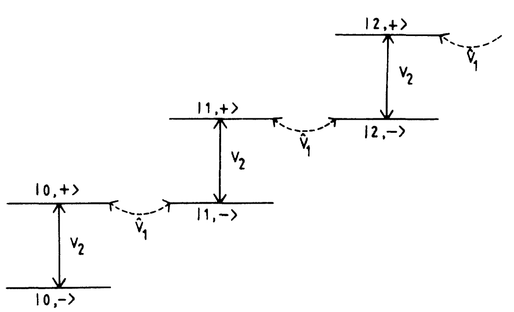

FIG. 1. Energy levels and coupling in the transformed JCM, where $\hat{V}_1$ is the fully quantized interaction which creates and destroys photons in the initially *empty* mode and V_2 is the interaction inducing semiclassical Rabi oscillations without changing the number of photons. Note that all of these transitions are reversible.

The leading features in our problem can be obtained by studying truncated forms of this matrix equation. If we restrict ourselves to the four states $|0\pm\rangle$ and $|1\pm\rangle$ (with two semiclassical couplings $\lambda\alpha$ and one quantum coupling $\lambda\sqrt{1}$) we find the four eigenfrequencies,

$$E_1=\frac{\lambda}{2}[1+(1+4\alpha^2)^{1/2}]=-E_3\,, \tag{10a}$$

$$E_2=\frac{\lambda}{2}[1-(1+4\alpha^2)^{1/2}]=-E_4\,, \tag{10b}$$

and inversion $W=|C_{0+}(t)|^2+|C_{1+}(t)|^2 - |C_{0-}(t)|^2-|C_{1-}(t)|^2$,

$$W=\frac{1}{(E_1-E_2)^2}[(E_1^2+\lambda^2\alpha^2)\cos 2E_1t+(E_2^2+\lambda^2\alpha^2)\cos 2E_2t] . \qquad (11)$$

We note that this is a sum of cosines as in the full JCM solution. We plot W from Eq. (11) in Fig. 2, where we have chosen a strong semiclassical field coupling $\lambda\alpha=1$ fixing $\lambda=0.02$. Figure 2 reveals that the competition between the semiclassical Rabi oscillation at $\lambda\alpha$ and the quantum oscillation at $\lambda\sqrt{1}$ leads to beats and an *exactly* periodic collapse and revival of population oscillations. For the values chosen in Fig. 2, the "branching ratio" from $|0+\rangle$ to $|0-\rangle$ or to $|1,-\rangle$ clearly favors the semiclassical transition: many Rabi oscillations occur before a dephasing collapse sets in as a simple beat phenomenon. This is demonstrated more explicity if $\alpha>>1$ so that we can expand the solution (11) for W as

$$W\simeq\cos\lambda t\cos[(\lambda^2+4\lambda^2\alpha^2)^{1/2}t] . \qquad (12)$$

If, however, $\alpha\lesssim 1$, the usual complicated beat structure is produced, and it is hard to recognize

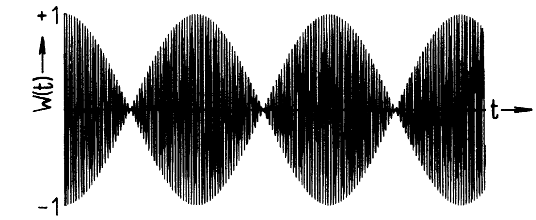

FIG. 2. Plot of the four-state result of the inversion $W(t)$ obtained from Eq. (11) demonstrating very simple collapse and exact revivals. We have chosen $\lambda=0.02$ and $\lambda\alpha=1$ ($\bar{n}^{1/2}=50$), and t varies from 0 to 500.

Rabi oscillations within the time evolution.

Eventually we must couple in the quantum interaction $\hat{V}_1$ a *second* time since population in $|1,+\rangle$ produced by the first coupling of $\hat{V}_1$ can now couple to $|2,-\rangle$ with quantum Rabi frequency $\sqrt{2}\lambda$. We have diagonalized the (6×6) matrix including $|0,\pm\rangle$, $|1,\pm\rangle$, $|2,\pm\rangle$ in terms of the roots of a cubic equation. The full six-level solution for $W(t)$ can then be found and a numerical evaluation of this is shown in Figs. 3(a) and 3(b),

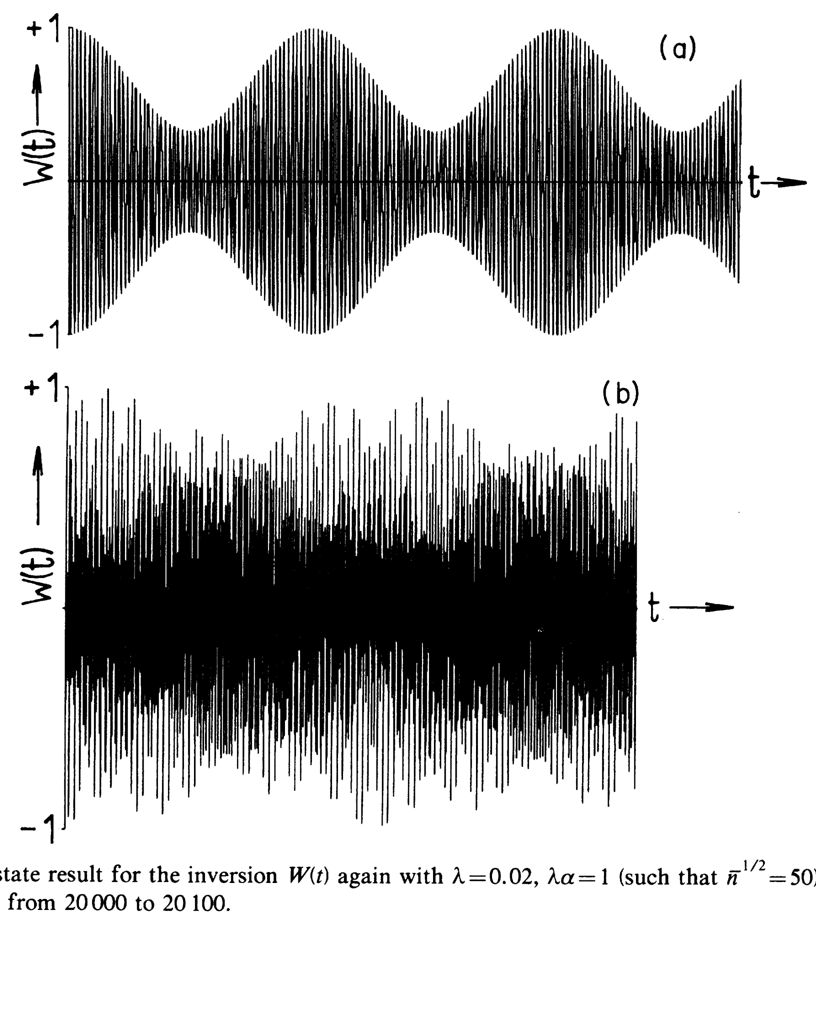

FIG. 3. Plot of the six-state result for the inversion $W(t)$ again with $\lambda=0.02$, $\lambda\alpha=1$ (such that $\bar{n}^{1/2}=50$). In (a), t varies from 0 to 500; in (b) from 20 000 to 20 100.

again for $\lambda\alpha=1$, $\lambda=0.02$ for different time ranges. The short-time behavior can be made more transparent by an expansion for small λ, and after considerable algebra we find (for $\lambda\alpha=1$),

$$W\simeq\tfrac{1}{3}(2+\cos\sqrt{3}\lambda t)\cos 2t\ . \tag{13}$$

For longer times, the evolution is considerably more complicated than in the four-state approximation with nearly chaotic behavior, as can be observed in Fig. 3(b).

To solve the coherent-state JCM exactly in our approach requires full (and painful) diagonalization of Eq. (9) which we know should reproduce ultimately the original JCM infinite series.[10] In our approach it is clear that the great richness of structure uncovered by the careful numerical and analytic work of Eberly *et al.*[6] has, in part, a physically appealing origin revealed by our "ladder of excitations" in Fig. 1. Using our approach one can see that as time progresses, increasingly more eigenfrequencies are coupled into the problem. If the mode is highly occupied initially ($\alpha>>1$) then the initial time evolution is just that of the first rung on the ladder. However, a small proportion of population (reversibly) leaks into the ground state emitting a photon. Thus we have a picture of a ladder with alternate strong (semiclassical $\lambda\alpha$) rungs and quantum rungs which start out weak ($\lambda\sqrt{1}$) but strengthen as n increases. When only one quantum rung is relevant we have collapse and almost exact revivals, but as the other quantum rungs are ascended the revivals are less exact (in the sense of reproducing earlier revivals) because of destructive interference. Had the mode been weakly occupied at $t=0$ ($\alpha<<1$), then quantum and semiclassical couplings are equally important, the semiclassical "branching ratio" will *not* permit many oscillations before a quantum "leakage," and the many important eigenfrequencies are quickly produced resulting in a nearly chaotic behavior. But in general we must wait the appropriate number of semiclassical Rabi cycles indicated by our ladder before coupling in a new (and destructively interfering) eigenfrequency. Finally, it is clear why no simple semiclassical result is obtained even for $|\alpha|>>1$ in that eventually large quantum Rabi frequencies $\lambda\sqrt{n}$ (but different from $\lambda\alpha$) compete for the atomic populations in a dynamical sense.

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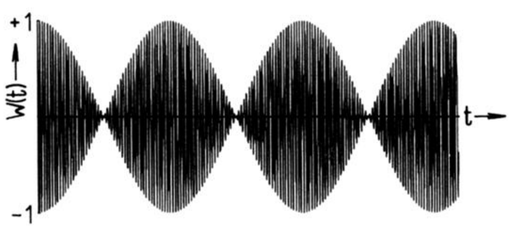
[10]It is clear from Eq. (1) that the JCM eigenfrequencies are $0,\pm\sqrt{1},\pm\sqrt{2},\ldots,$ and independent of α; this is presumably true of the eigenfrequencies of our *infinite* tridiagonal matrix in Eq. (10). If this is so, we do in fact find eigenfrequencies $0,\pm\sqrt{1},\pm\sqrt{2},\ldots,$ by setting $\alpha=0$.



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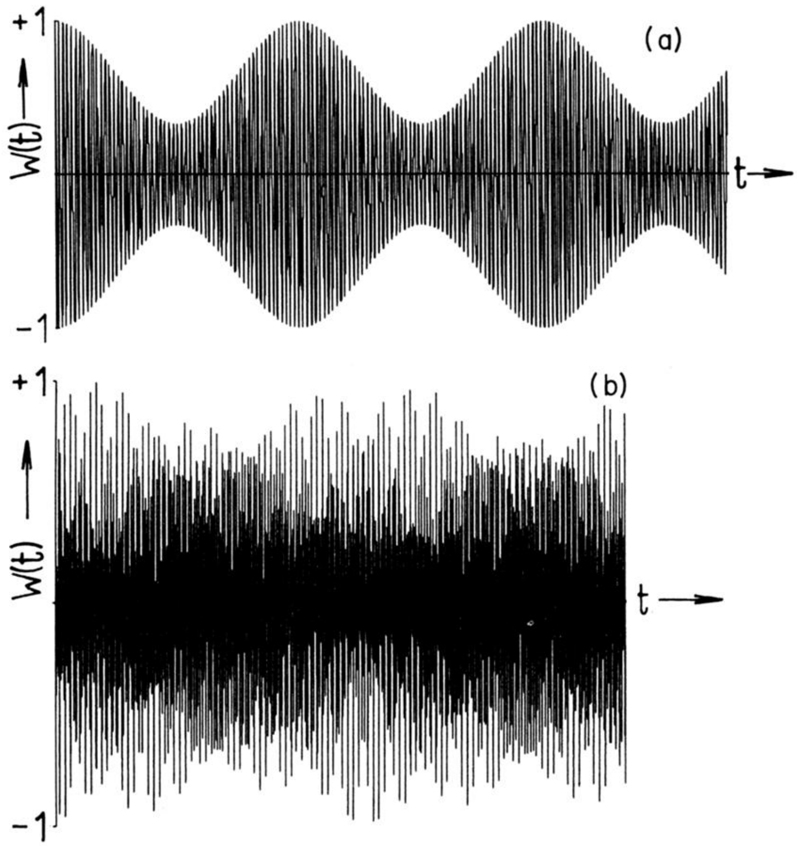

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