Path-integral approach to problems in quantum optics

Mark Hillery and M. S. Zubairy
Institute for Modern Optics, Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131 and Max-Planck-Institut für Quantenoptik, D-8046 Garching bei München, West Germany
(Received 11 December 1981)

A formalism for applying path integrals to certain problems in nonlinear optics is considered. The properties of a coherent-state propagator are discussed and a path-integral representation for the propagator is presented. This representation is then employed in evaluating the propagator for general single-mode and multimode Hamiltonians which are at most quadratic in the creation and destruction operators of the field. Some examples involving parametric processes are given.

I. INTRODUCTION

Path integrals and the approximations to which they have led have been used very much in quantum field theory in recent years. The path-integral representation of the propagator allows one to see more clearly than the standard operator approach, the connection between the classical and quantum dynamics of a system. Semiclassical approximations can then be derived in a natural way.[1] So far, however, these techniques have not found much use in quantum optics.[2] In this paper we will develop some of the formalism which will be of use in applying path-integral techniques to certain problems in nonlinear optics.

The types of problems to which we would like to apply these techniques are those in which the medium with which the light interacts can be described by a nonlinear susceptibility tensor.[3] These include such processes as parametric amplification and harmonic generation. The interaction between the different modes is then described by products of various powers (depending upon the specific process) of the creation and destruction operators of the modes involved.

The type of path integral which we will consider is not the one usually used in quantum field theory in which one makes use of a coordinate representation of the field. We will be interested in problems in which only a few of the modes of the field are important and we will use a path integral which makes use of a representation of these modes in terms of coherent states. Because the Hamiltonians which we will consider will be expressed in terms of creation and destruction operators, and not the corresponding position and momentum operators, coherent states, which are eigenstates of the destruction operator, are natural objects to use. The coherent-state path integral can be used to calculate the matrix element of the time development transformation between two coherent states. This matrix element can be regarded as a type of propagator. This form of the path integral was first discussed by Klauder[4] and was subsequently examined by Schweber[5] in the context of Bargmann spaces. Klauder[6] in later work showed that the coherent-state path integral is but one example of a more general class of objects known as continuous representation path integrals.

In Sec. II, we discuss some properties of the propagator and show how it can be used to calculate various quantities of interest in quantum optics. In Sec. III, we derive formulas which can be used to calculate the propagator for single-mode systems with Hamiltonians at most quadratic in the creation and destruction operators. These are then used to calculate the propagator for the case of second subharmonic generation when the pump field is classical. In Sec. IV, we generalize our results and calculate the propagator for an N-mode system whose Hamiltonian is quadratic. This result is then used to calculate the propagator for a parametric amplifier with a classical pump field.

II. COHERENT-STATE PROPAGATOR

We consider a system which consists of one mode of the radiation field. Let the corresponding time-evolution operator be $U(t_2,t_1)$, i.e., if $|\psi(t_1)\rangle$ is the state of the system at time t_1 then the state at time t_2 is

$$|\psi(t_2)\rangle = U(t_2,t_1)\,|\psi(t_1)\rangle \ . \qquad (1)$$

If the Hamiltonian governing the system is given by $H(t)$ then the time-evolution operator is (where we have chosen units such that $\hbar=1$)

$$U(t_2,t_1)=T\exp\left[-i\int_{t_1}^{t_2} H(t')dt'\right] , \qquad (2)$$

where T is the Dyson time-ordering operator.

We will consider the propagator

$$K(\alpha_2,t_2;\alpha_1,t_1)=\langle\alpha_2\,|\,U(t_2,t_1)\,|\,\alpha_1\rangle \ , \qquad (3)$$

where the coherent states $|\alpha_i\rangle$ are the eigenstates of the destruction operator a with eigenvalue α_i, at time $t=0$. Another expression for the propagator $K(\alpha_2,t_2;\alpha_1,t_1)$ can be derived by noting that the coherent state, at time t [i.e., the eigenstate of $a(t)$] is given by

$$|\alpha,t\rangle = U(t,0)^{-1}\,|\,\alpha\rangle \ . \qquad (4)$$

We then obtain

$$K(\alpha_2,t_2;\alpha_1,t_1)=\langle\alpha_2,t_2\,|\,\alpha_1,t_1\rangle = \langle\alpha_2\,|\,U(t_2,0)U(t_1,0)^{-1}\,|\,\alpha_1\rangle \ . \qquad (5)$$

In quantum optics, one is usually interested in evaluating certain correlation functions of the field. For a one-mode field these are proportional to the expectation values of products of the creation and destruction operators. These correlation functions can be expressed in terms of the propagator $K(\alpha_2,t_2;\alpha_1,t_1)$. We assume that, at $t=0$, the density matrix has a P representation, i.e.,

$$\rho=\int d^2\alpha\, P(\alpha)\,|\,\alpha\rangle\langle\alpha\,| \ , \qquad (6)$$

so that the expectation value of any operator, $O(t)$, in the Heisenberg picture is given by

$$\langle O(t)\rangle = \mathrm{Tr}[\rho O(t)] = \int d^2\alpha\, P(\alpha)\langle\alpha\,|\,O(t)\,|\,\alpha\rangle \ . \qquad (7)$$

On using the completeness property of the coherent states, namely,

$$\frac{1}{\pi}\int d^2\alpha\,|\,\alpha,t\rangle\langle\alpha,t\,| = 1 \ , \qquad (8)$$

it can be easily shown that

$$\langle a(t)\rangle = \frac{1}{\pi}\int\int d^2\alpha_1 d^2\alpha_2 P(\alpha_2)\,|\,K(\alpha_1,t_1;\alpha_2,0)\,|^2\alpha_1 \ , \qquad (9)$$

$$\langle a^\dagger(t_1)a(t_2)\rangle = \frac{1}{\pi^2}\int\int\int d^2\alpha_1 d^2\alpha_2 d^2\alpha_3 P(\alpha_3)K(\alpha_1,t_1;\alpha_2,t_2)\,K(\alpha_2,t_2;\alpha_3,0)K(\alpha_3,0;\alpha_1,t_1)\alpha_1^*\alpha_2 \ , \qquad (10)$$

$$\langle a^\dagger(0)a^\dagger(t)a(t)a(0)\rangle = \frac{1}{\pi^2}\int\int\int d^2\alpha_1 d^2\alpha_2 d^2\alpha_3 P(\alpha_3)K^*(\alpha_2,t;\alpha_3,0)K(\alpha_1,t;\alpha_3,0)\,|\,\alpha_3\,|^2\alpha_2^*\alpha_1 \ . \qquad (11)$$

The determination of the propagator thus enables us to calculate any correlation function of the field operators.

The propagator $K(\alpha_2,t_2;\alpha_1,t_1)$ is related to the Q representation of the radiation field, i.e.,

$$Q(\alpha,t)=\frac{1}{\pi}\langle\alpha,t\,|\,\rho\,|\,\alpha,t\rangle \ , \qquad (12)$$

in a natural way. On substituting for ρ from Eq. (6), we obtain

$$Q(\alpha,t)=\frac{1}{\pi}\int d^2\alpha_1 P(\alpha_1)\,|\,K(\alpha,t;\alpha_1,0)\,|^2 \ . \qquad (13)$$

In particular, for an initial coherent state, $P(\alpha_1)=\delta^2(\alpha_1-\alpha_0)$, and it follows from Eq. (13) that

$$Q(\alpha,t)=\frac{1}{\pi}\,|\,K(\alpha,t;\alpha_0,0)\,|^2 \ . \qquad (14)$$

The Q representation has the property that the expectation value, at time t, of any antinormally ordered function $O_A(a,a^\dagger)$ of a and $a^\dagger$ may be determined via the relation

$$\langle O_A(a,a^\dagger)\rangle = \int d^2\alpha\, O_A(\alpha,\alpha^*)Q(\alpha,t) \ . \qquad (15)$$

The close relation of propagator to the Q representation makes it easier to evaluate the expectation values of antinormally ordered products than the normally ordered products. For example, the mean number of photons at time t is most easily evaluated by using the commutation relation $[a,a^\dagger]=1$, as follows:

$$\langle a^\dagger(t)a(t)\rangle=\langle a(t)a^\dagger(t)\rangle-1=\frac{1}{\pi}\int d^2\alpha_1\int d^2\alpha_2 P(\alpha_2)\,|K(\alpha_1,t;\alpha_2,0)|^2\,|\alpha_1|^2-1\;. \tag{16}$$

Finally, we note that the Q and P representations are related to each other via the following relationship[7]:

$$Q(\alpha,t)=\int d^2\alpha_1 P(\alpha_1,t)\,|K(\alpha,0;\alpha_1,0)|^2\;. \tag{17}$$

We now turn to the calculation of the propagator itself for a particular set of systems.

III. REPRESENTATION OF THE PROPAGATOR

A. Path integral for the propagator

It is possible to express the coherent-state propagator in terms of a path integral. Here we outline the derivation of the path-integral representation which was first obtained by Klauder.[4]

We consider a system which is described by a Hamiltonian, $H(a^\dagger,a;t)$, which is expressed in terms of the creation and destruction operators $a^\dagger$ and a. We suppose further that $H(a^\dagger,a;t)$ is normally ordered. By inserting n resolutions of the identity into Eq. (5) we find that

$$K(\alpha_f,t_f;\alpha_i,t_i)=\left[\frac{1}{\pi}\right]^n\int d^2\alpha_1\cdots\int d^2\alpha_n\langle\alpha_f,t_f\,|\,\alpha_n,t_n\rangle\langle\alpha_n,t_n\,|\,\alpha_{n-1},t_{n-1}\rangle\cdots\langle\alpha_1,t_1\,|\,\alpha_i,t_i\rangle\;. \tag{18}$$

We also have that

$$\begin{aligned}\langle\alpha_j,t_j\,|\,\alpha_{j-1},t_{j-1}\rangle&=\left\langle\alpha_j\left|T\exp\left[-i\int_{t_{j-1}}^{t_j}d\tau H(\tau)\right]\right|\alpha_{j-1}\right\rangle\\&\cong\left\langle\alpha_j\left|\left[1-i\int_{t_{j-1}}^{t_j}d\tau H(a^\dagger,a;\tau)\right]\right|\alpha_{j-1}\right\rangle\\&\cong\langle\alpha_j\,|\,\alpha_{j-1}\rangle[1-i\epsilon H(\alpha_j^*,\alpha_{j-1};t_{j-1})]\\&\cong\exp[-\tfrac{1}{2}(|\alpha_j|^2+|\alpha_{j-1}|^2)+\alpha_j^*\alpha_{j-1}-i\epsilon H(\alpha_j^*,\alpha_{j-1};t_{j-1})]\;,\end{aligned} \tag{19}$$

where $\epsilon=(t_f-t_i)/n+1$, $t_j=t_i+j\epsilon$, and the function $H(\alpha''^*,\alpha',t)$ is defined as

$$H(\alpha''^*,\alpha';t)=\frac{\langle\alpha''\,|\,H(a^\dagger,a;t)\,|\,\alpha'\rangle}{\langle\alpha''\,|\,\alpha'\rangle}\;. \tag{20}$$

Inserting Eq. (19) into Eq. (18) immediately yields

$$K(\alpha_f,t_f;\alpha_i,t_i)=\lim_{n\to\infty}\left[\frac{1}{\pi}\right]^n\int d^2\alpha_1\cdots\int d^2\alpha_n\exp\left[\sum_{j=1}^{n+1}[-\tfrac{1}{2}(|\alpha_j|^2+|\alpha_{j-1}|^2)+\alpha_j^*\alpha_{j-1}-i\epsilon H(\alpha_j^*,\alpha_{j-1};t_{j-1})]\right]\;. \tag{21}$$

We note that

$$\begin{aligned}&\sum_{j=1}^{n+1}[-\tfrac{1}{2}(|\alpha_j|^2+|\alpha_{j-1}|^2)+\alpha_j^*\alpha_{j-1}-i\epsilon H(\alpha_j^*,\alpha_{j-1};t_{j-1})]\\&\qquad=\sum_{j=1}^{n+1}\left[-\tfrac{1}{2}\alpha_j^*\left[\frac{\alpha_j-\alpha_{j-1}}{\epsilon}\right]\epsilon+\tfrac{1}{2}\alpha_{j-1}\left[\frac{\alpha_j^*-\alpha_{j-1}^*}{\epsilon}\right]\epsilon-i\epsilon H(\alpha_j^*,\alpha_{j-1};t_{j-1})\right]\\&\qquad\to\int_{t_i}^{t_f}d\tau[\tfrac{1}{2}(\alpha\dot{\alpha}^*-\alpha^*\dot{\alpha})-iH(\alpha^*,\alpha;\tau)]\;,\end{aligned} \tag{22}$$

as $\epsilon\to 0$. It then follows that

$$K(\alpha_f,t_f;\alpha_i,t_i)=\int \mathscr{D}[\alpha(\tau)]\exp\left[\int_{t_i}^{t_f} d\tau[\tfrac{1}{2}(\alpha\dot{\alpha}^*-\alpha^*\dot{\alpha})-iH(\alpha^*,\alpha;\tau)]\right], \tag{23}$$

where $\int \mathscr{D}[\alpha(\tau)]$ designates the integration over all paths $\alpha(\tau)$, such that $\alpha(t_i)=\alpha_i$ and $\alpha(t_f)=\alpha_f$.

B. Quadratic Hamiltonian

If the Hamiltonian is at most quadratic in a and $a^\dagger$, it is possible to evaluate the path integral explicitly (Yuen[8] has calculated this propagator using a different method). The most general quadratic Hamiltonian is given by

$$H(a^\dagger,a;t)=\omega(t)a^\dagger a+f(t)a^2+f^*(t)a^{\dagger 2}+g(t)a+g^*(t)a^\dagger , \tag{24}$$

where $f(t)$ and $g(t)$ are arbitrary time-dependent functions. The evaluation of the path integral (21) corresponding to this Hamiltonian is outlined in Appendix A. The resulting expression for the propagator is

$$\begin{aligned}K(\alpha_f,t_f;\alpha_i,t_i)=\exp\Bigg[&-i\int_{t_i}^{t_f} d\tau[2f(\tau)X(\tau)+f(\tau)Z^2(\tau)+g(\tau)Z(\tau)]\\ &-\tfrac{1}{2}(|\alpha_f|^2+|\alpha_i|^2)+Y(t_f)\alpha_f^*\alpha_i+X(t_f)(\alpha_f^*)^2-i\alpha_i^2\int_{t_i}^{t_f} d\tau f(\tau)Y^2(\tau)+Z(t_f)\alpha_f^*\\ &-i\alpha_i\int_{t_i}^{t_f} d\tau[g(\tau)+2f(\tau)Z(\tau)]Y(\tau)\Bigg], \end{aligned}\tag{25}$$

where $X(t)$ satisfies the differential equation

$$\frac{dX}{dt}=-2i\omega(t)X-4if(t)X^2-if^*(t), \tag{26}$$

with $X(t_i)=0$ and

$$Y(t)=\exp\left[-i\int_{t_i}^{t} d\tau[\omega(\tau)+4f(\tau)X(\tau)]\right], \tag{27}$$

$$Z(t)=-i\int_{t_i}^{t} d\tau[g^*(\tau)+2g(\tau)X(\tau)]\exp\left[-i\int_{\tau}^{t} d\tau'[\omega(\tau')+4f(\tau')X(\tau')]\right]. \tag{28}$$

The nonlinear differential Eq. (26) for $X(t)$ can be solved if we can express $f(t)$ as

$$f(t)=\tilde{f}(t)\exp\left[2i\int_{t_i}^{t} d\tau\,\omega(\tau)\right], \tag{29}$$

where $\tilde{f}(t)$ is real or imaginary. We now consider a simple example where this condition is satisfied.

C. Degenerate parametric amplifier

The quantum statistical properties of the degenerate parametric amplifier have received considerable attention in recent years.[9] This nonlinear device is predicted to exhibit photon antibunching[10] which is a strictly quantum-mechanical effect. Squeezed states, which could prove to be useful in the efforts to detect gravitational waves, are also predicted to be generated in a degenerate parametric amplifier.[8,11]

The Hamiltonian that governs this nonlinear optical device is given by

$$H(t)=\omega a^\dagger a+\kappa(e^{2i\omega t}a^2+e^{-2i\omega t}a^{\dagger 2}), \tag{30}$$

where κ is a coupling constant and ω is the mode frequency. The Hamiltonian (30) is the same as that given by Eq. (24) if we make the following identifications:

$$\omega(t)=\omega,\quad f(t)=\kappa e^{2i\omega t},\quad g(t)=0. \tag{31}$$

Under these conditions Eq. (26) can be solved and we obtain

$$X(t)=\frac{1}{2i}e^{-2i\omega t}\tanh[2\kappa(t-t_i)], \tag{32a}$$

$$Y(t)=e^{-i\omega(t-t_i)}\operatorname{sech}[2\kappa(t-t_i)], \tag{32b}$$

$$Z(t)=0. \tag{32c}$$

On substituting from Eqs. (32a)–(32c) into Eq. (25) we obtain

$$K(\alpha_f,t_f;\alpha_i,t_i)=\{\text{sech}[2\kappa(t_f-t_i)]\}^{1/2} \times\exp\{-\tfrac{1}{2}(|\alpha_f|^2+|\alpha_i|^2)+\alpha_f^*\alpha_i e^{-i\omega(t_f-t_i)}\text{sech}[2\kappa(t_f-t_i)] -\tfrac{1}{2}i(\alpha_f^*)^2e^{-2i\omega t_f}\tanh[2\kappa(t_f-t_i)]-\tfrac{1}{2}i\alpha_i^2e^{2i\omega t_i}\tanh[2\kappa(t_f-t_i)]\} . \quad (33)$$

This expression for the propagator which we have derived using a path-integral approach can also be derived using a more conventional approach.[10]

IV. MULTIMODE PROBLEMS

A. Path integral

It is also possible to apply these techniques to problems involving more than one mode. If one is dealing with N modes the propagator becomes a function of $2N$ complex variables. In particular we have

$$K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)=\langle\vec{\alpha}_f\,|\,U(t_f,t_i)\,|\,\vec{\alpha}_i\rangle , \quad (34)$$

where $\vec{\alpha}_i$ and $\vec{\alpha}_f$ are N-component vectors with components denoted by $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_N^{(i)}$ (similarly for $\vec{\alpha}_f$), and

$$|\,\vec{\alpha}_i\rangle=|\,\alpha_1^{(i)}\rangle\otimes|\,\alpha_2^{(i)}\rangle\otimes\cdots\otimes|\,\alpha_N^{(i)}\rangle .$$

Correlation functions can be computed from this propagator in ways similar to those used in the one-mode case. One must simply evaluate more integrals.

There is also a path-integral representation for the N-mode propagator. One has

$$K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)=\int\mathscr{D}[\vec{\alpha}(\tau)]e^{iS} =\int\mathscr{D}[\alpha_1(\tau)]\cdots\int\mathscr{D}[\alpha_N(\tau)]e^{iS} , \quad (35)$$

where

$$iS=\int_{t_i}^{t_f}d\tau\left[\sum_{n=1}^{N}\tfrac{1}{2}(\dot{\alpha}_n^*\alpha_n-\alpha_n^*\dot{\alpha}_n)-iH(\vec{\alpha}^*,\vec{\alpha};\tau)\right] , \quad (36)$$

$\vec{\alpha}(t_i)=\vec{\alpha}_i$, $\vec{\alpha}(t_f)=\vec{\alpha}_f$, and if $H(a_1^\dagger,\ldots,a_n^\dagger,a_1,\ldots,a_n;\tau)$ is the normally ordered Hamiltonian for the system

$$H(\vec{\alpha}''^*,\vec{\alpha}';\tau)=\langle\vec{\alpha}''\,|H(a_1^\dagger,\ldots,a_n^\dagger,a_1,\ldots,a_n;\tau)|\,\vec{\alpha}'\rangle/\langle\vec{\alpha}''\,|\,\vec{\alpha}'\rangle . \quad (37)$$

B. Quadratic Hamiltonian

If the Hamiltonian is quadratic in $a_1,\ldots,a_N$ and $a_1^\dagger,\ldots,a_N^\dagger$ one can again explicitly evaluate the path integral. We express the Hamiltonian as

$$H=\sum_{i=1}^{N}\sum_{j=1}^{N}[\omega_{ij}(t)a_i^\dagger a_j+f_{ij}(t)a_ia_j+f_{ij}^*(t)a_i^\dagger a_j^\dagger] \quad (38)$$

and we assume that f has been chosen so that $f_{ij}(t)=f_{ji}(t)$. The detailed calculation of the propagator for this Hamiltonian is performed in Appendix B. We find that

$$K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)=\exp\left[-2i\int_{t_i}^{t_f}d\tau\,\text{Tr}[X(\tau)f(\tau)]-\tfrac{1}{2}[(\vec{\alpha}_f^*)\cdot\vec{\alpha}_f+(\vec{\alpha}_i^*)\cdot\vec{\alpha}_i]+(\vec{\alpha}_f^*)^TY(t_f)\vec{\alpha}_i +(\vec{\alpha}_f^*)^TX(t_f)\vec{\alpha}_f^*-i\int_{t_i}^{t_f}d\tau\,\vec{\alpha}_i^TY^T(\tau)f(\tau)Y(\tau)\vec{\alpha}_i\right] . \quad (39)$$

In the above equation $X(t)$ and $f(\tau)$ are $N\times N$ symmetric matrices. The elements of $f(t)$ are simply the functions $f_{ij}(t)$ which appear in the Hamiltonian. The matrix $X(t)$ satisfies the equation

$$\frac{dX}{dt}=-i(\omega X+X\omega+f^*+4XfX) , \quad (40)$$

where $\omega(t)$ is an $N\times N$ matrix whose elements are $\omega_{ij}(t)$, and $X(t_i)=0$. The $N\times N$ matrix $Y(t)$ is given by

$$Y(t)=T\exp\left[-i\int_{t_i}^{t_f}d\tau[\omega(\tau)+4X(\tau)f(\tau)]\right] . \quad (41)$$

The superscript T appearing on some of the vectors and matrices in Eq. (39) denotes transpose.

C. Parametric amplifier

The parametric amplifier with a classical pump field is a system which has been much studied in quantum optics.[12] Here we would like to use the formulas developed in the preceding section to find the propagator for this system.

The Hamiltonian we wish to consider is

$$H=\omega_1 a_1^\dagger a_1+\omega_2 a_2^\dagger a_2 +\kappa(e^{i\omega_3 t}a_1 a_2+e^{-i\omega_3 t}a_1^\dagger a_2^\dagger)\ , \tag{42}$$

where $\omega_3=\omega_1+\omega_2$. The matrices $\omega(t)$ and $f(t)$ are

$$\omega(t)=\begin{bmatrix}\omega_1 & 0\\ 0 & \omega_2\end{bmatrix}\ ,\qquad f(t)=\tfrac{1}{2}\kappa e^{i\omega_3 t}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\ . \tag{43}$$

Considering first the equation for $X(t)$, Eq. (40), we find that

$$X(t)=-\tfrac{1}{2}ie^{-i\omega_3 t}\tanh[\kappa(t-t_i)]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\ . \tag{44}$$

Rather than solve for $Y(t)$, we instead solve for the vector

$$\tilde{u}(t)=Y(t)\vec{\alpha}_i\ . \tag{45}$$

The vector $\tilde{u}$ satisfies the equation

$$\frac{d\tilde{u}}{dt}=-i(\omega\tilde{u}+4Xf\tilde{u})\ , \tag{46}$$

where $\tilde{u}(t_i)=\vec{\alpha}_i$. One finds that

$$\tilde{u}(t)=\operatorname{sech}[\kappa(t-t_i)]\begin{bmatrix}e^{-i\omega_1(t-t_i)}\alpha_1^{(i)}\\ e^{-i\omega_2(t-t_i)}\alpha_2^{(i)}\end{bmatrix}\ . \tag{47}$$

The final result for the propagator is then

$$K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)=[\operatorname{sech}\kappa(t_f-t_i)]\exp\Bigg[-\tfrac{1}{2}[(\vec{\alpha}_f^*)\cdot\vec{\alpha}_f+(\vec{\alpha}_i^*)\cdot\vec{\alpha}_i]-\tfrac{1}{2}ie^{-i\omega_3 t_f}\tanh[\kappa(t_f-t_i)](\vec{\alpha}_f^*)^T\sigma_1\vec{\alpha}_f^* +\operatorname{sech}[\kappa(t_f-t_i)](\vec{\alpha}_f^*)^T\begin{bmatrix}e^{-i\omega_1(t_f-t_i)} & 0\\ 0 & e^{-i\omega_2(t_f-t_i)}\end{bmatrix}\vec{\alpha}_i -\tfrac{1}{2}ie^{i\omega_3 t_i}\tanh[\kappa(t_f-t_i)]\vec{\alpha}_i^T\sigma_1\vec{\alpha}_i\Bigg]\ , \tag{48}$$

where $\sigma_1=\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$.

V. CONCLUSION

We have shown how a formalism incorporating coherent-state propagators and path integrals can be of use in the consideration of certain problems in nonlinear optics. Here we concentrated on the formalism itself and certain basic results for the path integrals. These are necessary steps toward the development of approximation schemes for more complicated systems. It is in these approximations that the promise of these techniques lies.

ACKNOWLEDGMENT

Research for this work was supported by Kirtland Air Force Base under Contract No. F29601-82-K-0017.

APPENDIX A

According to Eq. (21) the propagator $K(\alpha_f,t_f;\alpha_i,t_i)$ corresponding to the Hamiltonian (24) is given by

$$K(\alpha_f,t_f;\alpha_i,t_i)=\lim_{n\to\infty}\left[\frac{1}{\pi}\right]^n\int\cdots\int\left[\prod_{j=1}^{n}d^2\alpha_j\right]e^{iS_n}\,, \tag{A1}$$

where

$$iS_n=\sum_{j=1}^{n+1}[-\tfrac{1}{2}(|\alpha_j|^2+|\alpha_{j-1}|^2)+(1-i\epsilon\omega_j)\alpha_j^*\alpha_{j-1}-i\epsilon f_{j-1}\alpha_{j-1}^2-i\epsilon f_j^*\alpha_j^{*2}-i\epsilon g_{j-1}\alpha_{j-1}-i\epsilon g_j^*\alpha_j]\,. \tag{A2}$$

The α_i integrations in Eq. (A1) are lengthy but straightforward. The resulting equation is

$$K(\alpha_f,t_f;\alpha_i,t_i)=\lim_{n\to\infty}\frac{1}{\left[\prod_{i=1}^{n}(1+4i\epsilon f_iX_i)^{1/2}\right]}\times\exp\left[\sum_{j=0}^{n}\left\{i\epsilon\left[\left[\frac{f_jZ_j^2+g_jZ_j-i\epsilon g_j^2X_j}{1+4i\epsilon f_jX_j}\right]+\left[\frac{f_jY_j^2}{1+4i\epsilon f_jX_j}\right]\alpha_i^2+\left[\frac{2f_jY_jZ_j+g_jY_j}{1+4i\epsilon f_jX_j}\right]\alpha_i\right]\right\}+X_{n+1}\alpha_f^{*2}+Y_{n+1}\alpha_i\alpha_f^*+Z_{n+1}\alpha_f^*\right]\,, \tag{A3}$$

where X_j, Y_j, and Z_j satisfy the following recursion relations:

$$Z_j=-i\epsilon f_j^*+\frac{(1-i\epsilon\omega_j)^2X_{j-1}}{1+4i\epsilon f_{j-1}X_{j-1}}\,, \tag{A4}$$

$$Y_j=\frac{(1-i\epsilon\omega_j)Y_{j-1}}{1+4i\epsilon f_{j-1}X_{j-1}}\,, \tag{A5}$$

$$Z_j=-i\epsilon g_j^*+\frac{(1-i\epsilon\omega_j)(Z_{j-1}-2i\epsilon g_jX_{j-1})}{1+4i\epsilon f_{j-1}X_{j-1}}\,, \tag{A6}$$

with $X_0=Z_0=0$ and $Y_0=1$. On taking the limit $n\to\infty$, we obtain

$$\prod_{i=1}^{n}(1+4i\epsilon f_iX_i)^{1/2}\to\exp\left[2i\int_{t_i}^{t_f}d\tau\,f(\tau)X(\tau)\right]\,, \tag{A7}$$

$$\sum_{j=0}^{n}\left[\frac{-i\epsilon(f_jZ_j^2+g_jZ_j-i\epsilon g_j^2X_j)}{1+4i\epsilon f_jX_j}\right]\to -i\int_{t_i}^{t_f}d\tau\,Z(\tau)[f(\tau)Z(\tau)+g(\tau)]\,, \tag{A8}$$

$$\sum_{j=0}^{n}\left[\frac{-i\epsilon f_jY_j^2}{1+4i\epsilon f_jX_j}\right]\to -i\int_{t_i}^{t_f}d\tau\,f(\tau)Y^2(\tau)\,, \tag{A9}$$

$$\sum_{j=0}^{n}\left[\frac{-i\epsilon(2f_jY_jZ_j+g_jY_j)}{1+4i\epsilon f_jX_j}\right]\to -i\int_{t_i}^{t_f}d\tau[2f(\tau)Y(\tau)Z(\tau)+g(\tau)Y(\tau)]\,, \tag{A10}$$

$$X_{n+1},Y_{n+1},Z_{n+1}\to X(t_f),Y(t_f),Z(t_f)\,, \tag{A11}$$

and, in view of the recursion relations (A4)–(A6), the functions $X(t)$, $Y(t)$, and $Z(t)$ satisfy the differential equation

$$\frac{dX}{dt}=-2i\omega(t)X-4if(t)X^2-if^*(t)\ , \tag{A12}$$

$$\frac{dY}{dt}=-i[\omega(t)+4f(t)X(t)]Y\ , \tag{A13}$$

$$\frac{dZ}{dt}=-i[\omega(t)+4f(t)X(t)]Z-i[g^*(t)+2g(t)X(t)]\ , \tag{A14}$$

where $X(t_i)=Z(t_i)=0$ and $Y(t_i)=1$.

On substituting from Eqs. (A7)–(A11) into Eq. (A3), we obtain

$$\begin{aligned}K(\alpha_f,t_f;\alpha_i,t_i)=\exp\Big[&-i\int_{t_i}^{t_f}d\tau[2f(\tau)X(\tau)+f(\tau)Z^2(\tau)+g(\tau)Z(\tau)]-\tfrac{1}{2}(|\alpha_f|^2+|\alpha_i|^2)\\&+Y(t_f)\alpha_f^*\alpha_i+X(t_f)(\alpha_f^*)^2-i\alpha_i^2\int_{t_i}^{t_f}d\tau f(\tau)Y^2(\tau)\\&-i\alpha_i\int_{t_i}^{t_f}d\tau[g(\tau)+2f(\tau)Z(\tau)]Y(\tau)+Z(t_f)\alpha_f^*\Big]\ .\end{aligned} \tag{A15}$$

Equations (A13) and (A14) can be integrated and the resulting solutions for $Y(t)$ and $Z(t)$ are

$$Y(t)=\exp\left[-i\int_{t_i}^{t}d\tau[\omega(\tau)+4f(\tau)X(\tau)]\right]\ , \tag{A16}$$

$$Z(t)=-i\int_{t_i}^{t}d\tau g^*(\tau)[1+2X(\tau)]\exp\left[-i\int_{\tau}^{t}d\tau'[\omega(\tau')+4f(\tau')X(\tau')]\right]\ , \tag{A17}$$

where $X(t)$ is determined by solving Eq. (A12) subject to $X(t_i)=0$.

APPENDIX B

We would like to compute the propagator for the system governed by the Hamiltonian given by Eq. (38). As in the one-mode case we have that the propagator is given by

$$K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)\ \lim_{n\to\infty}\left[\frac{1}{\pi^N}\right]^n\int d\vec{\alpha}_1\cdots d\vec{\alpha}_n e^{iS_n}\ , \tag{B1}$$

where $d\vec{\alpha}_j=d^2\alpha_1^{(j)}d^2\alpha_2^{(j)}\cdots d^2\alpha_N^{(j)}$ and

$$iS_n=\sum_{l=1}^{n+1}\{-\tfrac{1}{2}[(\vec{\alpha}_l^*)\cdot\vec{\alpha}_l+(\vec{\alpha}_{l-1}^*)\cdot\vec{\alpha}_{l-1}]+(\vec{\alpha}_l^*)\cdot\vec{\alpha}_{l-1}-i\epsilon[(\vec{\alpha}_l^*)^T\omega_l\vec{\alpha}_{l-1}+\vec{\alpha}_{l-1}^Tf_{l-1}\vec{\alpha}_{l-1}+(\vec{\alpha}_l^*)^Tf_l^*(\vec{\alpha}_l^*)]\}\ . \tag{B2}$$

In the above equation $\vec{\alpha}^T$ designates the transpose of $\vec{\alpha}$ and $f_l=f(t_l)$ is an $N\times N$ matrix where $t_l=t_i+l\epsilon$.

To perform the integrations it is necessary to split each $\alpha_j^{(l)}$ into real and imaginary parts. That is, for each l we must go from a N-dimensional space, C^N (of which $\vec{\alpha}_l$ is a member), to a $2N$-dimensional space. It is best to view this space as a tensor product space $C^N\otimes C^2$. If $\eta_i\in C^N$ is the vector whose ith component is 1 and whose other components are 0, and $\nu_j\in C^2$ is the vector whose jth component is 1 and whose other component is 0, then $\vec{\alpha}\in C^N\to z\in C^N\otimes C^2$, where

$$z=\sum_{j=1}^{N}(x_j\eta_j\otimes\nu_1+y_j\eta_j\otimes\nu_2) \tag{B3}$$

and the components of $\vec{\alpha}$ are $\alpha_j=x_j+iy_j$. It is then possible to express the action as

$$iS_n=-\sum_{l=1}^{n}z_l^TM_lz_l+\sum_{l=1}^{n+1}z_l^TL_lz_{l-1}-\tfrac{1}{2}[(\vec{\alpha}_f^*)\cdot\vec{\alpha}_f+(\vec{\alpha}_i^*)\cdot\vec{\alpha}_i]-i\epsilon[\vec{\alpha}_i^Tf_i\vec{\alpha}_i+(\vec{\alpha}_f^*)^Tf_f^*\vec{\alpha}_f^*]\ , \tag{B4}$$

where $M_l = I + i\epsilon(f_l \otimes \gamma_1 + f_l^* \otimes \gamma_2)$, $L_l = (I_N - i\epsilon\omega_l) \otimes \mu$, I is the identity on $C^N \otimes C^2$, I_N is the identity on C^N, and

$$\mu = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}. \tag{B5}$$

We now want to do the integrations starting with $l=1$, then going to $l=2$ and so on. To do this we make use of the formula for the integral (assuming that it exists)

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n e^{-\vec{x}^T A \vec{x} + \vec{y}^T \vec{x}} = \frac{\pi^{n/2}}{(\det A)^{1/2}} e^{(1/4)\vec{y}^T A^{-1} \vec{y}}, \tag{B6}$$

where A is a symmetric $n \times n$ matrix and $\vec{y}$ is an n-component vector. Using this formula to do the $l=1$ integration we pick up a factor of

$$\pi^N (\det M_1)^{-1/2} \exp(\tfrac{1}{4} z_0^T L_1^T M_1^{-1} L_1 z_0)$$

and terms in the exponent which are linear and quadratic in z_2. We can express the part of the action containing z_2 (after having done the $l=1$ integration) as

$$-z_2^T M_2' z_2 + z_3^T L_2 z_2 + v_2^T z_2 , \tag{B7}$$

where

$$M_2' = M_2 - \tfrac{1}{4} L_2 M_1^{-1} L_2^T \tag{B8}$$

and

$$v_2 = \tfrac{1}{4}[L_2 M_1^{-1} L_1 z_0 + L_2 (M_1^{-1})^T L_1 z_0] . \tag{B9}$$

In general, if one has done $l-1$ of the integrations the part of the action containing z_l can be expressed as

$$-z_l^T M_l' z_l + z_{l+1}^T L_l z_l + v_l^T z_l , \tag{B10}$$

where M_l' and v_l obey the recurrence relations

$$M_{l+1}' = M_{l+1} - \tfrac{1}{4} L_{l+1} (M_l')^{-1} L_{l+1}^T , \tag{B11}$$

$$v_l = \tfrac{1}{4}[L_{l+1}(M_l'^{-1})^T + L_{l+1}(M_l'^{-1})] v_l . \tag{B12}$$

Note also that each integration contributes a factor of

$$\pi^N (\det M_l')^{-1/2} \exp(\tfrac{1}{4} v_l^T M_l'^{-1} v_l) .$$

One can show from the above recursion relations that it is possible to express M_l' and v_l in the form

$$M_l' = M_l - X_l \otimes \gamma_2 , \quad v_l = u_l \otimes \hat{e}_1 ,$$

where $\hat{e}_1 = (1/\sqrt{2})(v_1 - i v_2)$ and, to first order in ϵ, X_l, and u_l obey the recursion relations

$$X_{l+1} = X_l - i\epsilon(\omega_{l+2} X_l + X_l \omega_{l+2} + f_{l+1}^* + 4 X_l f_{l+1} X_l) , \tag{B13}$$

$$u_{l+1} = u_l - i\epsilon(\omega_{l+1} u_l + 4 X_{l-1} f_l u_l) . \tag{B14}$$

Upon taking the $\epsilon \to 0$ limit these equations become

$$\frac{dX}{dt} = -i(\omega X + X\omega + f^* + 4XfX) , \tag{B15}$$

$$\frac{du}{dt} = -i(\omega u + 4Xfu) , \tag{B16}$$

where $X(t_i) = 0$ and $u(t_i) = \sqrt{2}\,\vec{\alpha}_i$.

Upon performing all n integrations we find that

$$K(\vec{\alpha}_f, t_f; \vec{\alpha}_i, t_i) = \lim_{n\to\infty} \prod_{j=1}^{n} \frac{1}{(\det M_j')^{1/2}} \times \exp\left[-\tfrac{1}{2}[(\vec{\alpha}_i^*)\cdot\vec{\alpha}_i + (\vec{\alpha}_f^*)\cdot\vec{\alpha}_f] - z_f^T (M_{n+1}' - I) z_f + v_{n+1}^T z_f + \tfrac{1}{4}\sum_{l=1}^{n} v_l^T (M_l'^{-1}) v_l - i\epsilon[\vec{\alpha}_i^T f_i \vec{\alpha}_i + (\vec{\alpha}_f^*)^T f_f^* \vec{\alpha}_f^*] \right] . \tag{B17}$$

We now take the limit $n \to \infty$ and find that

$$\prod_{l=1}^{n} \frac{1}{(\det M')^{1/2}} \to \exp\left[-2i \int_{t_i}^{t_f} d\tau \, \mathrm{Tr}[X(\tau) f(\tau)] \right] , \tag{B18}$$

$$-z_f^T (M_{n+1}' - I) z_f \to (\vec{\alpha}_f^*)^T X(t_f) \vec{\alpha}_f^* , \tag{B19}$$

$$v_{n+1}^T z_f \to \frac{1}{\sqrt{2}} (\vec{\alpha}_f^*)^T u(t_f) , \tag{B20}$$

$$\frac{1}{4}\sum_{l=1}^{n} v_l^T(M_l'^{-1})v_l \rightarrow -\frac{1}{2}i\int_{t_i}^{t_f} d\tau\, u^T(\tau)f(\tau)u(\tau)\ . \tag{B21}$$

We can reexpress the terms involving $u(\tau)$ by defining a matrix

$$Y(t)=T\exp\left[-i\int_{t_i}^{t} d\tau[\omega(\tau)+4X(\tau)f(\tau)]\right] \tag{B22}$$

and noting that

$$u(t)=\sqrt{2}Y(t)\vec{\alpha}_i\ , \tag{B23}$$

so that $K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)$ is given by the expression in Eq. (48).

One can check that this expression is correct by observing that $K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)$ satisfies the equation

$$\begin{aligned} i\frac{\partial}{\partial t}K(\vec{\alpha},t;\vec{\beta},t_i) &= \langle\vec{\alpha}\,|\,H(t)U(t,t_i)\,|\,\vec{\beta}\rangle \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\left[\omega_{ij}\alpha_i^*\left[\frac{\partial}{\partial\alpha_j^*}+\frac{1}{2}\alpha_j\right]+f_{ij}\left[\frac{\partial}{\partial\alpha_i^*}+\frac{1}{2}\alpha_i\right]\left[\frac{\partial}{\partial\alpha_j^*}+\frac{1}{2}\alpha_j\right]+f_{ij}^*\alpha_i^*\alpha_j^*\right]K(\vec{\alpha},t;\vec{\beta},t_i) \end{aligned} \tag{B24}$$

and verifying that, indeed, the expression given by Eq. (48) does satisfy this equation.

[1]R. Rajaraman, Phys. Rep. 21C, (1975).
[2]A. Zardecki, Phys. Rev. A 22, 1664 (1980).
[3]N. Bloembergen, *Nonlinear Optics* (Benjamin, New York, 1965).
[4]J. R. Klauder, Ann. Phys. (N.Y.) 11, 123 (1960).
[5]S. S. Schweber, J. Math. Phys. (N.Y.) 3, 831 (1962).
[6]J. R. Klauder, in *Path Integrals,* proceedings of the NATO Advanced Summer Institute, edited by G. J. Papadopoulos and J. T. Devreese (Plenum, New York, 1978), p. 5.
[7]M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, Mass., 1974).
[8]Horace P. Yuen, Phys. Rev. A 13, 2226 (1976).
[9]M. Schubert and P. Wilhelmi, in *Progress in Optics,* edited by E. Wolf (North-Holland, Amsterdam, 1980), Vol. 17, p. 163; J. Perina, in *Progress in Optics,* edited by E. Wolf (North-Holland, Amsterdam, 1980), Vol. 18, p. 127.
[10]D. Stoler, Phys. Rev. Lett. 33, 1397 (1974).
[11]C. Caves, Phys. Rev. D 23, 1693 (1981).
[12]B. R. Mollow and R. J. Glauber, Phys. Rev. 160, 1076 (1967); 160, 1097 (1967).