Stochastic particle instability for electron motion in combined helical wiggler, radiation, and longitudinal wave fields

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The relativistic motion of an electron is calculated in the combined fields of a transverse helical wiggler field (axial wavelength is $\lambda_0=2\pi/k_0$) and the constant-amplitude, circularly polarized primary electromagnetic wave ($\hat{\delta}B_T,\omega,k$) propagating in the z direction. For particle velocity near the beat-wave phase velocity $\omega/(k+k_0)$ of the primary wave, it is shown that the presence of a second, moderate-amplitude longitudinal wave ($\hat{\delta}E_L,\omega,k$) or transverse electromagnetic wave ($\hat{\delta}B_2,\omega_2,k_2$) can lead to stochastic particle instability in which particles trapped near the separatrix of the primary wave undergo a systematic departure from the potential well. The condition for onset of instability is calculated, and the importance of these results for free-electron-laser (FEL) application is discussed. For development of long-pulse or steady-state free-electron lasers, the maintenance of beam integrity for an extended period of time will be of considerable practical importance. The fact that the presence of secondary, moderate-amplitude longitudinal or transverse electromagnetic waves can destroy coherent motion for certain classes of beam particles moving with velocity near $\omega/(k+k_0)$ may lead to a degradation of beam quality and concomitant modification of FEL emission properties.

I. INTRODUCTION

It is well known that stochastic instabilities can develop in systems where the particle motion is described by certain classes of nonlinear oscillator equations. Indeed, during the past several years, powerful analytic and numerical techniques have been developed that describe important features of stochastic instabilities[1-6] that occur under a wide range of physical circumstances. Particularly noteworthy is the development of systematic (secular) variations of particle action and/or energy for classes of particles that in the absence of the appropriate perturbation force undergo coherent (e.g., nonlinear periodic) motion. Moreover, the "normal" coherent particle motion can be drastically modified by the stochastic instability and develop several chaotic features.

In the present article, we consider the possible development of stochastic instability in circumstances relevant to sustained free-electron-laser (FEL) radiation generation by an electron beam in a helical wiggler field.[7-12]. In particular, we consider a tenuous relativistic electron beam with negligibly small equilibrium self-fields propagating in the z direction through a steady, monochromatic radiation field. The relativistic dynamics of a typical beam electron is investigated for particle motion in combined, constant-amplitude, electromagnetic fields consisting of (a) an equilibrium transverse helical wiggler field with axial wavelength $\lambda_0=2\pi/k_0$ [Eq. (2)], (b) circularly polarized transverse electromagnetic wave propagating in the z direction [Eqs. (3) and (6)], and (c) longitudinal electrostatic wave propagating in the z direction [Eq. (7)]. Both the transverse and longitudinal waves are assumed to have frequency ω and wave number k and could represent the nonlinear saturated state of a FEL instability. For zero transverse canonical momenta[7] $P_x=0=P_y$, the exact equation of motion for the axial coordinate $\zeta=(k+k_0)z-\omega t$ reduces to Eq. (27), where $v_p=\omega/(k+k_0)$ is the (beat-wave) phase velocity of the combined wiggler field and transverse electromagnetic wave. For moderate values of field amplitude and particle velocity dz/dt in the neighborhood of $v_p=\omega/(k+k_0)$ [Eq. (36)], the dynamical equation (27) can be approximated to leading order by [Eq. (45)]

$$\frac{d^2\zeta}{d\tau^2}+\sin\zeta=-3\frac{\omega}{c(k+k_0)}\epsilon_T^{1/2}\frac{d\zeta}{d\tau}\frac{d^2\zeta}{d\tau^2}-\delta_L\frac{k+k_0}{k}\times\sin\left[\frac{k}{k+k_0}\left[\zeta-\frac{k_0\omega}{k\hat{\omega}_T}\tau\right]\right],$$

where $\tau=\hat{\omega}_T t$, the small parameter ϵ_T [Eq. (23)] measures the strength of the transverse electromagnetic field, and $\delta_L=\hat{\omega}_L^2/\hat{\omega}_T^2$ [Eq. (34)] measures the strength of the longitudinal field. Here, $\hat{\omega}_T=\text{const}$ [Eq. (28)] is the bounce frequency of a particle near the bottom of the beat-wave potential in the limit where $\epsilon_T\to 0$ and $\delta_L\to 0$.

The assumptions and analysis leading to the approximate dynamical equation (45) are presented in Secs. II and III. In Sec. IV, we investigate the stochastic particle instability associated with the δ_L driving term in Eq. (45) assuming that $\delta_L\ll 1$. In the absence of longitudinal wave ($\delta_L=0$), it is clear that the equation of motion is conservative with $(d/d\tau)(H_0+H_1)=0$, where $H_0=(\frac{1}{2})(d\zeta/d\tau)^2-\cos\zeta$ is the zeroth-order pendulum energy, and $H_1=[\omega/c(k+k_0)]\epsilon_T^{1/2}(d\zeta/d\tau)^3$ is the (small) conservative energy modulation produced by the $\epsilon_T^{1/2}$ driving term in Eq. (45). On the other hand, for $\delta_L\neq 0$, the right-hand side of Eq. (45) appropriately averaged over the zeroth-order pendulum motion can lead to systematic (secular) changes in the energy H or action J for a selected range of system parameters. The associated stochastic instability is examined in detail in Sec. IV. Introducing the action J [Eq. (59)] and bounce frequency $\omega_T(J)$ [Eq. (61)] associated with the zeroth-order pendulum motion $d^2\zeta/d\tau^2+\sin\zeta=0$, it is shown for $\delta_L\ll 1$ and $k_0\omega/k\hat{\omega}_T\gg 1$ that stochastic instability develops for (low) values of bounce frequency satisfying [Eq. (81)]

$$\omega_T(J)\lesssim(\omega_T)_{\text{cr}}=\pi\hat{\omega}_T\left[\ln\frac{16\pi^2}{\delta_L}+\frac{\pi k_0\omega}{2(k+k_0)\hat{\omega}_T}\right]^{-1}.$$

That is, stochastic instability develops in a narrow energy band $(\Delta H)_{\text{cr}}=(1-H_0)_{\text{cr}}$ near the separatrix, and particles in this region undergo a systematic departure from their "trapped" zeroth-order pendulum motion.

For analytic simplicity, the parameter δ_L is assumed to be small ($\delta_L\ll 1$) in the analysis in Sec. IV. Therefore, the energy range of particles experiencing stochastic instability is correspondingly small and located near the separatrix of the primary beat wave. As δ_L is increased to values approaching unity, however, the instability range is expected to increase significantly, and deeply trapped particles will also undergo a systematic departure from the potential well. The dynamical equation (45) is currently under investigation numerically in this parameter range.

An analogous stochastic instability can also develop in circumstances where the longitudinal electric field is negligibly small, but a second, moderate-amplitude electromagnetic wave is present.[6] The relevant assumptions and features of the final dynamical equation are outlined in Sec. V in circumstances where $\delta E_z=0$ and two constant-amplitude, circularly polarized electromagnetic waves $(\hat{\delta B}_1,\omega_1,k_1)$ and $(\hat{\delta B}_2,\omega_2,k_2)$ are present. For particle velocity dz/dt near to the beat-wave phase velocity $\omega_1/(k_0+k_1)$ of the primary wave, the exact dynamical equation (85) can be approximated by [Eq. (89)]

$$\frac{d^2\zeta}{d\tau^2}+\sin\zeta=-\frac{k_1+k_0}{k_2+k_0}\delta_2\sin\left[\frac{k_2+k_0}{k_1+k_0}\zeta-\frac{\Delta\omega}{\hat{\omega}_{T1}}\tau\right],$$

where $\zeta=(k_1+k_0)z-\omega_1 t$, $\tau=\hat{\omega}_{T1}t$, $\delta_2=\hat{\omega}_{T2}^2/\hat{\omega}_{T1}^2$, $\Delta\omega=[(k_1+k_0)\omega_2-(k_2+k_0)\omega_1]/(k_1+k_0)$, and $\hat{\omega}_{T1}$ and $\hat{\omega}_{T2}$ are the bounce frequencies [Eq. (88)] in the troughs of the two beat waves. Apart from the (conservative) $\epsilon_T^{1/2}$ term in Eq. (45), the dynamical equation (89) is similar in form to Eq. (45), and can also lead to stochastic instability for particles near the separatrix of the primary beat wave. Moreover, for δ_2 of order unity, deeply trapped particles in the primary wave can be "untrapped" by the second electromagnetic wave.

In summary, we have considered electron motion in the combined fields of a helical wiggler and constant-amplitude, circularly polarized primary electromagnetic wave. For particle velocity near the beat-wave phase velocity of the primary wave, it is shown that the presence of a second, moderate-amplitude longitudinal wave or transverse electromagnetic wave can lead to stochastic particle instability in which particles trapped near the separatrix of the primary wave undergo a systematic departure from the potential well. The condition for onset of instability has been calculated [Eq. (80)]. The importance of these results for FEL applications is evident. For development of long-pulse or steady-state free-electron lasers, the maintenance of beam integrity over an extended period of time will be of considerable practical importance. The fact that the presence of secondary, moderate-amplitude longitudinal or transverse electromagnetic waves can destroy coherent motion for certain classes of beam particles moving with velocity near $\omega/(k+k_0)$ may lead to a degradation of beam quality and concommitant modification of FEL emission properties.

II. ELECTROMAGNETIC-FIELD CONFIGURATION AND BASIC ASSUMPTIONS

A. Electromagnetic-field configuration

Consider a tenuous relativistic electron beam with negligibly small equilibrium self-fields propagating in the z direction. In the present analysis, we examine the relativistic motion of a typical beam electron in the presence of a helical equilibrium wiggler field and a constant-amplitude circularly polarized transverse electromagnetic wave propagating in the z direction. All spatial variations of field quantities are assumed to be in the z direction. The total magnetic field $\vec{B}(\vec{x},t)$ is expressed as

$$\vec{B}(\vec{x},t)=\vec{B}_0(\vec{x})+\delta\vec{B}_T(\vec{x},t)\ , \tag{1}$$

where the helical wiggler field $\vec{B}_0(\vec{x})$ is given by

$$\vec{B}_0(\vec{x})=B_w(\cos k_0 z\,\hat{e}_x+\sin k_0 z\,\hat{e}_y)\ , \tag{2}$$

and the magnetic-field components of the transverse electromagnetic wave are expressed as

$$\delta\vec{B}_T(\vec{x},t)=\hat{\delta}B_T[\cos(kz-\omega t)\hat{e}_x-\sin(kz-\omega t)\hat{e}_y]\ . \tag{3}$$

In Eqs. (2) and (3), the wiggler amplitude B_w and the amplitude $\hat{\delta}B_T$ of the circularly polarized electromagnetic wave are assumed to be constant (independent of $\vec{x}$ and t). In this regard, we emphasize that $B_w=\text{const}$ is only a valid approximation, strictly speaking, close to the magnetic axis where[13]

$$k_0^2(x^2+y^2)\ll 1\ . \tag{4}$$

Throughout the present analysis, it is assumed that Eq. (4) is satisfied.

With regard to the wave electric field $\delta\vec{E}(\vec{x},t)$ we allow for both transverse and longitudinal components, i.e.,

$$\delta\vec{E}(\vec{x},t)=\delta\vec{E}_T(\vec{x},t)+\delta\vec{E}_L(\vec{x},t)\ . \tag{5}$$

The transverse electric field $\delta\vec{E}_T(\vec{x},t)$ consistent with Eq. (3) and Maxwell's equation $\vec{\nabla}\times\delta\vec{E}_T=-(1/c)(\partial/\partial t)\delta\vec{B}_T$ is given by

$$\delta\vec{E}_T(\vec{x},t)=-\frac{\omega}{ck}\hat{\delta}B_T[\sin(kz-\omega t)\hat{e}_x+\cos(kz-\omega t)\hat{e}_y]\ , \tag{6}$$

where $\hat{\delta}B_T=\text{const}$. In addition, it is assumed that a constant-amplitude longitudinal wave component exists with

$$\delta\vec{E}_L(\vec{x},t)=\hat{e}_z\hat{\delta}E_L\sin(kz-\omega t)\ , \tag{7}$$

where $\hat{\delta}E_L=\text{const}$.

The electromagnetic wave fields described by Eqs. (3), (6), and (7) correspond to a circularly polarized transverse electromagnetic wave propagating in the z direction with $\hat{\delta}B_T=\text{const}$, combined with a constant-amplitude longitudinal wave with $\hat{\delta}E_L=\text{const}$, also propagating in the z direction. Both waves are assumed to have frequency ω and wave number k and could represent the nonlinear saturated monochromatic wave state of a FEL instability.

B. Transverse electron motion

For the electromagnetic-field configuration described in Sec. II A, the transverse canonical momenta P_x and P_y are exact single-particle invariants with[7]

$$P_x=p_x-\frac{e}{c}A_x(z,t)=\text{const}\ , \tag{8}$$

$$P_y=p_y-\frac{e}{c}A_y(z,t)=\text{const}\ . \tag{9}$$

In Eqs. (8) and (9), the vector potential $\vec{A}=A_x\hat{e}_x+A_y\hat{e}_y$ satisfies $\vec{\nabla}\times\vec{A}=\vec{B}_0+\delta\vec{B}_T$, where $\vec{B}_0(\vec{x})$ and $\delta\vec{B}_T(\vec{x},t)$ are defined in Eqs. (2) and (3), i.e.,

$$A_x(z,t)=-(B_w/k_0)\cos k_0 z+(\hat{\delta}B_T/k)\cos(kz-\omega t)\ , \tag{10}$$

$$A_y(z,t)=-(B_w/k_0)\sin k_0 z-(\hat{\delta}B_T/k)\sin(kz-\omega t)\ . \tag{11}$$

Moreover, the mechanical momentum $\vec{p}$ and particle velocity $\vec{v}=d\vec{x}/dt$ are related by $\vec{p}=\gamma m\vec{v}$, where the relativistic mass factor γ is defined by

$$\gamma=\left[1+\frac{p_x^2}{m^2c^2}+\frac{p_y^2}{m^2c^2}+\frac{p_z^2}{m^2c^2}\right]^{1/2}\ , \tag{12}$$

where m is the electron rest mass and c is the speed of light in vacua.

Throughout the present analysis, we assume that the transverse electron motion is characterized by the cold-beam constraints,[7,9] $P_x=0=P_y$, so that Eqs. (8) and (9) give for the transverse particle momentum

$$p_x=\gamma m v_x=-(eB_w/ck_0)\cos k_0 z+(e\hat{\delta}B_T/ck)\cos(kz-\omega t)\ , \tag{13}$$

$$p_y=\gamma m v_y=-(eB_w/ck_0)\sin k_0 z - (e\hat{\delta}B_T/ck)\sin(kz-\omega t) \ . \qquad (14)$$

In Sec. III, Eqs. (13) and (14) will be used to eliminate the transverse particle dynamics in the axial equation of motion for dp_z/dt. Substituting Eqs. (13) and (14) into Eq. (12), the relativistic mass factor γ can be expressed as

$$\gamma=\left[1+\left[\frac{eB_w}{mc^2k_0}\right]^2+\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]^2-2\left[\frac{eB_w}{mc^2k_0}\right]\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]\cos[(k+k_0)z-\omega t]+\frac{p_z^2}{m^2c^2}\right]^{1/2} . \qquad (15)$$

In deriving Eqs. (13)–(15), no approximation has been made regarding the size of the dimensionless parameters $b_w^2=(eB_w/mc^2k_0)^2$ and $b_T^2=(e\hat{\delta}B_T/mc^2k)$.[2] In typical applications, however, $b_T^2 \ll 1$ and $b_w^2 \lesssim 1$.

For future reference, Eq. (15) can be used to express γ in terms of z and dz/dt. Defining $\zeta=(k+k_0)z-\omega t$, and making use of $d\zeta/dt=(k+k_0)dz/dt-\omega$ and $p_z=\gamma m\, dz/dt$, Eq. (15) readily gives

$$\gamma^2=\left[1+\left[\frac{eB_w}{mc^2k_0}\right]^2+\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]^2-2\left[\frac{eB_w}{mc^2k_0}\right]\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]\cos\zeta\right]\left[1-\frac{1}{c^2(k+k_0)^2}\left[\frac{d\zeta}{dt}+\omega\right]^2\right]^{-1} . \qquad (16)$$

III. AXIAL EQUATION OF MOTION

A. Exact equation of motion

The axial equation of motion for an electron moving in the electromagnetic-field configuration described in Sec. II A is given by

$$\frac{dp_z}{dt}=-\frac{e}{c}\{v_x[B_{0y}(z)+\delta B_y(z,t)]-v_y[B_{0x}(z)+\delta B_x(z,t)]\}-e\delta E_z(z,t) \ , \qquad (17)$$

where $\vec{B}_0(\vec{x})$, $\delta\vec{B}(\vec{x},t)$, and $\delta E_z(z,t)$ are defined in Eqs. (2), (3), and (7). Making use of Eqs. (13) and (14) to eliminate $v_x=p_x/\gamma m$ and $v_y=p_y/\gamma m$, and combining all magnetic-field terms in Eq. (17), the axial equation of motion can be expressed as

$$\frac{dp_z}{dt}=\frac{-mc^2(k+k_0)}{\gamma}\left[\frac{eB_w}{mc^2k_0}\right]\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]\sin[(k+k_0)z-\omega t]-e\hat{\delta}E_L\sin(kz-\omega t) \ . \qquad (18)$$

It is clear from Eq. (18) that the wiggler and transverse electromagnetic-field terms have combined to form a beat wave with effective phase velocity $v_p=\omega/(k+k_0)$. In the special case where $\omega\simeq kc$ and the axial motion is nearly resonant with the beat wave $(dz/dt=v_z\simeq v_p)$, we obtain the familiar consistency condition $k\simeq k_0/(1-v_z/c)$ for the upshifted wave number.

For present purposes, it is convenient to rewrite Eq. (18) in the frame of reference of the beat wave. We define the dimensionless axial coordinate

$$\zeta=(k+k_0)z-\omega t \ , \qquad (19)$$

where $d\zeta/dt=(k+k_0)dz/dt-\omega$. Then, expressing

$$\frac{dp_z}{dt}=\left[\frac{d}{dt}\right]\left[\frac{\gamma m\,dz}{dt}\right]=(k+k_0)^{-1}\left[\frac{d}{dt}\right]\left[\gamma m\left[\frac{d\zeta}{dt}+\omega\right]\right] ,$$

Eq. (18) can be rewritten in the equivalent form

$$\frac{d^2\zeta}{dt^2}+\frac{1}{\gamma}\frac{d\gamma}{dt}\left[\frac{d\zeta}{dt}+\omega\right]=-\frac{c^2(k+k_0)^2}{\gamma^2}\left[\frac{eB_w}{mc^2k_0}\right]\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]\sin\zeta - \frac{(k+k_0)}{k\gamma}\left[\frac{ek\hat{\delta}E_L}{m}\right]\sin\left[\frac{k}{k+k_0}\left[\zeta-\frac{k_0}{k}\omega t\right]\right] . \qquad (20)$$

The expression for γ in Eq. (16) is used to eliminate $d\gamma/dt$ in favor of (ζ, $d\zeta/dt$, $d^2\zeta/dt^2$). After some straightforward algebra that makes twofold use of Eq. (16), we find

$$\frac{1}{\gamma}\frac{d\gamma}{dt}=\left[\frac{1}{c^2(k+k_0)^2}\left[\frac{d\zeta}{dt}+\omega\right]\frac{d^2\zeta}{dt^2}+\frac{\sin\zeta}{\gamma^2}\left[\frac{eB_w}{mc^2k_0}\right]\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]\frac{d\zeta}{dt}\right]\left[1-\frac{1}{c^2(k+k_0)^2}\left[\frac{d\zeta}{dt}+\omega\right]^2\right]^{-1}. \tag{21}$$

Making use of Eq. (21) to eliminate $(1/\gamma)(d\gamma/dt)(d\zeta/dt+\omega)$ in Eq. (20) gives

$$\frac{d^2\zeta}{dt^2}=\frac{-c^2(k+k_o)^2}{\gamma^2}\left[\frac{eB_w}{mc^2k_0}\right]\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]\left[1-\frac{\omega}{c^2(k+k_0)^2}\left[\frac{d\zeta}{dt}+\omega\right]\right]\sin\zeta$$
$$-\frac{(k+k_0)}{k\gamma}\left[\frac{ek\hat{\delta}E_L}{m}\right]\left[1-\frac{1}{c^2(k+k_0)^2}\left[\frac{d\zeta}{dt}+\omega\right]^2\right]\sin\left[\frac{k}{k+k_0}\left[\zeta-\frac{k_0}{k}\omega t\right]\right], \tag{22}$$

where $\gamma(\zeta,d\zeta/dt)$ is defined in Eq. (16).

Introducing the dimensionless parameter ϵ_T defined by

$$\epsilon_T\equiv\left[\frac{eB_w}{mc^2k_0}\right]\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]\left[1+\left[\frac{eB_w}{mc^2k_0}\right]^2+\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]^2\right]^{-1}, \tag{23}$$

the expression for γ in Eq. (16) readily reduces to

$$\frac{1}{\gamma^2}=\left[1-\frac{1}{c^2(k+k_0)^2}\left[\frac{d\zeta}{dt}+\omega\right]\right]^2(1-2\epsilon_T\cos\zeta)^{-1}\left[1+\left[\frac{eB_w}{mc^2k_0}\right]^2+\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]^2\right]^{-1}. \tag{24}$$

The (small) dimensionless parameter ϵ_T defined in Eq. (23) is clearly a measure of the strength of the combined transverse electromagnetic and wiggler fields in the equation of motion (22). It is also useful to introduce the dimensionless parameter ϵ_L defined by

$$\epsilon_L\equiv\left[\frac{ek\hat{\delta}E_L}{mc^2(k+k_0)^2}\right]\left[1+\left[\frac{eB_w}{mc^2k_0}\right]^2+\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]^2\right]^{-1/2}, \tag{25}$$

which characterizes the strength of the longitudinal field contribution in Eq. (22). Introducing the normalized frequency Ω,

$$\Omega\equiv\frac{\omega}{c(k+k_0)}, \tag{26}$$

the axial equation of motion (22) can be expressed as

$$\frac{d^2\zeta}{dt'^2}+\frac{\epsilon_T\sin\zeta}{(1-2\epsilon_T\cos\zeta)}\left[1-\Omega\left[\frac{d\zeta}{dt'}+\Omega\right]\right]\left[1-\left[\frac{d\zeta}{dt'}+\Omega\right]^2\right]$$
$$+\frac{\epsilon_L}{(1-2\epsilon_T\cos\zeta)^{1/2}}\left[1-\left[\frac{d\zeta}{dt'}+\Omega\right]^2\right]^{3/2}\frac{(k+k_0)}{k}\sin\left[\frac{k}{k+k_0}\left[\zeta-\frac{k_0}{k}\Omega t'\right]\right]=0, \tag{27}$$

where $t'=c(k+k_0)t$, $\zeta=(k+k_0)z-\Omega t'$, and ϵ_T and ϵ_L are defined in Eqs. (23) and (25).

Equation (27) is the *exact* dynamical equation for the axial motion assuming that the transverse electromagnetic wave [Eqs. (3) and (6)] and the longitudinal electrostatic wave [Eq. (7)] have constant amplitudes $\hat{\delta}B_T=$const and $\hat{\delta}E_L=$const. No assumption has been made in deriving Eq. (27) from Eq. (17) that ϵ_T and ϵ_L are small parameters. Moreover, the factors in Eq. (27) proportional to powers of $[1-(d\zeta/dt'+\omega)^2]^{1/2}=(1-v_z^2/c^2)^{1/2}$ are related to mass modifications associated with the relativistic axial motion. Here $v_z = dz/dt$ is the axial velocity. Equation (27) can be solved analytically (in an approximate sense) or numerically for a broad range of system parameters of practical interest. In Secs. III B and IV, we will solve Eq. (27) iteratively in circumstances where the axial velocity v_z is close to resonance with the beat-wave phase velocity $v_p = \omega/(k+k_0)$, i.e., in circumstances where the normalized axial velocity $d\zeta/dt'$ is small with $|\Omega\, d\zeta/dt'| \ll (1-\Omega^2)$ [Eq. (36)]. In this case, it is useful to rewrite Eq. (27) in terms of the effective transverse and longitudinal bounce frequencies defined by

$$\hat{\omega}_T^2 \equiv c^2(k+k_0)^2(1-\Omega^2)^2\epsilon_T$$

$$= c^2(k+k_0)^2\left[1-\frac{\omega^2}{c^2(k+k_0)^2}\right]^2\left[\frac{eB_w}{mc^2k_0}\right]\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]\left[1+\left[\frac{eB_w}{mc^2k_0}\right]^2+\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]^2\right]^{-1}, \tag{28}$$

and

$$\hat{\omega}_L^2 \equiv c^2(k+k_0)^2(1-\Omega^2)^{3/2}\epsilon_L$$

$$= \left[1-\frac{\omega^2}{c^2(k+k_0)^2}\right]^{3/2}\left[\frac{ek\hat{\delta}E_L}{m}\right]\left[1+\left[\frac{eB_w}{mc^2k_0}\right]^2+\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]^2\right]^{-1/2}. \tag{29}$$

Substituting Eqs. (28) and (29) in Eq. (27) gives the exact dynamical equation

$$\frac{d^2\zeta}{dt^2}+\omega_T^2(\zeta,\dot{\zeta})\sin\zeta+\omega_L^2(\zeta,\dot{\zeta})\frac{(k+k_0)}{k}\sin\left[\frac{k}{k+k_0}\left[\zeta-\frac{k_0}{k}\omega t\right]\right]=0\,, \tag{30}$$

where $\omega_T^2(\zeta,\dot{\zeta})$ and $\omega_L^2(\zeta,\dot{\zeta})$ are defined by

$$\omega_T^2(\zeta,\dot{\zeta}) \equiv \frac{\hat{\omega}_T^2}{(1-2\epsilon_T\cos\zeta)}\left[1-\Omega\epsilon_T^{1/2}\frac{d\zeta}{d\tau}\right]\left[1-2\Omega\epsilon_T^{1/2}\frac{d\zeta}{dt}-(1-\Omega^2)\epsilon_T\left[\frac{d\zeta}{dt}\right]^2\right], \tag{31}$$

$$\omega_L^2(\zeta,\dot{\zeta}) \equiv \frac{\hat{\omega}_L^2}{(1-2\epsilon_T\cos\zeta)^{1/2}}\left[1-2\Omega\epsilon_T^{1/2}\frac{d\zeta}{d\tau}-(1-\Omega^2)\epsilon_T\left[\frac{d\zeta}{d\tau}\right]^2\right]^{3/2}. \tag{32}$$

In Eqs. (31) and (32), $\Omega = \omega/c(k+k_0)$, ϵ_T is defined in Eq. (23), and $\tau = \hat{\omega}_T t$.

It is clear from Eq. (30) that the exact axial equation of motion has the form of a nonlinear equation for coupled pendula with amplitude- and velocity-dependent frequencies $\omega_T(\zeta,\dot{\zeta})$ and $\omega_L(\zeta,\dot{\zeta})$.

B. Approximate equation of motion

For present purposes, we now impose the (weak) restriction that the amplitude $\hat{\delta}B_T$ of the radiation field be sufficiently weak that

$$\epsilon_T \ll 1\,, \tag{33}$$

where ϵ_T is defined in Eq. (23). We further assume that the longitudinal electric field $\hat{\delta}E_L$ is weak in comparison with the transverse electromagnetic field in the sense that

$$\delta_L \equiv \frac{\hat{\omega}_L^2}{\hat{\omega}_T^2} = \frac{\epsilon_L}{(1-\Omega^2)^{1/2}\epsilon_T} \ll 1\,. \tag{34}$$

Substituting Eqs. (28) and (29) into Eq. (34) readily gives the requirement

$$\delta_L = \left[\frac{ek\hat{\delta}E_L}{mc^2(k+k_0)^2}\right]\left[\frac{mc^2k_0}{eB_w}\right]\left[\frac{mc^2k}{e\hat{\delta}B_T}\right]$$

$$\times\left[1+\left[\frac{eB_w}{mc^2k_0}\right]^2+\left[\frac{e\hat{\delta}B_T}{mc^2k}\right]^2\right]^{1/2}\left[1-\frac{\omega^2}{c^2(k+k_0)^2}\right]^{-1/2} \ll 1\,. \tag{35}$$

Finally, for present purposes, we also assume that the axial electron velocity $v_z = dz/dt$ is relatively close to the beat-wave phase velocity $v_p = \omega/(k+k_0)$. Specifically, in Eq. (27) [or Eq. (30)] it is assumed that

$$\left| \Omega \frac{d\zeta}{dt'} \right| \ll (1-\Omega^2) , \tag{36}$$

where $t' = c(k+k_0)t$. Equivalently, defining

$$\tau \equiv \hat{\omega}_T t = c(k+k_0)(1-\Omega^2)\epsilon_T^{1/2} t ,$$

Eq. (36) can be expressed as

$$\left| \epsilon_T^{1/2} \Omega \frac{d\zeta}{d\tau} \right| \ll 1 , \tag{37}$$

where $\Omega = \omega/c(k+k_0)$.

The exact dynamical equation (30) is now simplified within the context of Eqs. (33), (34), and (37). To leading order, we approximate $\omega_L^2(\zeta,\dot{\zeta}) = \hat{\omega}_L^2 \equiv \delta_L \hat{\omega}_T^2$ and $\omega_T^2(\zeta,\dot{\zeta}) = \hat{\omega}_T^2(1 - 3\Omega\epsilon_T^{1/2} d\zeta/d\tau)$ in Eqs. (31) and (32). Equation (30) then reduces to

$$\frac{d^2\zeta}{dt^2} + \hat{\omega}_T^2 \sin\zeta = 3\Omega\hat{\omega}_T^2 \epsilon_T^{1/2} \frac{d\zeta}{d\tau} \sin\zeta - \delta_L \hat{\omega}_T^2 \frac{(k+k_0)}{k} \sin\left[\frac{k}{k+k_0} \left[\zeta - \frac{k_0}{k}\omega t \right] \right] . \tag{38}$$

Introducing the dimensionless time variable

$$\tau = \hat{\omega}_T t , \tag{39}$$

Eq. (38) can be expressed as

$$\frac{d^2\zeta}{d\tau^2} + \sin\zeta = 3\Omega\epsilon_T^{1/2} \frac{d\zeta}{d\tau} \sin\zeta - \delta_L \frac{(k+k_0)}{k} \sin\left[\frac{k}{k+k_0} \left[\zeta - \frac{k_0}{k} \frac{\omega}{\hat{\omega}_T} \tau \right] \right] , \tag{40}$$

where $\Omega = \omega/c(k+k_0)$, and $\epsilon_T \ll 1$ and $\delta_L \ll 1$ have been assumed. Since $\tau = \hat{\omega}_T t$, we note that time is measured in the basic unit $\hat{\tau}_T = \hat{\omega}_T^{-1}$, which corresponds to the bounce time of an electron near the bottom of the beat-wave potential well in Eq. (40).

Since $\epsilon_T \ll 1$ and $\delta_L \ll 1$ are assumed in Eq. (40), the lowest-order axial motion is determined from the pendulum equation $d^2\zeta/d\tau^2 + \sin\zeta = 0$. In an iterative sense, replacing $\sin\zeta$ on the right-hand side of Eq. (40) by $-d^2\zeta/d\tau^2$, the equation of motion (40) can be approximated by

$$\frac{d^2\zeta}{d\tau^2} + \sin\zeta = -3\Omega\epsilon_T^{1/2} \frac{d\zeta}{d\tau} \frac{d^2\zeta}{d\tau^2} - \delta_L \frac{(k+k_0)}{k} \sin\left[\frac{k}{k+k_0} \left[\zeta - \frac{k_0}{k} \frac{\omega}{\hat{\omega}_T} \tau \right] \right] . \tag{41}$$

Defining an effective energy H by

$$H = H_0 + H_1$$
$$= \frac{1}{2} \left[\frac{d\zeta}{d\tau} \right]^2 - \cos\zeta + \Omega\epsilon_T^{1/2} \left[\frac{d\zeta}{d\tau} \right]^3 , \tag{42}$$

and multiplying Eq. (41) by $d\zeta/d\tau$, we obtain

$$\frac{dH}{d\tau} = -\delta_L \frac{(k+k_0)}{k} \frac{d\zeta}{d\tau} \times \sin\left[\frac{k}{k+k_0} \left[\zeta - \frac{k_0}{k} \frac{\omega}{\hat{\omega}_T} \tau \right] \right] . \tag{43}$$

In Eq. (42), $H_0 \equiv (\frac{1}{2})(d\zeta/d\tau)^2 - \cos\zeta$ is the zeroth-order pendulum energy, and H_1 represents the small conservative energy modulation proportional to $\epsilon_T^{1/2}$.

In the absence of longitudinal wave ($\delta_L = 0$), it is clear from Eqs. (41)–(43) that the equation of motion (41) is conservative with $dH/d\tau = 0$. On the other hand, for $\delta_L \neq 0$, the right-hand side of Eq. (43), appropriately averaged over the zeroth-order motion, can lead to systematic (secular) changes in the energy H for a selected range of system parameters. This property and the associated stochastic particle motion are discussed in Sec. IV. For future reference, it is useful to simplify the notation in Eqs. (41)–(43). Defining $k' = k + k_0$, and introducing the dimensionless phase velocity V_p,

$$V_p \equiv \frac{k_0}{k}\frac{\omega}{\hat{\omega}_T} , \tag{44}$$

the equation of motion (41) becomes

$$\frac{d^2\zeta}{d\tau^2}+\sin\zeta=-3\frac{\omega}{ck'}\epsilon_T^{1/2}\frac{d\zeta}{d\tau}\frac{d^2\zeta}{d\tau^2} -\delta_L\frac{k'}{k}\sin\left[\frac{k}{k'}(\zeta-V_p\tau)\right] \tag{45}$$

and the time rate of change of energy can be expressed as

$$\frac{dH}{d\tau}=-\delta_L\frac{k'}{k}\frac{d\zeta}{d\tau}\sin\left[\frac{k}{k'}(\zeta-V_p\tau)\right] , \tag{46}$$

where

$$H=H_0+H_1 = \frac{1}{2}\left[\frac{d\zeta}{d\tau}\right]^2-\cos\zeta+\frac{\omega}{ck'}\epsilon_T^{1/2}\left[\frac{d\zeta}{d\tau}\right]^3 . \tag{47}$$

For $\omega \simeq kc$, note from Eq. (44) that $V_p \simeq k_0 c/\hat{\omega}_T \gg 1$.

IV. STOCHASTIC INSTABILITY

A. Zeroth-order pendulum equation

In this section, we briefly summarize properties of the solutions to the approximate dynamical equation (45) in the limit $\epsilon_T \to 0$ and $\delta_L \to 0$, which gives the pendulum equation[11,12]

$$\frac{d^2\zeta}{d\tau^2}+\sin\zeta=0 , \tag{48}$$

where $\tau=\hat{\omega}_T t$ and $\hat{\omega}_T$ is defined in Eq. (28). The energy-conservation relation associated with Eq. (48) is given by

$$\frac{1}{2}\left[\frac{d\zeta}{d\tau}\right]^2-\cos\zeta=H_0 , \tag{49}$$

where $H_0=$ const [Eq. (46)]. Equation (49) can also be expressed as

$$\frac{1}{4}\left[\frac{d\zeta}{d\tau}\right]^2=\kappa^2-\sin^2\frac{\zeta}{2} , \tag{50}$$

where

$$\kappa^2 \equiv \tfrac{1}{2}(1+H_0) . \tag{51}$$

The solution to Eq. (50) can be expressed in terms of the elliptic integrals $F(\eta,\kappa)$ and $E(\eta,\kappa)$ where

$$F(\eta,\kappa)=\int_0^\eta \frac{d\eta'}{(1-\kappa^2\sin^2\eta')^{1/2}} , \tag{52}$$

$$E(\eta,\kappa)=\int_0^\eta d\eta'(1-\kappa^2\sin^2\eta')^{1/2} . \tag{53}$$

We now solve Eq. (50), distinguishing two cases: (1) trapped-particle orbits ($\kappa^2<1$) and (2) untrapped orbits ($\kappa^2>1$).

(1) Trapped particle orbits ($\kappa^2<1$). Introducing the coordinate η defined by

$$\kappa\sin\eta=\sin\frac{\zeta}{2} , \tag{54}$$

Eq. (50) can be expressed as

$$\left[\frac{d\eta}{d\tau}\right]^2=(1-\kappa^2\sin^2\eta) , \tag{55}$$

which has the solution for $\eta(\tau)$

$$F(\eta,\kappa)=F_0+\tau , \tag{56}$$

where $\eta=\sin^{-1}[(1/\kappa)\sin\zeta/2]$, $F_0 \equiv F(\eta(\tau=0),\kappa)$, and $F(\eta,\kappa)$ is the elliptic integral of the first kind defined in Eq. (52). Several properties of the (periodic) trapped particle motion can be determined directly from Eqs. (50), (54), and (56). For example, it is readily shown that the normalized velocity in the beat-wave frame is given by

$$\frac{d\zeta}{d\tau}=2\kappa\,\mathrm{cn}(F_0+\tau) , \tag{57}$$

where $\mathrm{cn}(F_0+\tau)=[1-\mathrm{sn}^2(F_0+\tau)]^{1/2}$, and $\mathrm{sn}(F_0+\tau)=\sin\eta\equiv(1/\kappa)\sin\zeta/2$ is the inverse function to the elliptic integral

$$F\left[\sin^{-1}\left[\frac{1}{\kappa}\sin\frac{\zeta}{2}\right],\kappa\right] .$$

For subsequent discussion of the stochastic particle instability in Sec. IV. B, it is useful to express properties of the trapped particle motion in terms of action-angle variables (J,θ). Defining, in the usual manner,

$$J=J(H_0)=\frac{1}{2\pi}\oint\frac{d\zeta}{d\tau}d\zeta ,$$
$$\theta(\zeta,J)=\frac{\partial}{\partial J}S(\zeta,J) , \tag{58}$$
$$S(\zeta,J)=\frac{1}{2\pi}\int^\zeta\frac{d\zeta}{d\tau}d\zeta ,$$

we find

$$J(H_0)=\frac{8}{\pi}\left[E\left[\frac{\pi}{2},\kappa\right]-(1-\kappa^2)F\left[\frac{\pi}{2},\kappa\right]\right], \tag{59}$$

where $\kappa^2=(1/2)(1+H_0)$, and $F(\eta,\kappa)$ and $E(\eta,\kappa)$ are defined in Eqs. (52) and (53). The unperturbed equation of motion (48) in new variables (J,θ) is given by

$$\frac{dJ}{d\tau}=0,\quad \frac{d\theta}{d\tau}=\omega_T(J)/\hat{\omega}_T\ , \tag{60}$$

where $\hat{\omega}_T$ is defined in Eq. (28), and the frequency $\omega_T(J)$ is determined from $\omega_T(J)/\hat{\omega}_T=\partial H_0(J)/\partial J$, i.e.,

$$\omega_T(J)=\frac{\pi}{2F(\pi/2,\kappa)}\hat{\omega}_T\ . \tag{61}$$

Near the bottom of the potential well, $H_0\to-1$, $\kappa^2\to0$, $F(\pi/2,\kappa)\to\pi/2$, and therefore $\omega_T(J)\to\hat{\omega}_T$, as expected from Eq. (48). On the other hand, near the top of the potential well, $H_0\to+1$, $\kappa^2\to1$, $F(\pi/2,\kappa)\to\infty$, and the period $2\pi/\omega_T(J)$ of the trapped particle motion becomes infinitely long.

For future reference, neglecting initial conditions in Eq. (57), the normalized velocity in the beat-wave frame can be expressed as

$$\begin{aligned}\frac{d\zeta}{d\tau}&=2\kappa\,\mathrm{cn}(\tau)\\&=8\frac{\omega_T}{\hat{\omega}_T}\sum_{n=1}^{\infty}\frac{a^{n-1/2}}{1+a^{2n-1}}\cos[(2n-1)\omega_T t]\ ,\end{aligned} \tag{62}$$

where $F_0=0$ is assumed, $\tau=\hat{\omega}_T t$, and $\omega_T=\omega_T(J)$ is defined in Eq. (61). Moreover, the quantity a in Eq. (62) is defined by

$$\begin{aligned}&a\equiv\exp(-\pi F'/F)\ ,\\&F'\equiv F[\pi/2,(1-\kappa^2)^{1/2}],\quad F\equiv F(\pi/2,\kappa)\ .\end{aligned} \tag{63}$$

Near the top of the potential well (i.e., near the separatrix) where $H_0\to1$, we will find in Sec. IV A that the particle motion becomes stochastic in the presence of the perturbation force in Eq. (45). Defining $H_0=1-\Delta H$, where $\Delta H\ll1$ near the separatrix, we find $\kappa^2\to1$, $\omega_T(J)\to0$, and

$$\begin{aligned}&F\simeq\tfrac{1}{2}\ln(32/\Delta H)\ ,\\&F'\simeq\frac{\pi}{2}\ ,\\&\omega_T\simeq\pi\hat{\omega}_T[\ln(32/\Delta H)]^{-1}\ ,\\&a\simeq\exp(-\pi\omega_T/\hat{\omega}_T)\ ,\end{aligned} \tag{64}$$

for small $\Delta H\ll1$.

(2) Untrapped particle motion ($\kappa^2>1$). Although the emphasis in Sec. IV B will be on the trapped particle motion, for completeness we summarize here properties of the solution to Eq. (50) when the orbits are untrapped ($\kappa^2>1$). Defining $\eta=\zeta/2$, Eq. (50) can be expressed as

$$\left[\frac{d\eta}{d\tau}\right]^2=\kappa^2\left[1-\frac{1}{\kappa^2}\sin^2\eta\right], \tag{65}$$

where $1/\kappa^2<1$. Solving Eq. (65) gives for $\zeta(\tau)=2\eta(\tau)$

$$F(\zeta/2,1/\kappa)=F_0+\kappa\tau\ , \tag{66}$$

where $F_0\equiv F(\zeta(\tau=0)/2,1/\kappa)$. The solutions (56) and (66) clearly match exactly at the separatrix where $\kappa^2=1$.

B. Stochastic instability

In Sec. IV A, we considered properties of the equation of motion in circumstances where the right-hand side of Eq. (45) is negligibly small ($\epsilon_T\to0$ and $\delta_L\to0$) and the lowest-order motion is described by the pendulum equation (48). In this section, leading-order corrections to the particle motion are retained on the right-hand side of Eq. (45) in an iterative sense. For consideration of the stochastic particle instability that develops near the separatrix, it is particularly convenient to examine the particle motion in action-angle variables.[1] Correct to order $\epsilon_T^{1/2}$ and δ_L, we find

$$\frac{dJ}{d\tau}=\frac{dJ}{dH_0}\frac{dH_0}{d\tau}=\frac{\hat{\omega}_T}{\omega_T}\frac{dH_0}{d\tau}\ , \tag{67}$$

where $\omega_T=\omega_T(J)$, and

$$\begin{aligned}\frac{dH_0}{d\tau}=&-\frac{\omega}{ck'}\epsilon_T^{1/2}\frac{d}{d\tau}\left[\frac{d\zeta}{d\tau}\right]^3\\&-\delta_L\frac{k'}{k}\frac{d\zeta}{d\tau}\sin\left[\frac{k}{k'}(\zeta-V_p\tau)\right]\end{aligned} \tag{68}$$

follows directly from Eqs. (46) and (47). Here, $k'=k+k_0$ and V_p is the dimensionless phase velocity $V_p=k_0\omega/k\hat{\omega}_T$. For $\omega\simeq kc$, note that

$$V_p=\frac{k_0\omega}{k\hat{\omega}_T}\simeq\frac{k_0 c}{\hat{\omega}_T}\gg1\ , \tag{69}$$

in parameter regimes of practical interest. The $\epsilon_T^{1/2}$ contribution to $dH_0/d\tau$ in Eq. (68) is expressed as a complete time derivative. Hence, correct to order $\epsilon_T^{1/2}$, we find from Eqs. (67) and

(68) that the $\epsilon_T^{1/2}$ contributions to $dJ/d\tau$ and $dH_0/d\tau$ are conservative and do not lead to a systematic (secular) change in action or energy when averaged over a cycle of the zeroth-order pendulum motion. Therefore, for purposes of investigating the stochastic particle motion associated with systematic changes in the action J, only the longitudinal wave contribution to $dH_0/d\tau$ is retained, and Eq. (67) is approximated by

$$\frac{dJ}{d\tau}=-\delta_L\frac{\hat{\omega}_T}{\omega_T}\frac{k'}{k}\frac{d\zeta}{d\tau}\sin\left[\frac{k}{k'}(\zeta-V_p\tau)\right] . \quad (70)$$

For present purposes, we consider particle orbits which are trapped and periodic ($\kappa^2<1$) in the absence of the longitudinal perturbation in Eq. (70). It is well known that near the separatrix ($H_0\to 1$ and $\kappa^2\to 1$) Eq. (70) can lead to a stochastic instability that is manifest by a secular change in the action J and a systematic departure of the particle from the potential well. Near the separatrix with $H_0\to 1$, it follows from Eqs. (49) and (62) that the particle is moving with an approximately constant normalized velocity $d\zeta/d\tau\simeq 2$ for a short time of order $\hat{\tau}_T=\hat{\omega}_T^{-1}$. Moreover, this feature of the particle motion recurs with frequency $\omega_T(J)\ll\hat{\omega}_T$, and can lead to a significant change in the action J in Eq. (70).

We now examine the implications of Eq. (70) near the separatrix, keeping in mind that $V_p\gg 1$ and that the sine term on the right-hand side generally represents a high-frequency modulation. Making use of the zeroth-order expression for the normalized velocity $d\zeta/d\tau$ in Eq. (62), it follows directly that $dJ/d\tau$ can be expressed as

$$\frac{dJ}{d\tau}=-4\delta_L\sum_{n=1}^{\infty}\frac{a^{n-1/2}}{1+a^{2n-1}}\left\{\sin\left[\frac{k}{k'}\zeta+\left[(2n-1)\frac{\omega_T}{\hat{\omega}_T}-\frac{k}{k'}V_p\right]\tau\right]-\sin\left[\frac{k}{k'}\zeta-\left[(2n-1)\frac{\omega_T}{\hat{\omega}_T}+\frac{k}{k'}V_p\right]\tau\right]\right\} , \quad (71)$$

where $k'=k+k_0$, $\omega_T=\omega_T(J)$, and a is defined in Eq. (63). Near the separatrix $d\zeta/d\tau\simeq 2\ll V_p$ in Eq. (71). Therefore, the first term on the right-hand side of Eq. (71) acts as a nearly constant driving term for some high harmonic numbers s ($\gg 1$) satisfying the resonance condition

$$2s\frac{\omega_T(J_s)}{\hat{\omega}_T}\simeq\frac{k}{k'}V_p ,$$

or equivalently,

$$\omega_T(J_s)\simeq\frac{\hat{\omega}_T}{2s}\frac{k}{k'}V_p=\frac{1}{2s}\frac{\omega k_0}{k+k_0} . \quad (72)$$

Here, J_s is the action corresponding to the resonance condition for resonance number s. From Eq. (72), it follows that the distance between the adjacent resonances s and $s+1$ is given by

$$\delta_s\equiv\omega_T(J_s)-\omega_T(J_{s+1})\simeq\frac{\hat{\omega}_T}{2s^2}\frac{k}{k'}V_p$$
$$\simeq\frac{2k'}{k\hat{\omega}_T V_p}\omega_T^2(J_s)=2\frac{(k+k_0)}{k_0\omega}\omega_T^2(J_s) . \quad (73)$$

On the other hand, for a (small) change in the action ΔJ_s, the characteristic frequency width of the sth resonance can be expressed as

$$\Delta\omega_T(J_s)=\left[\frac{d\omega_T(J_s)}{dJ_s}\right]\Delta J_s ,$$

where $\Delta\omega_T(J_s)\ll\omega_T(J_s)$ is assumed. The condition for appearance of stochastic instability[1] is $\Delta\omega_T(J_s)\gg\delta_s$, or equivalently,

$$\left|\frac{d\omega_T(J_s)}{dJ_s}\Delta J_s\right|\gg 2\frac{k'}{k\hat{\omega}_T V_p}\omega_T^2(J_s) . \quad (74)$$

To estimate the size of ΔJ_s, we express $\omega_T(J)$ as $\omega_T(J_s)+\Delta\omega_T(J_s)$ and integrate Eq. (71) over a time interval of order $\hat{\tau}_T=\hat{\omega}_T^{-1}$ ($\Delta\tau\approx 1$) in the vicinity of the sth resonance defined by Eq. (72). In an order-of-magnitude sense, this gives for the characteristic magnitude of ΔJ_s

$$\Delta J_s\approx 2\delta_L\hat{\omega}_T\frac{a^{s-1/2}}{1+a^{2s-1}}\left|s\frac{d\omega_T(J_s)}{dJ_s}\Delta J_s\right|^{-1} .$$

Solving for ΔJ_s and eliminating s by means of Eq. (72) gives

$$\Delta J_s\approx\left[\frac{4\delta_L a^{s-1/2}/(1+a^{2s-1})}{|d\omega_T(J_s)/dJ_s|}\frac{\omega_T(J_s)}{kV_p/k'}\right]^{1/2} . \quad (75)$$

Substituting Eq. (75) into Eq. (74) then gives as the condition for stochastic instability,

$$\delta_L \left| \frac{d\omega_T(J_s)}{dJ_s} \right| \left| \frac{a^{s-1/2}}{1+a^{2s-1}} \right| \gg \frac{\omega_T^3(J_s)}{\omega\hat{\omega}_T} \left| \frac{k+k_0}{k_0} \right| , \quad (76)$$

where use has been made of $k'=k+k_0$ and $V_p=k_0\omega/k\hat{\omega}_T$.

We now estimate that various factors in Eq. (76) near the separatrix where $H_0 \to 1$ and $\omega_T(J_s) \gg \hat{\omega}_T$. From Eqs. (64) and (72), it follows that $a^s \simeq \exp(-\pi s \omega_T/\hat{\omega}_T)$ and

$$a^s \simeq \exp \left[-\frac{\pi}{2} \frac{k_0}{k+k_0} \frac{\omega}{\hat{\omega}_T} \right] , \quad (77)$$

where $a^s \ll 1$ and $a^{s-1/2}/(1+a^{2s-1}) \simeq a^{s-1/2}$. Also from Eq. (64), $\ln[32/(1-H_0)] \simeq \pi\hat{\omega}_T/\omega_T$ gives

$$\frac{\partial H_0/\partial J}{1-H_0} = -\pi \frac{\hat{\omega}_T}{\omega_T^2(J)} \frac{d\omega_T(J)}{dJ} .$$

Making use of $\partial H_0/\partial J = \omega_T(J)/\hat{\omega}_T$, we obtain

$$\frac{\hat{\omega}_T^2}{\omega_T^3(J)} \frac{d\omega_T(J)}{dJ} = -\frac{1}{32\pi} \exp[\pi\hat{\omega}_T/\omega_T(J)] . \quad (78)$$

Substituting Eqs. (77) and (78) into Eq. (76) gives

$$\frac{\delta_L}{32\pi} \frac{k_0\omega}{(k+k_0)\hat{\omega}_T} \exp \left[\pi \frac{\hat{\omega}_T}{\omega_T} - \frac{\pi k_0 \omega}{2(k+k_0)\hat{\omega}_T} \right] \gg 1 \quad (79)$$

as the condition for stochastic instability. Equation (79) can also be expressed in the equivalent form

$$\omega_T \ll \pi\hat{\omega}_T \left[\ln \frac{16\pi^2}{\delta_L} + \frac{\pi k_0 \omega}{2(k+k_0)\hat{\omega}_T} - \ln \frac{\pi k_0 \omega}{2(k+k_0)\hat{\omega}_T} \right]^{-1} \equiv (\omega_T)_{\text{cr}} . \quad (80)$$

In Eq. (80), $(\omega_T)_{\text{cr}}$ is the critical bounce frequency for onset of stochastic instability when $\omega_T \lesssim (\omega_T)_{\text{cr}}$. Since $\alpha \gg \ln\alpha$ for $\alpha \gg 1$, Eq. (80) gives

$$\pi \frac{\hat{\omega}_T}{(\omega_T)_{\text{cr}}} = \left[\ln \frac{16\pi^2}{\delta_L} + \frac{\pi k_0 \omega}{2(k+k_0)\hat{\omega}_T} \right] \quad (81)$$

to good accuracy. In a regime where

$$\ln(16\pi^2/\delta_L) \gg \pi k_0 \omega / 2(k+k_0)\hat{\omega}_T ,$$

Eq. (81) gives $\pi\hat{\omega}_T/(\omega_T)_{\text{cr}} \simeq \ln(16\pi^2/\delta_L)$. From $\ln[32/(1-H_0)] \simeq \pi\hat{\omega}_T/\omega_T$, the condition for onset of stochastic instability can then be expressed as

$$\Delta H \lesssim (1-H_0)_{\text{cr}} \simeq 2\delta_L/\pi^2 , \quad (82)$$

where $\Delta H = 1-H_0$. On the other hand, in a regime where

$$\ln(16\pi^2/\delta_L) \ll \pi k_0 \omega / 2(k+k_0)\hat{\omega}_T ,$$

Eq. (81) gives $\pi\hat{\omega}_T/(\omega_T)_{\text{cr}} \simeq \pi k_0\omega/2(k+k_0)\hat{\omega}_T$, and the condition for onset of stochastic instability can be expressed as

$$\Delta H \lesssim (1-H_0)_{\text{cr}} \simeq 32 \exp \left[-\frac{\pi k_0 \omega}{2(k+k_0)\hat{\omega}_T} \right] . \quad (83)$$

Unlike Eq. (82), the energy band for instability in Eq. (83) is exponentially small.

V. STOCHASTIC INSTABILITY FOR TWO MODERATE-AMPLITUDE ELECTROMAGNETIC WAVES

An analogous stochastic instability can also develop in circumstances where the longitudinal electric field δE_z is negligibly small but a second, moderate-amplitude electromagnetic wave is present.[6] In this section, we briefly outline the assumptions and relevant features of the final dynamical equation. We consider circumstances where $\delta E_z = 0$ and two, constant-amplitude, circularly polarized electromagnetic waves $(\hat{\delta B}_1, \omega_1, k_1)$ and $(\hat{\delta B}_2, \omega_2, k_2)$ are present with polarization similar to Eqs. (5) and (6). The second electromagnetic wave $(\hat{\delta B}_2, \omega_2, k_2)$ may also be a consequence of the FEL amplification process, with frequency and wave number (ω_2, k_2) nearby to (ω_1, k_1). Defining

$$b_w \equiv \left| \frac{eB_w}{mc^2 k_0} \right| , \quad b_1 \equiv \left| \frac{e\hat{\delta B}_1}{mc^2 k_1} \right| , \quad b_2 \equiv \left| \frac{e\hat{\delta B}_2}{mc^2 k_2} \right| , \quad (84)$$

after some straightforward algebra, it can be shown that the axial equation of motion can be expressed as

$$\frac{d^2z}{dt^2}=-\frac{1}{\gamma^2}\left[b_w b_1\left[c^2(k_1+k_0)-\omega_1\frac{dz}{dt}\right]\sin[(k_1+k_0)z-\omega_1 t]\right.$$
$$+b_w b_2\left[c^2(k_2+k_0)-\omega_2\frac{dz}{dt}\right]\sin[(k_2+k_0)z-\omega_2 t]$$
$$\left.-b_1 b_2\left[c^2(k_2-k_1)-(\omega_2-\omega_1)\frac{dz}{dt}\right]\sin[(k_2-k_1)z-(\omega_2-\omega_1)t]\right], \tag{85}$$

where

$$\frac{1}{\gamma^2}=\left[1-\frac{1}{c^2}\left[\frac{dz}{dt}\right]^2\right]\{1+b_w^2+b_1^2+b_2^2$$
$$-2b_1b_w\cos[(k_1+k_0)z-\omega_1 t]-2b_2b_w\cos[(k_2+k_0)z-\omega_2 t]$$
$$+2b_1b_2\cos[(k_2-k_1)z-(\omega_2-\omega_1)t]\}^{-1}. \tag{86}$$

The form of Eq. (86) is somewhat analogous to Eq. (27). If we neglect the b_1b_2 terms in comparison with b_1b_w and b_2b_w, and examine particle motion with axial velocity dz/dt in the vicinity of the beat-wave phase velocity $\omega_1/(k_1+k_0)$ of the primary wave, then for $|\omega_1 dz/dt-\omega_1/(k_1+k_0)|\ll c^2(k_1+k_0)^2-\omega_1^2$, Eq. (85) can be approximated by

$$(k_1+k_0)\frac{d^2z}{dt^2}+\hat{\omega}_{T1}^2\sin[(k_1+k_0)z-\omega_1 t]+\frac{k_1+k_0}{k_2+k_0}\hat{\omega}_{T2}^2\sin[(k_2+k_0)z-\omega_2 t]=0\ , \tag{87}$$

where

$$\hat{\omega}_{T1}^2=\left[1-\frac{\omega_1^2}{c^2(k_1+k_0)^2}\right]\frac{[c^2(k_1+k_0)^2-\omega_1^2]}{(1+b_w^2)}b_wb_1\ ,$$
$$\hat{\omega}_{T2}^2=\left[1-\frac{\omega_1^2}{c^2(k_1+k_0)^2}\right]\frac{[c^2(k_2+k_0)^2-\omega_2\omega_1(k_2+k_0)/(k_1+k_0)]}{(1+b_w^2)}b_wb_2\ , \tag{88}$$

and only leading-order terms proportional to b_wb_1 and b_wb_2 are retained in Eq. (87). Introducing $\zeta=(k_1+k_0)z-\omega_1 t$, Eq. (87) can be expressed as

$$\frac{d^2\zeta}{dt^2}+\hat{\omega}_{T1}^2\sin\zeta+\frac{k_1'}{k_2'}\hat{\omega}_{T2}^2\sin\left[\frac{k_2'}{k_1'}\zeta-\Delta\omega t\right]=0\ , \tag{89}$$

where $k_1'=k_1+k_0$, $k_2'=k_2+k_0$, and $\Delta\omega=(k_1'\omega_2-k_2'\omega_1)/k_1'$.

Analogous to Eq. (41), if $\delta=\hat{\omega}_{T2}^2/\hat{\omega}_{T1}^2$ is treated as a small parameter, the dynamical equation (89) can lead to stochastic instability for particles near the separatrix. For $\Delta\omega/\hat{\omega}_{T1}\gg 1$, the general features of the instability are similar to those discussed in Sec. IV. For $\Delta\omega/\hat{\omega}_{T1}\lesssim 1$, using techniques similar to Zaslavskii and Filonenko,[1] it can be shown that the energy band ΔH corresponding to instability can be much wider than in the case $\Delta\omega/\hat{\omega}_T\gg 1$.

VI. CONCLUSIONS

In summary, we have considered electron motion in the combined fields of a helical wiggler and constant-amplitude, circularly polarized primary electromagnetic wave $(\hat{\delta B}_T,\omega,k)$. For particle velocity near the beat-wave phase velocity $\omega/(k+k_0)$ of the primary wave, it was shown that the presence of a second, moderate-amplitude longitudinal wave $(\hat{\delta E}_L,\omega,k)$ or transverse electromagnetic wave $(\hat{\delta B}_2,\omega_2,k_2)$ can lead to stochastic particle instability in which particles trapped near the separatrix of the primary wave undergo a systematic departure from the potential well. The condition for onset of instability has been calculated [Eq. (80)]. The importance of these results for FEL applications is evident. For development of long-pulse or steady-state free-electron lasers, the maintenance of beam integrity of an extended period of time will be of considerable practical importance. The fact that the presence of secondary, moderate-amplitude longitudinal or transverse electromagnetic waves can destroy coherent motion for certain classes of beam particles moving with velocity near $\omega/(k+k_0)$ may lead to a degradation of beam quality and concommitant modification of FEL emission properties.

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