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Integrable Hamiltonian systems and the Lax pair formalism

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The Lax pair formalism for finite degree-of-freedom Hamiltonian systems is extended to include systems for which the eigenvalues of the L operator are degenerate. This extension is applied to a system of restricted multiple three-wave interactions which describes the interaction of a single, low-frequency, internal ocean wave with higher-frequency external waves.

In recent years, there has been a remarkable resurgence of interest in the classical problem of determining when Hamiltonian systems are integrable and when not. Modern-day interest has centered about applications to plasma, biological, chemical, and statistical physics, as well as the traditional domain of celestial mechanics.[1] One powerful and elegant means for demonstrating the integrability of Hamiltonian systems is the Lax formalism, which was first applied to an infinite-dimensional Hamiltonian system, i.e., the Korteweg–de Vries equation.[2] This approach has since been applied with great success to finite-dimensional systems by Flaschka[3] and Moser,[4] among others.

The method works as follows: Let $L(a_1, \ldots, a_m)$ and $A(a_1, \ldots, a_m)$ be two $M\times M$ matrices, where $a_1, \ldots, a_m$ are the variables of the system under consideration. These matrices act on the M-dimensional vector space whose elements we will designate ψ. Suppose now that $[A,L]=L_t$, with the brackets indicating commutation and the subscript t indicating the derivative with respect to time, and consider the eigenvalue equation $L\psi=\lambda\psi$, along with the evolution equation $A\psi=\psi_t$. It follows immediately from the commutation relation that $\lambda_t=0$, implying that the eigenvalues are constants of the motion. Resolving the equation $\det(L-\lambda I)=0$, one finds that the coefficient of each power of λ is a constant of the motion. For the cases considered to date,[3,4] one finds that these coefficients are independent and in involution. Moreover, unless the Lax pairs do not contain the full information from the equations of motion, one finds just the total number of constants needed to demonstrate the system's integrability.

However, as we shall show, there exist systems for which the Lax pairs do contain the full amount of information needed, but for which the procedure just described fails to yield the required number of constants. The question then immediately arises: How can one extend the Lax pair formalism to obtain constants of the motion in these cases as well? It is this question we address here.

We concentrate principally on the comparison of two examples. The first is the well-known Toda lattice, for which the equations of motion are

$$\dot{a}_n = a_n(b_{n+1}-b_n), \quad \dot{b}_n = 2(a_n^2 - a_{n-1}^2), \qquad n=1,2,\ldots,N, \tag{1}$$

where $a_n=\frac{1}{2}\exp[-(Q_n-Q_{n-1})/2]$ is related to the displacement of the lattice elements and $b_n=-P_{n-1}/2$ is related to their momentum. We are assuming periodicity, so that $a_1=a_{N+1}$ and $b_1=b_{N+1}$. The Lax pairs, which were first found by Flaschka,[3] are given by

$$\begin{aligned} &L_{n,n}=b_n, \quad n=1,\ldots,N, \\ &L_{n,n+1}=L_{n+1,n}=A_{n,n+1}=-A_{n+1,n}=a_n, \quad n=1,\ldots,N-1, \\ &L_{1,N}=L_{N,1}=-A_{1,N}=A_{N,1}=a_N, \end{aligned} \tag{2}$$

and all other elements are zero. It can be readily seen that the Lax formalism described earlier yields the N independent constants in involution needed to demonstrate integrability.

The second example we consider is a special case of restricted multiple three-wave interactions, in which a set of wave triads has only one common wave, and all interacting waves are coupled with equal strength. This system was first used by Watson, West, and Cohen[5] to model the interaction of a low-frequency internal ocean wave with higher-frequency surface waves. It has since been studied by Meiss[6] and by us.[7] The equations of motion are

$$\begin{aligned} &\dot{b}_0 = i\sum_{n=1}^{N} b_n b_n'^*, \\ &\dot{b}_n = i b_0 b_n', \quad \dot{b}_n' = i b_0^* b_n, \quad n=1,2,\ldots,N, \end{aligned} \tag{3}$$

and are generated by the Hamiltonian

$$H = i\sum_{n=1}^{N}(b_0 b_n^* b_n' + b_0^* b_n b_n'^*), \tag{4}$$

where each of the variables is canonically conjugate to its complex conjugate. Lax pairs are given by

$$L=\begin{pmatrix} 0 & -b_0 & -\frac{b_1}{2} & -\frac{b_1'^*}{2} & \cdots & -\frac{b_N}{2} & -\frac{b_N'^*}{2} \\ b_0^* & 0 & -\frac{b_1'}{2} & \frac{b_1^*}{2} & \cdots & -\frac{b_N'}{2} & \frac{b_N^*}{2} \\ \frac{1}{2}b_1^* & -\frac{1}{2}b_1'^* & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2}b_1' & -\frac{1}{2}b_1 & 0 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ \frac{1}{2}b_N^* & -\frac{1}{2}b_N'^* & \vdots & \vdots & & & \vdots \\ -\frac{1}{2}b_N' & -\frac{1}{2}b_N & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad A=\begin{pmatrix} 0 & 0 & \frac{i}{2}b_1 & \frac{i}{2}b_1'^* & \cdots & \frac{i}{2}b_N & \frac{i}{2}b_N'^* \\ 0 & 0 & -\frac{i}{2}b_1' & \frac{i}{2}b_1^* & \cdots & -\frac{i}{2}b_N' & \frac{i}{2}b_N^* \\ -\frac{i}{2}b_1^* & -\frac{i}{2}b_1'^* & 0 & 0 & \cdots & 0 & 0 \\ \frac{i}{2}b_1' & -\frac{i}{2}b_1 & 0 & 0 & \cdots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ -\frac{i}{2}b_N^* & -\frac{i}{2}b_N'^* & \vdots & \vdots & & & \vdots \\ \frac{i}{2}b_N' & -\frac{i}{2}b_N & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \qquad (5)$$

If one attempts to use the Lax formalism described earlier, one finds only the following three constants:

$$I_1=b_0b_0^*+\frac{1}{2}\sum_{n=1}^{N}(b_nb_n^*-b_n'b_n'^*) \ ,$$

$$I_2=\frac{1}{2}\sum_{n=1}^{N}(b_0b_n'^*b_n'+b_0^*b_nb_n'^*) \ , \qquad (6)$$

$$I_3=\frac{1}{16}\left[\sum_{n=1}^{N}(b_nb_n^*+b_n'b_n'^*)\right]^2 \ ,$$

and no more.

In resolving this situation, we begin by recalling that

$$\psi(t)=\sum_{m=1}^{M}c_m\psi_m(t) \ , \qquad (7)$$

where we recall that M is the dimensionality of the Lax pair vector space, ψ is any vector in this space, c_m is a constant, and $\psi_m(t)$ is the eigenvector corresponding to the eigenvalue λ_m, determined by solving the equation $\det(L-\lambda I)=0$. If the eigenvalues are all different, then each eigenvector is determined to within a constant at any given time; this situation is just that of the Toda lattice. However, if any of the eigenvalues are degenerate, then the eigenvectors corresponding to that eigenvalue span a space of the same dimensionality as the degeneracy; this situation is just that of the restricted multiple three-wave interactions, and we may use the arbitrariness in our choice of eigenvectors to extract additional constants of the motion. In the case of restricted multiple three-wave interactions the eigenvalue $\lambda=0$ has an $M-4$ degeneracy, and the corresponding eigenvector ψ_0 has the form $\psi_0=(0,0,\phi_1,\phi_1',\ldots,\phi_N,\phi_N')^T$, where the first two rows of L will allow us to eliminate two of the elements of ψ_0. Ignoring the first two columns of A, we note that the first two rows of A are just multiples of the corresponding rows of L, which immediately implies $\psi_{0,t}=A\psi_0=0$, so that $\phi_1,\phi_1',\ldots,\phi_N,\phi_N'$ are constants of the motion. Using the first two rows of L to eliminate ϕ_1, we find

$$\phi_1'=-\sum_{n=2}^{N}\left[\frac{(b_1b_n'-b_1'b_n)}{(b_1b_1^*+b_1'b_1'^*)}\phi_n+\frac{(b_1b_n^*+b_1'b_n'^*)}{(b_1b_1^*+b_1'b_1'^*)}\phi_n'\right] \ , \qquad (8)$$

and using the constancy of $\phi_1',\phi_2,\ldots,\phi_N'$, as well as the arbitrariness of $\phi_2,\phi_2',\ldots,\phi_N'$, we immediately infer that each of the coefficients of $\phi_2,\ldots,\phi_N'$ must be constant. Eliminating in turn ϕ_1',ϕ_2, etc., one obtains different sets of constants, and using the constancy of $b_1b_1^*+b_1'b_1'^*+\cdots+b_Nb_N^*+b_N'b_N'^*$, obtained from Eq. (6), one concludes that all quadratic combinations of the form $b_ib_j^*+b_i'b_j'^*$, $b_ib_j'-b_i'b_j$, or $b_i^*b_j'^*-b_i'^*b_j^*$ are constant, where i and j are arbitrary. From these quadratic constants, it is not at all difficult to construct the needed number of independent constants in involution, as Meiss[8] has previously shown. One acceptable combination is

$$I_1 \text{ and } I_2 \text{ [from Eq. (6)]} \ ,$$

$$b_nb_n^*+b_n'b_n'^*, \quad n=1,2,\ldots,N \ , \qquad (9)$$

$$\sum_{j=1}^{n-1}(b_jb_n^*+b_j'b_n'^*)(b_j^*b_n+b_j'^*b_n'), \quad n=2,3,\ldots,N \ .$$

In general, it may not be true that $\psi_t=0$. To deal with the general case, we begin by supposing that L is degenerate for the eigenvalue μ. Noting that $L_t=[A,L]$ implies $(L\psi_\mu)_t=\mu\psi_{\mu,t}$, where ψ_μ indicates any eigenvector corresponding to the eigenvalue μ, we find that an eigenvector of μ at the initial time remains an eigenvector of μ for all time. As a result, we may define A_μ, the restriction of A to the vector space corresponding to μ, such that $\psi_{\mu,t}=A_\mu\psi_\mu$. Diagonalizing this equation, we will obtain d relations, where d is the dimensionality of the vector space corresponding to μ, in the form $\dot\theta_i=\nu_i(a_1,a_2,\ldots,a_m)\theta_i$, which may be immediately integrated to yield

$$\theta_i=c_i\exp\left[\int_0^t\nu_i(a_1,\ldots,a_n)\,dt'\right] \ ,$$

where c_i is an arbitrary constant. Returning to the L operator, one finds that there is at least one relationship in the form

$$\phi(t)=\sum_{i=1}^{d} c_i f_i(a_1,\ldots,a_m) \times \exp\left(\int_0^t \nu_i(a_1,\ldots,a_m)\,dt'\right) , \quad (10)$$

and noting that $c_1,\ldots,c_d$ are arbitrary, one immediately concludes that all relations of the form

$$I_{ij}=\frac{f_i(a_1,\ldots,a_m)}{f_j(a_1,\ldots,a_m)} \times \exp\left(\int_0^t [\nu_i(a_1,\ldots,a_m)-\nu_j(a_1,\ldots,a_m)]\,dt'\right) \quad (11)$$

are constants of the motion. It is evident that these constants are not useful for demonstrating the integrability of the original Hamiltonian system unless the integral inside the exponential can be explicitly expressed in terms of $a_1,a_2,\ldots,a_m$. In the example that we considered previously, we have $\nu_i=0$ for all i, and this condition is trivially met.

In this Communication, we have shown how to extend the usual Lax pair formalism to deal with cases where the eigenvalues of the L operator are degenerate. We have illustrated this extension by applying it to restricted multiple three-wave interactions.

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