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Transition to chaos in the Duffing oscillator

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It is observed in the Duffing oscillator that a bifurcation from a solution composed of only odd harmonics to one composed of both even and odd harmonics precedes the period-doubling bifurcations. Keeping all parameters fixed except for the amplitude of the driving force F, we determine the value of F at which the bifurcation occurs. Results are compared with experiment. A mechanism for the period-doubling bifurcations is suggested.

It has been found that in some nonlinear systems the transition to chaos can occur via consecutive period doublings.[1–6] We, among other investigators, have observed period doubling in an anharmonic oscillator with a cubic restoring force, a harmonic driving force, and nonzero damping.[7–11] Our experiments were performed on an analog circuit[12] which simulates Duffing's equation[13–15]

$$\ddot{x}+\alpha\dot{x}+\omega_0^2x+\beta x^3=F\cos\Omega t;\quad \alpha>0,\ \beta>0\ . \tag{1}$$

The input voltage is $V_{\text{in}}=(F/\omega_0^2)\cos\Omega t$ and the output voltage is the solution $V_{\text{out}}=x(t)$. (In what follows we will express F in units of volts.) Our critical parameter here is the amplitude F of the driving force. All other parameters are kept fixed. The solution $x(t)$ is analyzed by determining its spectral density.

For small nonlinearity, x may be approximated by $x\approx A_1\cos(\Omega t+\phi_1)$, where A_1 and ϕ_1 are given by

$$\begin{aligned}\beta A_1^2&=\beta F^2/[(\omega_0^2+\tfrac{3}{4}\beta A_1^2-\Omega^2)^2+\alpha^2\Omega^2]\ ,\\ \tan\phi_1&=-\alpha\Omega/(\omega_0^2+\tfrac{3}{4}\beta A_1^2-\Omega^2)\ .\end{aligned} \tag{2}$$

This is Duffing's approximation.[13–15] In order to determine the first-order correction to x, we let

$$x\approx A_1\cos(\Omega t+\phi_1)+A_3\cos(3\Omega t+\phi_3)\ . \tag{3}$$

Neglecting terms involving higher-order harmonics and keeping only those terms linear in A_3, we determine the amplitude A_3 and phase ϕ_3 to be

$$\begin{aligned}\beta A_3^2&=\tfrac{1}{16}\beta^3A_1^6/[(\omega_0^2+\tfrac{3}{2}\beta A_1^2-9\Omega^2)^2+9\Omega^2\alpha^2]\\ \tan(3\phi_1-\phi_3)&=3\alpha\Omega/(\omega_0^2+\tfrac{3}{2}\beta A_1^2-9\Omega^2)\ .\end{aligned} \tag{4}$$

All the odd harmonics contributing to x may be generated by perturbation theory.[13–15]

For large damping, we observed only odd harmonics in the output voltage. On the other hand, period doubling was observed to occur only for small damping. (Here $2\zeta=\alpha/\omega_0=0.1$.) In the experiment F was continuously increased starting from $F=0$. For small F the output voltage agrees with the solution $x(t)$ given by Eq. (3) with A_1, ϕ_1, and A_3, ϕ_3 being defined by Eqs. (2) and (4), respectively. However, when F exceeds some value F_E, even harmonics of Ω are observed in the output voltage. As F is further increased, period doubling bifurcations occur as evidenced by the consecutive appearance of $\frac{1}{2}$ and $\frac{1}{4}$ subharmonics (see Fig. 1). Generally chaotic behavior was observed soon after the appearance of the $\frac{1}{4}$ subharmonic.

The observed period-doubling bifurcations are always found to be preceded by a bifurcation from a solution containing only odd harmonics to one containing both odd and even harmonics. This suggests that the solutions containing both odd and even harmonics are, in this system at least, precursors for the period-doubled solutions. In what follows we analyze the onset of the even harmonics, and on the basis of these results we suggest a mechanism for the period-doubling bifurcations.

We start with the solution $x(t)$ given by Eq. (3) and perturb it by η, where $\eta\ll x$. Neglecting terms of order η^3, we obtain from Eq. (1) the following equation for η:

$$\ddot{\eta}+\alpha\dot{\eta}+\gamma^2\left[1+\frac{H}{\gamma^2}\cos(2\Omega t+\lambda)+\frac{G}{\gamma^2}\cos(4\Omega t+\phi_1+\phi_3)\right]\eta=0\ . \tag{5}$$

In the coefficient of η a term containing 6Ω has also been neglected since its contribution to Eq. (5) is small.

γ is a renormalized frequency defined by

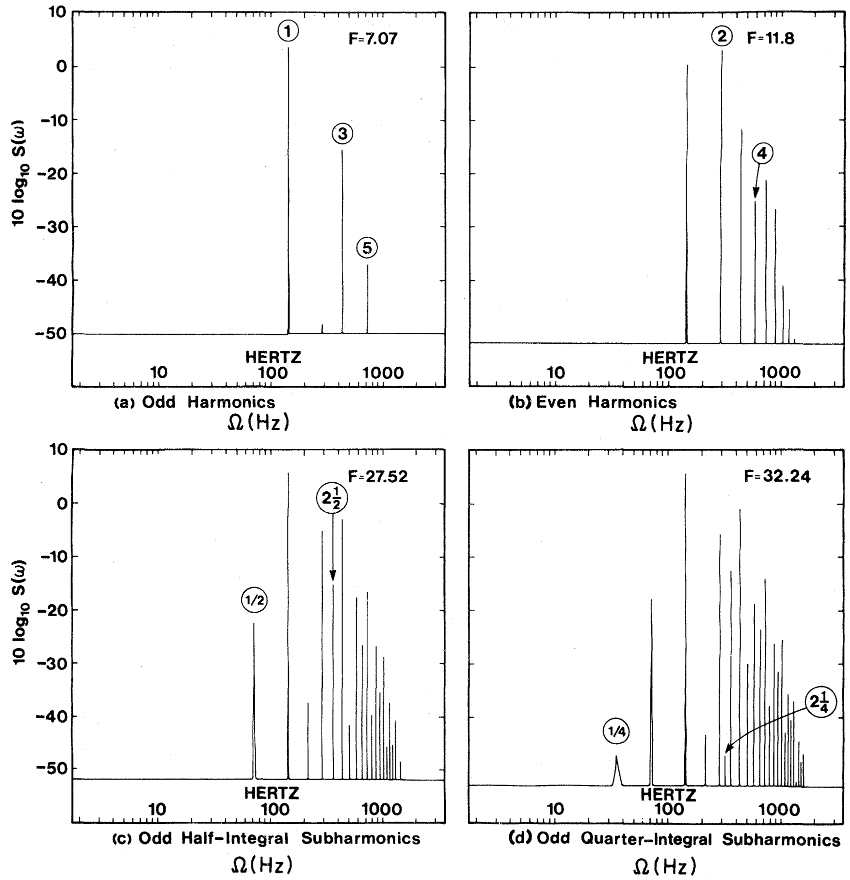

FIG. 1. Log-log plots of spectral density vs frequency. Primary response labeled ① is at driving frequency $\Omega=144$ Hz; $\omega_0=109$ Hz. Parameters $2\zeta=\alpha/\omega_0$ and $B=\beta/\omega_0^2$ are fixed at 0.1 and $0.5/\mathrm{V}^2$, respectively. Spectra indicate that bifurcation at $\gamma\approx 2\Omega$ from solution composed of (a) only odd harmonics to solution composed of (b) both odd and even harmonics preceded period-doubling bifurcations, (c) and (d).

$\gamma^2=\omega_0^2+\frac{3}{2}\beta A_1^2+\frac{3}{2}\beta A_3^2$. H, G, and λ are functions of A_1, ϕ_1, and A_3, ϕ_3 being defined by

$$H=[\,9\beta^2A_1^2A_3^2+\tfrac{9}{4}(\beta A_1^2)^2+9\beta^2A_1^3A_3\cos(\phi_3-3\phi_1)\,]^{1/2}\ ,$$

$$G=3\beta A_1A_3\ , \qquad (6)$$

$$\tan\lambda=\frac{3\beta A_1A_3\sin(\phi_3-\phi_1)+\frac{3}{2}\beta A_1^2\sin 2\phi_1}{3\beta A_1A_3\cos(\phi_3-\phi_1)+\frac{3}{2}\beta A_1^2\cos 2\phi_1}\ .$$

Notice that $\eta=0$ is a solution to Eq. (5). However, we will show that for certain parameter regimes this solution is unstable. There exist even harmonic solutions to Eq. (5) that grow exponentially in time. Of course Eq. (5) is only valid for small η. Thus, when $F>F_E$ and the solution to the linear equation diverges, it is necessary to include the nonlinear term $\beta\eta^3$ in Eq. (5) in order to determine the *stable* even harmonic solution in that regime. Nevertheless, the linear analysis is sufficient to determine the value F_E at which the solution $\eta=0$ first becomes unstable. We proceed now with the linear analysis.

First, it is convenient to transform Eq. (5) into the form of Hill's equation. This is accomplished by using the transformation $\eta(t)=\mu(t)^{-(\alpha/2)t}$ to eliminate the damping term $\alpha\dot\eta$. The transformation also results in a frequency shift, $\gamma'^2=\gamma^2-\frac{1}{4}\alpha^2$. When either $H=0$ or $G=0$, the equation reduces to Mathieu's equation. Reference 13 gives a concise explanation of parametric resonance as described by Mathieu's equation. Here we generalize their results.

A system parametrically driven at frequency ν exhibits resonance whenever $\nu\approx 2\gamma'/k$, where k is an integer. In Eq. (5) two driving frequencies occur, $\nu_1=4\Omega$ and $\nu_2=2\Omega$. When $\gamma'\approx 2\Omega$ resonance occurs, the order of the resonance being $k_1=1$ and $k_2=2$ for driving frequencies $\nu_1=4\Omega$ and $\nu_2=2\Omega$, respectively. For $\gamma' \approx 2\Omega$, μ is of the form

$$\mu = A_0(t) + A_2(t)\cos\left(2\Omega t + \frac{\phi_1+\phi_3}{2}\right) + A_2^*(t)\sin\left(2\Omega t + \frac{\phi_1+\phi_3}{2}\right) + \cdots \ , \tag{7}$$

where the ellipsis represents higher-order even harmonics and $A_0(t)$, $A_2(t)$, $A_2^*(t) \sim e^{\rho t}$. Notice that, although the resonance occurs at $\gamma' \approx 2\Omega$, the zero frequency mode is coupled to lowest order to the 2Ω mode through the term $(H/\gamma^2)\cos(2\Omega t + \lambda)$ in Eq. (5).

Since resonance occurs for γ' in the neighborhood of 2Ω, it is convenient to define $\epsilon = (4\Omega^2 - \gamma^2)/4\Omega$ to be the difference between these two frequencies. Then from Eqs. (5) and (7) one obtains the following equations for A_0, A_2, and A_2^*.

$$A_0 = -\frac{H}{2\gamma'^2}\left[A_2\cos\left(\frac{\phi_1+\phi_3}{2} - \lambda\right) + A_2^*\sin\left(\frac{\phi_1+\phi_3}{2} - \lambda\right)\right] \ , \tag{8a}$$

$$\begin{bmatrix} -4\Omega\rho + \frac{H^2}{4\gamma'^2}\sin(2\lambda - \phi_1 - \phi_3) & -4\Omega\epsilon - \frac{G}{2} - \frac{H^2}{4\gamma'^2}[1 - \cos(2\lambda - \phi_1 - \phi_3)] \\ -4\Omega\epsilon + \frac{G}{2} - \frac{H^2}{4\gamma'^2}[1 + \cos(2\lambda - \phi_1 - \phi_3)] & 4\Omega\rho + \frac{H^2}{4\gamma'^2}\sin(2\lambda - \phi_1 - \phi_3) \end{bmatrix} \begin{bmatrix} A_2 \\ A_2^* \end{bmatrix} = 0 \ . \tag{8b}$$

$A_0(t)$, $A_2(t)$, and $A_2^*(t)$ are assumed to be slowly varying functions of time. Thus, second derivatives of these quantities with respect to time have been neglected. In order for nontrivial solutions of Eq. (8) to exist, it is necessary that the determinant of the matrix multiplying (A_2, A_2^*) vanish. It therefore follows that

$$(4\Omega\rho)^2 = -(-4\Omega\epsilon)^2 - \frac{2H^2\Omega\epsilon}{\gamma'^2} + \frac{G^2}{4} - \frac{GH^2}{4\gamma'^2}\cos(2\lambda - \phi_1 - \phi_3) \ . \tag{9}$$

Since $\mu \sim e^{\rho t}$, it follows that $\eta \sim e^{(\rho - \alpha/2)t}$. For real ρ, η will be unstable if $\rho \geq \alpha/2$. We are interested in determining F_E, the value of F at which the instability first sets in. Therefore, we set $\rho = \alpha/2$ in Eq. (9). F_E is defined to be the smallest F that satisfied Eq. (9) with $\rho = \alpha/2$.

In Table I, we compare our predictions for F_E with experiment. Here $\omega_0 = 109$ Hz, $2\zeta = \alpha/\omega_0 = 0.1$ and $B = \beta/\omega_0^2 = 1.0/\text{V}^2$. For odd solutions the output voltage has zero mean. However, when F exceeds F_E, the solution acquires a zero-frequency component A_0, and the output voltage has a dc offset. We use a voltmeter to determine when the offset first deviates from zero. As can be seen in Table I, the experimental values of F_E are in good agreement with our theoretical predictions over the range 90–300 Hz.

The bifurcation to solutions containing both odd and even harmonics occurs as a result of a parametric excitation of an even harmonic. In the simplest case which is treated here (Fig. 1) the excitation is at $\gamma' \approx 2\Omega$. From Figs. 1(c) and 1(d) the period-doubled solutions appear to arise from parametric excitations of the $2\frac{1}{2}$ and $2\frac{1}{4}$ subharmonics, respectively. Generalizing this result, we shall assume that the kth period-doubling bifurcation occurs as a result of a parametric excitation of the $(2+1/2^k)$ subharmonic at F_k. Notice also from Fig. 1 that the response of the system at $(1/2^k)\Omega$ is comparable to that at $(2+1/2^k)\Omega$. This suggests that the two modes are coupled to lowest order as was previously the case for the zero-frequency and 2Ω modes. Based on these observations we suggest a mechanism for the period-doubling bifurcations.

Let ξ_k be that component of $x(t)$ composed of odd $1/2^k$ subharmonics. To lowest order we assume that ξ_k contains a $1/2^k$ subharmonic and a $(2+1/2^k)$ sub-

TABLE I. F_E is the minimum value of the input voltage at which even harmonics first occur. Theoretical predictions of F_E are compared with experimental values. The damping parameter $2\zeta = \alpha/\omega_0 = 0.1$, the nonlinearity parameter $B = \beta/\omega_0^2 = 1/\text{V}^2$, and $\omega_0 = 109$ Hz.

Ω (Hz)	F_E (V) Experiment	Theory
90	1.4–1.5	1.50
120	3.52	3.39
150	6.28	6.48
180	11.03	11.1
210	17.5	17.5
240	25.5	25.9
270	36.5	36.8
300	47.7	50.4

harmonic and that ξ_k satisfies

$$\ddot{\xi}_k + \gamma'^2 \left\{ 1 + \frac{H}{\gamma'^2}\cos(2\Omega t + \lambda) + \frac{6\beta\xi_{k-1}}{\gamma'^2} A_2 \cos(2\Omega t + \phi_2) \right\} \xi_k = 0 \quad . \quad (10)$$

ξ_{k-1}, contained in the last term in Eq. (10), is determined by solving the appropriate *nonlinear* equation for ξ_{k-1} including the cubic term. The above equation, linear in ξ_k, determines F_k. ξ_0 is defined to be

$$\xi_0 \approx A_1 \cos(\Omega t + \phi_1) + A_3 \cos(3\Omega t + \phi_3) \quad ,$$

and, in general, the *stable* solution ξ_k is of the form

$$\xi_k \approx C_k \cos\left[2\frac{1}{2^k}\Omega t + \delta_k \right] + D_k \cos\left[\frac{1}{2^k}\Omega t + \sigma_k \right] \quad . \quad (11)$$

The damping term is removed from Eq. (10) by the transformation used earlier in our discussion of the even harmonics. H and λ are defined by Eq. (6) and

$$\gamma'^2 = \omega_0^2 - \frac{\alpha^2}{4} + \tfrac{3}{2}\beta \left[2A_0^2 + A_1^2 + A_2^2 + A_3^2 + \sum_j (C_j^2 + D_j^2) \right] \quad . \quad (12)$$

For $F < F_E$, γ' reduces to the γ' defined earlier. For $F < F_k$, the upper limit on the summation is $k-1$, since modes whith $j > k-1$ have not yet appeared.

There are four combination frequencies formed from the parametric driving term

$$(6\beta\xi_{k-1}/\gamma'^2) A_2 \cos(2\Omega t + \phi_2)$$

in Eq. (10). Of these four, we single out $\nu_1 = (4 + 1/2^{k-1})\,\Omega$ because for $\gamma' \approx (2 + 1/2^k)\,\Omega$, ν_1 results in a first-order resonance that triggers the $(2 + 1/2^k)\,\Omega$ mode. The $(1/2^k)\,\Omega$ mode is also triggered because it is coupled to the $(2 + 1/2^k)\,\Omega$ mode via the term $H/\gamma'^2 \cos(2\Omega t + \lambda)$ in Eq. (10).

Note from Eq. (12) that if $A_2 = 0$, the driving term for a parametric excitation of the $(2 + 1/2^k)\,\Omega$ mode is zero, and no period doubling bifurcations can occur. This agrees with our observation that the bifurcation to a solution containing even harmonics must precede the period-doubling bifurcations. Note also that the occurrence of the $(k-1)$st bifurcation is a necessary condition for the kth bifurcation since ξ_{k-1} is included in the parametric driving term for ξ_k. Thus, this mechanism explains the observed sequence of bifurcations:

$$T'/T = 2^0, 2, 2^2, 2^3, \ldots, 2^k, \ldots \quad ,$$

where T' is the period of $x(t)$ and T is the period of $F(t)$.

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