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Hard-core square-well fermions

George A. Baker, Jr. and L. P. Benofy*
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

M. Fortes, M. de Llano, S. M. Peltier,† and A. Plastino‡
Instituto de Física, Universidad Nacional Autónoma de México, 01000 Distrito Federal, México
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Based on the well-known low-density expansion for the ground-state energy per particle of a many-fermion system interacting pairwise via a central potential we develop a perturbation scheme in which the zero-order state is that of the purely-repulsive-spheres fluid. For the hard-core plus square-well pair-interaction case we have deduced, for two-species fermion matter, perturbation formulas up to sixth order in the attractive coupling parameter. We explicitly report results up to fourth order here. Padé-approximant analyses in the density are carried out in each order with the intent of providing the basis necessary for extensive equation-of-state calculations of various many-fermion systems. In particular, for zero order (the fermion hard-sphere fluid) we predict an uncertainty-principle divergence in the energy, and hence pressure, corresponding to the random close packing of spheres, at a density of about 0.5 times the recently obtained value for boson hard spheres and about 0.2 times the well-known empirical value for classical hard spheres.

I. INTRODUCTION

There are essentially two general approaches to the calculation of the ground-state properties of a many-body system based on first principles, i.e., on, for example, the two-body interaction: (1) Variational trial function techniques[1] and (2) perturbation theory.[2]

Within the *second* approach, one usually considers the ideal, i.e., noninteracting, gas of particles as the unperturbed problem which can be solved and then applies Rayleigh-Schrödinger perturbation theory to the interaction potential. For fermions with singular (e.g., hard-core) interactions, one must carry out infinite order partial summations of selected classes of contributions in order to achieve finite values for, for example, the energy per particle. In this way, the various techniques of quantum-field theory have given for the ground-state energy E of a system of N fermions of mass m, contained in a volume V, the clearly *low-density* result[3]

$$\frac{10Em}{3\hbar^2k_F^2N}\equiv\epsilon=1+C_1k_Fa+C_2(k_Fa)^2+C_3\tfrac{1}{2}r_0a^2k_F^3+C_4A_1(0)k_F^3+C_5(k_Fa)^3+C_6(k_Fa)^4\ln|k_Fa|+C_7\tfrac{1}{2}r_0a^3k_F^4+C_8aA_0''(0)k_F^4+C_9(k_Fa)^4+\cdots\ . \quad (1)$$

Note the singular nature of the expansion because of the log term. The $C_1,\ldots,C_9$ are dimensionless coefficients depending on ν; the latter together with the Fermi-momentum $\hbar k_F$ is defined through the particle number density

$$\rho=N/V=\nu k_F^3/6\pi^2\ , \quad (2)$$

ν being the number of intrinsic degrees of freedom associated with each fermion and a, r_0, $A_1(0)$ and $A_0''(0)$ are parameters containing information related to *two*-body scattering due to a central potential $V(r)$. Specifically, if $\delta_l(k)$ is the l-wave scattering phase shift in relative momentum $\hbar k$ then, as in "effective-range theory,"[4] the well-known expressions

$$k\cot\delta_0(k)\underset{k\to0}{\simeq}-\frac{1}{a}+\tfrac{1}{2}r_0k^2+O(k^4)\ , \quad (3)$$

$$k^3\cot\delta_1(k)\underset{k\to0}{\simeq}-\frac{1}{A_1(0)}+O(k^2) \quad (4)$$

define the S-wave scattering length a and effective range r_0, and the P-wave scattering length (cubed) $A_1(0)$. The first quantity appearing in (1) which is *not* determined by scattering phase shifts at all is

TABLE I. Dimensionless coefficients in energy formula Eq. (1).

ν	Case	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
2	Complete	0.353 678	0.185 537	0.106 103	0.954 930	0.030 467	0	0.164 207	0.052 267	−0.062 013
2	Ladder	0.353 678	0.185 537	0.106 103	0.954 930	0.106 770	0	0.089 153	0.127 320	0.063 853
4	Complete	1.061 033	0.566 610	0.318 310	1.591 549	0.664 000	1.408 598	?	?	?
4	Ladder	1.061 033	0.556 610	0.318 310	1.591 549	0.320 310	0	0.267 460	0.381 960	0.191 560

$$A_0''(0) \equiv -\frac{m}{3\hbar^2}\int_0^\infty dr\, r^4 V(r)\psi_0(0,r)\,, \tag{5}$$

where $\psi_0(0,r)$ is the $l=0$, $k=0$ radial scattering wave function.

Table I contains the coefficients C_1 to C_9 of (1) for both $\nu=2$ and 4 for both the "ladder approximation" (Bethe-Goldstone equation) as well as the "complete" theory, as discussed in Ref. 3. [In elaborating Table I, two arithmetic errors in Ref. 3, Eq. (4.88), have been corrected, namely, the coefficients of $a^2 r_0 k_F^6$ and $k_F^6 a^4$ should be, respectively, 0.024 631 and −0.018 604, instead of 0.045 899 and −0.022 16.]

In Sec. II we give explicit formulas for a, r_0, $A_1(0)$, and $A_0''(0)$ for the hard-core plus square-well pair potential. Section III transforms the perturbation scheme about the ideal gas into a scheme about the hard-sphere fluid. Section IV examines the latter and reports results of a Padé-approximant analysis in the density, predicting the random close-packing density for fermion hard spheres. In Sec. V we study the first four perturbative orders in the strength of a pure attractive well pair potential in terms of a density Padé analysis and compare with the corresponding exact (Monte Carlo) results available in the literature for both the ladder and complete theories with the objective of selecting out the best extrapolant in each order. Section VI has our conclusions.

II. HARD-CORE SQUARE-WELL POTENTIAL

As we are initially mainly interested in how close one can approach the ground-state energy of the many-fermion Schrödinger equation—for intermediate density and coupling—we pick the simple (yet sufficiently rich in physical content) pair potential

$$V(r) = \begin{cases} \infty, & r < c \\ -V_0, & c < r < R \\ 0, & r > R \end{cases} \tag{6}$$

whose parameters for liquid helium[5] and nucleonic matter[3] are found in the literature. We define, respectively, the dimensionless density x, attractive well strength λ and attractive well range α parameters by

$$x \equiv k_F c, \quad \lambda \equiv \frac{mV_0}{\hbar^2}(R-c)^2, \quad \alpha \equiv \frac{R-c}{c} . \tag{7}$$

Using effective-range theory[4] and matching the radial wave function and its first derivative at $r=R$, we finally obtain the four expressions

$$k_F a = x\left[1+\alpha\left[1-\frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}}\right]\right] \underset{\lambda\to 0}{\longrightarrow} x , \tag{8}$$

$$\tfrac{1}{2} r_0 a^2 k_F^3 = \tfrac{1}{3}\alpha^3 x^3\left[\left[\frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}}\right]^3+\left[1+\frac{1}{\alpha}-\frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}}\right]^3-\tfrac{3}{2}\left[\frac{1+\tan^2\sqrt{\lambda}}{\lambda}-\frac{\tan\sqrt{\lambda}}{(\sqrt{\lambda})^3}\right]\right] \underset{\lambda\to 0}{\longrightarrow} \tfrac{1}{3}x^3 , \tag{9}$$

$$A_1(0)k_F^3 = \tfrac{1}{3}(\alpha+1)^3 x^3\left[1-\frac{3\alpha^3}{(1+\alpha)^2\lambda}\left[\frac{\left[1+\dfrac{1+\alpha}{\alpha^2}\lambda\right]\dfrac{\tan\sqrt{\lambda}}{\sqrt{\lambda}}-1}{1+\alpha\dfrac{\tan\sqrt{\lambda}}{\sqrt{\lambda}}}\right]\right] \underset{\lambda\to 0}{\longrightarrow} \tfrac{1}{3}x^3 , \tag{10}$$

$$A_0''(0)k_F^3 = \left[\frac{x}{\alpha}\right]^3\left\{-\tfrac{1}{3}\alpha^3+\frac{\alpha^6}{3(\sqrt{\lambda})^3}\left[(3\lambda-6)\tan\sqrt{\lambda}+\sqrt{\lambda}(6-\lambda)+\frac{3}{\alpha}\sqrt{\lambda}(2\sqrt{\lambda}\tan\sqrt{\lambda}-\lambda+2-2\sec\sqrt{\lambda}) + \frac{3}{\alpha^2}\lambda(\tan\sqrt{\lambda}-\sqrt{\lambda})-\left[\frac{\sqrt{\lambda}}{\alpha}\right]^3(1-\sec\sqrt{\lambda})\right]\right\} \underset{\lambda\to 0}{\longrightarrow} -\tfrac{1}{3}x^3 . \tag{11}$$

We note that the pure hard-core limits indicated yield the well-known values for a, r_0, $A_1(0)$, and $A_0''(0)$ as c, $\frac{2}{3}c$, $\frac{1}{3}c^3$, and $-\frac{1}{3}c^3$, respectively. Also, for $c\to 0$ and $\lambda\to -\lambda$, Eqs. (8) to (11) reduce, as they should, to those of the repulsive barrier potential, Eqs. (4.90) to (4.94), of Ref. 4.

III. NEW PERTURBATION SCHEME

The four expressions (8) to (11) appear in the energy (1) in the nine different combinations shown there. These nine terms have been expanded in powers of λ, around $\lambda=0$, to sixth order by two different computer algorithms: (i) a FORTRAN subroutine that manipulates double series and (ii) a LISP program for algebraic manipulation called REDUCE,[6] which carries out and evaluates nth-order derivatives. The results of these two schemes appear in Appendix A.

Substituting into (1) there results the double series, in x and λ, for the energy

$$\epsilon = \sum_{j=0}^{\infty} \epsilon_j(x)\lambda^j = \sum_{i=0}^{4}\sum_{j=0}^{\infty} \epsilon_{ij}(\alpha)x^i\lambda^j , \tag{12}$$

valid for ladder, $\nu=2$ and 4, and complete $\nu=2$, theories. Clearly, $\epsilon_j(x)\lambda^j$ represents the jth-order perturbation correction to the energy of the hard-sphere fluid, at density x, due to attractions between particles. For the complete theory with $\nu=4$ we have an expression identical to (12) *except* that the $i=4$ term is *not* x^4 *but rather* $x^4\ln x$, the coefficient of the x^4 term (of smaller order than the $x^4\ln x$ one) being unknown at present. The coefficients $\epsilon_{ij}(\alpha)$ are listed in Appendix B up to 4th order, i.e., $j=1,2,3,4$. In Secs. IV and V we treat the term $\epsilon_0(x)$ in (12), namely, the energy per particle of the hard-sphere fluid, in both the ladder and complete pictures.

IV. HARD-SPHERE FERMIONS IN LADDER APPROXIMATIONS

We will henceforth restrict ourselves to $\nu=2$ fermion matter. In this section we try our methods on the ladder approximations because there is an exact representation of the sum of all the diagrams in this approximation in terms of the solution of an integral equation, the Bethe-Goldstone equation. Numerical solutions for the hard-sphere case are reported by Baker *et al.*[7] These provide the opportunity to compare the results of our methods with an exact answer. Instead of the energy becoming infinite at a finite value of k_F, as we expect in the physical problem, the ladder approximation remains finite at *all* values of k_F. An examination of the numerical solution reveals that the ladder energy behaves approximately like $(k_F c)^6$ as $k_F\to\infty$.

By putting $\lambda=0$ in Eq. (B3) of Appendix B, we obtain

$$\epsilon_0(x) \equiv \frac{10E_L m}{3N\hbar^2 k_F^2} = 1+\sum_{i=1}^{4} D_i x^i + O(x^5) \tag{13}$$

TABLE II. Expansion coefficients for $\epsilon_0(x)$, Eqs. (13) and (15).

Case	D_1	D_2	D_3	D_4
Ladder	0.353 678	0.185 537	0.460 448	0.051 131
Complete	0.353 678	0.185 537	0.384 144	−0.024 700

in units of the ideal Fermi-gas kinetic energy. The D_i are listed in Table II. For reasons explained in Sec. V in regard to the physical energy, it is better to study $\epsilon_0(x)^{-1/2}$ instead of $\epsilon_0(x)$ directly. Here we have, by the observation made above, the result that $\epsilon_0(x)^{-1/2}$ is proportional to x^{-2} as $x \equiv k_F c \to \infty$. The procedure is to extrapolate the function values to positive values of x from the series coefficients given in Table III. We give our best results based on the Padé approximant method[8] as

$$\epsilon_0(x) \simeq [1/3](x)^{-2} = \left[\frac{1+0.249\,3688x+0.089\,958\,608x^2+0.221\,7122x^3}{1+0.072\,527\,551x} \right]^2 . \tag{14}$$

A comparison is given in Table IV with the results from the solution of the integral equation. The solution of the integral equation is thought to be accurate to within about 0.1%. We find the accuracy of this result to be satisfactory for our purposes.

V. HARD-SPHERE FERMIONS IN THE COMPLETE THEORY

Setting $\lambda=0$ in the energy formulas of Appendix B we quickly arrive at the fourth-order polynomial

$$\epsilon_0(x) = 1 + D_1 x + D_2 x^2 + D_3 x^3 + D_4 x^4 + \cdots \tag{15}$$

with further terms being unknown at present. The coefficients D_i are given in Table II. Equation (15) is our only concrete knowledge of the gound-state energy of a two-species fermion hard-sphere fluid. On physical grounds we expect that the energy should diverge at a value of x, say x_P, corresponding to a *random* close packing of the spheres. For the *classical* fluid of hard spheres, it is known[9] from experiments (with ball bearings, for instance) that the random-close-packing density occurs at about 0.86 times the *regular* close-packing density $\rho_0 \equiv \sqrt{2}/c^3$, the latter occurring when the spheres of diameter c are packed in a face-centered-cubic or hexagonal-close-packing arrangement. We expect *quantum* hard spheres to random close pack at a density *smaller* than the classical value of $0.86\sqrt{2}/c^3$, because of well-known diffraction effects. (The latter are responsible, e.g., for making the total cross section for the scattering of two identical hard spheres equal to *four* times the geometrical cross section of πc^2, for zero energy, and *twice*[10] that value for infinite energy.) For *bosons,* the random-close-packing density found in a recent study,[11] which agrees very well with available Green's-function Monte Carlo data points, is about 0.35 times $\rho_0 \equiv \sqrt{2}/c^3$; this is not inconsistent with our expectation as described above. For the present problem we expect the random-close-packing density for *fermions* to be *strictly less* than the boson value, since Pauli exclusion effects provide additional repulsion between the hard spheres. The pole $\epsilon_0(x_P)=\infty$ must, moreover, be a second-order pole, since it is essentially due to uncertainty principle arguments, namely, that the energy will diverge as the inverse of an available "length" squared. This property is clearly absent from (15) and one could not expect more, given that it is a *low-density* expansion.

We found it convenient to convert (15) into

TABLE III. Expansion coefficients for $\epsilon_0(x)^{-1/2}$, Eq. (16).

Case	F_1	F_2	F_3	F_4
Ladder	−0.176 833	−0.045 863	−0.194 827	0.091 995
Complete	−0.176 833	−0.045 863	−0.156 677	0.109 672

TABLE IV. Comparison of the solution of the integral equation and the prediction of the [1/3] Padé approximant for $\epsilon_0(x)^{-1/2}$.

x	[1/3]	integral equation
0.25	0.950 25	0.949 81
0.50	0.881 97	0.881 24
0.75	0.791 97	0.793 91
1.00	0.686 86	0.694 491
1.50	0.476 64	0.497 58
2.00	0.314 928	0.342 11
3.00	0.142 302	0.147 060

$$\epsilon_0(x)^{-1/2}=1+F_1x+F_2x^2+F_3x^3+F_4x^4+O(x^5)\ , \qquad (16)$$

where the coefficients F_i are listed in Table III, and construct to (16) the Padé table, i.e., all $[L/M](x);L,M=0,1,2,\ldots$ for $L+M\leq 4$ and look for those having a zero at real, positive x. The *only* such solutions appear in Table V where, in addition, we list the corresponding density ρ_P, in units of ρ_0, where $\epsilon_0(x)$ will have the sought-after double pole, recalling from (2) that

$$\rho/\rho_0=x^3/3\sqrt{2}\pi^2 \quad (\nu=2)\ . \qquad (17)$$

The best candidate is clearly the one given by the [3/1] Padé, which yields a random-close-packing density of about one half the boson value of Ref. 11. The suggested extrapolant to (15) is therefore

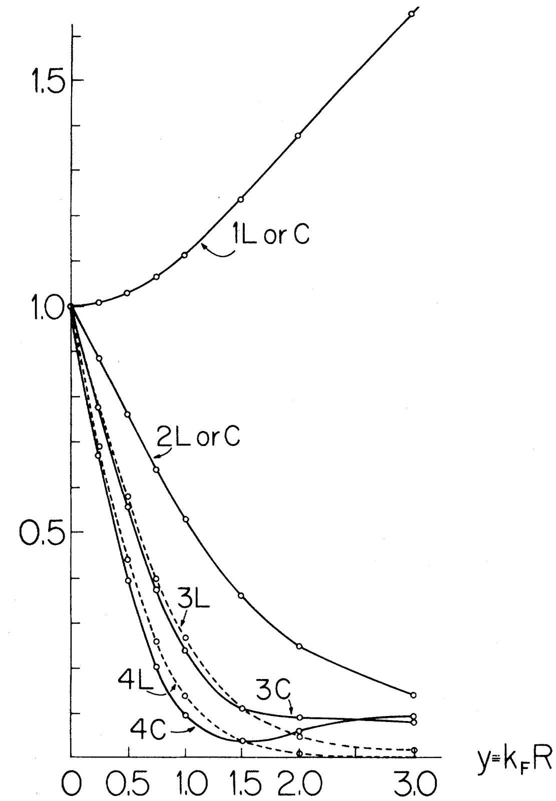

FIG. 1. First- (1) through fourth- (4) order energies, Eqs. (21), for the soft repulsive barrier potential of height unity and range R as function of $y\equiv k_FR$, for both the complete (*C*) and ladder approximation (*L*) theory, as taken from Table III of Ref. 3. Both theories are identical in first and second order.

$$\epsilon_0(x)\simeq[3/1](x)^{-2}=\left[\frac{1+0.699\,986\,10x}{1+0.523\,152\,76x-0.169\,644\,17x^2-0.188\,780\,69x^3}\right]^2 . \qquad (18)$$

We next discuss the first four orders in the perturbative correction to this energy, due to the presence of attractive forces.

VI. TREATMENT OF ATTRACTIVE FORCES

In order to compare with exact results previously carried out with Monte Carlo techniques up to fourth order for both the ladder and complete theories, we shall study only the purely attractive square-well-potential problem. (The exact results are listed in Table III of Ref. 3. Note, however, that the opposite sign convention from ours for the potential is used there, i.e., in Ref. 3 plus is repulsive, and here plus is attractive.) This will permit

TABLE V. Location of zero of $[L/M](x)$ Padé approximants to $\epsilon_0(x)^{-1/2}$, Eq. (16).

L,M	1,1	1,2	2,1	2,2	3,1
x_p	2.2926	0.4323	0.2915 6.1449	1.2159	1.939 15
ρ_P/ρ_0	0.2878	0.0019	0.0006 5.541	0.0429	0.1741

TABLE VI. Coefficients of the form Eq. (20) in orders 1–4. Suffixes c and l refer to complete and ladder theories, respectively.

Order	f_0	f_1	f_2	f_3
1	1	0.0	0.120 00	0.0
2	1	−0.437 161	−0.142 857	0.102 011
3c	1	−0.864 029	−0.238 029	0.360 693
3l	1	−0.864 029	−0.090 079	0.285 792
4c	1	−1.289 476	−0.251 810	0.871 143
4l	1	−1.289 476	0.186 645	0.450 267

us to select from the complete Padé approximant which best fits the data and thus suggest specific approximants for each order. Thus, we take, according to (7),

$$c=0,\quad \alpha\equiv\frac{R-c}{c}\to\infty\ ,$$

$$\alpha x\equiv k_F(R-c)=k_FR\equiv y\ , \tag{19}$$

so that our density variable is now y. Substituting in the energy formulas of Appendix B we get, for the first four orders, a form

$$f_0+f_1y+f_2y^2+f_3y^3 \tag{20}$$

for each expression

$$-\tfrac{3}{10}\epsilon_1(y)/0.035\,368y\ , \tag{21a}$$

$$-\tfrac{3}{10}\epsilon_2(y)/0.014\,147\,11y\ , \tag{21b}$$

$$-\tfrac{3}{10}\epsilon_3(y)/0.005\,7262y\ , \tag{21c}$$

$$-\tfrac{3}{10}\epsilon_4(y)/0.002\,3204y\ . \tag{21d}$$

The coefficients f_0 to f_3 appear in Table VI. (Note that ladder and complete theories are identical in orders one and two.) In Fig. 1 we have plotted, using the Monte Carlo data of Ref. 3, the quantities of Eq. (21). We observe the following features. In first order, the behavior is quite mild and varies by less than a factor of 2. In second to fourth order the dominant feature is that these higher-order terms drop precipitously with increasing density. This effect means that, for the same strength of attraction, the higher-order terms become rapidly less important as the density increases. A minor exception is that the complete fourth-order term (labeled 4C in Fig. 1) goes through a rather small minimum at about $y=1.5$ and then increases slowly.

Table VII gives the best-fitting Padé approximant of (20), which turned out to be the [3/0] for first and the [0/3] for second order. These are listed up to $y=k_FR=3$. (For comparison, the saturation density of liquid helium three is $x=1.3$.)

Finally, Tables VIII and IX give the corresponding best-fitting extrapolant to third and fourth orders, where we found it necessary, because of the sharp decrease in y, to introduce the exponent $1/n$ to achieve better fits, in both (19c) and (19d), and consequently the general form (18). In third order, Table VIII, we find that $n=3$ gives a good fit nearly to $y=2$ for the ladder case, but only up to $y=1$ for the complete. On the other hand, $n=2$ gives a poorer quality fit but not too bad to about $y=1.5$. In fourth order, Table IX, the ladder case, $n=8$, gives a good fit to about $y=1.5$. For the complete there is a "knee" in the curve at about $y=1.5$ and we cannot go beyond this point with the number of available coefficients in the y series and this method. We find that $n=5$ gives a good fit to $y=1$ and perhaps even to $y=1.25$.

TABLE VII. Comparison of best Padé approximant, for first and second orders, to Monte Carlo (MC) data for several values of $y\equiv k_FR$ for the purely attractive square well of range R, depth unity.

	$-\frac{3}{10}y^2\epsilon_1(y)$		$-\frac{3}{10}y^2\epsilon_2(y)$	
y	MC	$0.035\,367\,76y^2[3/0](y)$	MC	$0.014\,147\,11y^3[0/3](y)$
0.25	5.567×10^{-4}	5.5677×10^{-4}	1.960×10^{-4}	$1.953\,03\times10^{-4}$
0.50	4.5516×10^{-3}	4.5536×10^{-3}	1.346×10^{-3}	$1.344\,40\times10^{-3}$
0.75	1.589×10^{-2}	1.5928×10^{-2}	3.827×10^{-3}	$3.824\,30\times10^{-3}$
1.00	3.936×10^{-2}	$3.961\,19\times10^{-2}$	7.495×10^{-3}	$7.534\,81\times10^{-2}$
1.50	$1.475\,17\times10^{-1}$	1.5159×10^{-1}	1.715×10^{-2}	$1.725\,95\times10^{-2}$
2.00	$3.901\,006\times10^{-1}$	4.1875×10^{-2}	2.868×10^{-2}	$2.786\,45\times10^{-2}$
3.00	1.574 696	1.968 25	5.367×10^{-2}	$4.663\,311\times10^{-2}$

TABLE VIII. Comparison of best Padé approximant to $[-\frac{3}{10}\epsilon_3(y)/0.005\,7262y]^{1/n}$ with MC data, for the ladder (lad) and complete (comp) theories.

y	MC(lad)	[0/3] $n=3$ (lad)	MC(comp)	[0/3] $n=3$ (comp)	MC(comp)	[0/3] $n=2$ (comp)
0.25	0.921 58	0.921 624	0.918 76	0.918 593	0.880 65	0.880 529
0.50	0.833 59	0.833 793	0.823 68	0.822 607	0.747 55	0.747 561
0.75	0.738 81	0.742 031	0.719 90	0.720 031	0.610 81	0.616 348
1.00	0.645 20	0.651 572	0.620 50	0.618 845	0.488 77	0.498 391
1.50	0.483 91	0.489 154	0.483 03	0.441 931	0.335 71	0.318 681
2.00	0.376 25	0.361 501	0.450 56	0.310 724	0.302 41	0.205 605
3.00	0.260 40	0.200 025	0.435 27	0.158 652	0.287 17	0.094 121

VII. CONCLUSIONS

The *raison d'etre* of this paper is contained in Sec. III: The new perturbation scheme for the ground-state energy of the many-fermion system. The idea is to develop a way to expand about the hard-sphere gas in terms of the interparticle attraction. Our approach is to make use of the low-density behavior, which is known to all orders in the interparticle potential, together with such other information as is available to develop expressions as functions of k_F for the first few derivatives with respect to the attractive part of the interparticle potential. With these expressions in hand, one can then use them to complete the ground-state energy as a function of k_F without great difficulty, and to determine the saturation density and energy. We have previously investigated, with considerable success, the many boson problem[5,11] by these methods. We were greatly assisted in that investigation by the Monte Carlo determination[12] of the hard-sphere boson-gas ground-state energy, and feel that similar results would be desirable for fermions.

Specifically, we conclude that our approach looks like a practical one. The higher derivatives in the attraction decrease (relative to the lower ones) quite rapidly in k_F and are well determined by the exact low-density results for small k_F so that for practical purposes our representations should do very well here. Fortunately, the low-order derivatives behave relatively simply so that we conclude that our expressions for them are quite satisfactory.

Therefore, in this paper we have developed a sound basis for the application of this method to physical problems and expect to do so in the future.

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APPENDIX A: SCATTERING-PARAMETER FORMULAS

This appendix summarizes expansions of the two-body scattering parameters a (scattering length), v_0 (effective range), $A_1(0)$, and $A_0''(0)$, as defined in Eqs (8)—(11) for the potential described in Eq. (6). The series expansions for these scattering parameters and for some related quantities, which appear in the ground-state en-

TABLE IX. Comparison of best Padé approximant to $[-\frac{3}{10}\epsilon_4(y)/0.002\,3204y]^{1/n}$ with MC data for the ladder (lad) and complete (comp) theories.

y	MC(lad)	[0/3] $n=8$ (lad)	MC(comp)	[0/3] $n=5$ (comp)
0.25	0.954 88	0.955 653	0.923 70	0.924 611
0.50	0.903 50	0.904 102	0.830 94	0.831 277
0.75	0.846 10	0.847 059	0.727 22	0.728 354
1.00	0.785 03	0.786 520	0.626 52	0.624 935
1.50	0.665 41	0.662 887	0.527 55	0.442 306
2.00	0.576 03	0.546 463	0.569 39	0.307 278
3.00	0.451 92	0.360 293	0.621 94	0.153 446

ergy, in the notation of Eq. (7) are the following:

$$k_F a = x\left[1-\alpha\left(\tfrac{1}{3}\lambda+\tfrac{2}{15}\lambda^2+\tfrac{17}{315}\lambda^3+\tfrac{62}{2835}\lambda^4+\tfrac{1382}{155\,925}\lambda^5+\tfrac{21\,844}{6081\,075}\lambda^6+\cdots\right)\right], \tag{A1}$$

$$\begin{aligned}(k_F a)^2 = x^2\Big\{1-\alpha\Big[&\tfrac{2}{3}\lambda+\left(\tfrac{4}{15}-\tfrac{1}{9}\alpha\right)\lambda^2+\left(\tfrac{34}{315}-\tfrac{4}{45}\alpha\right)\lambda^3+\left(\tfrac{124}{2835}-\tfrac{254}{4725}\alpha\right)\lambda^4\\ &+\left(\tfrac{2764}{155\,925}-\tfrac{176}{6075}\alpha\right)\lambda^5+\left(\tfrac{43\,688}{6\,081\,075}-\tfrac{47\,981}{3274\,425}\alpha\right)\lambda^6+\cdots\Big]\Big\},\end{aligned} \tag{A2}$$

$$\begin{aligned}(k_F a)^3 = x^3\Big\{1-\alpha\Big[&\lambda+\left(\tfrac{2}{5}-\tfrac{1}{3}\alpha\right)\lambda^2+\left(\tfrac{17}{105}-\tfrac{4}{15}\alpha+\tfrac{1}{27}\alpha^2\right)\lambda^3+\left(\tfrac{62}{945}-\tfrac{254}{1575}\alpha+\tfrac{2}{45}\alpha^2\right)\lambda^4\\ &+\left(\tfrac{1382}{51\,975}-\tfrac{176}{2025}\alpha+\tfrac{169}{4725}\alpha^2\right)\lambda^5\\ &+\left(\tfrac{21\,844}{2075\,025}-\tfrac{47\,981}{1091\,475}\alpha+\tfrac{5114}{212\,625}\alpha^2\right)\lambda^6+\cdots\Big]\Big\},\end{aligned} \tag{A3}$$

$$\begin{aligned}(k_F a)^4 = x^4\Big\{1-\alpha\Big[&\tfrac{4}{3}\lambda+\left(\tfrac{8}{15}-\tfrac{2}{3}\alpha\right)\lambda^2+\left(\tfrac{68}{315}-\tfrac{8}{15}\alpha+\tfrac{4}{27}\alpha^2\right)\lambda^3+\left(\tfrac{248}{2835}-\tfrac{508}{1575}+\tfrac{8}{45}\alpha^2-\tfrac{1}{81}\alpha^3\right)\lambda^4\\ &+\left(\tfrac{5528}{155\,925}-\tfrac{352}{2025}\alpha+\tfrac{676}{4725}\alpha^2-\tfrac{8}{405}\alpha^3\right)\lambda^5\\ &+\left(\tfrac{87\,376}{6081\,075}-\tfrac{95\,962}{1091\,475}\alpha+\tfrac{20\,456}{212\,625}\alpha^2-\tfrac{844}{42\,525}\alpha^3\right)\lambda^6+\cdots\Big]\Big\},\end{aligned} \tag{A4}$$

$$(k_F a)^4\ln|k_F a| = (k_F a)^4\ln x \quad \text{to} \quad O[(k_F a)^4], \tag{A5}$$

$$\begin{aligned}\tfrac{1}{2}r_0 a^2 k_F^3 = x^3\Big\{\tfrac{1}{3}-\alpha\Big[&\left(\tfrac{1}{3}-\tfrac{1}{15}\alpha^2\right)\lambda+\left(\tfrac{2}{15}-\tfrac{1}{9}\alpha-\tfrac{26}{315}\alpha^2\right)\lambda^2+\left(\tfrac{17}{315}-\tfrac{4}{45}\alpha-\tfrac{157}{2835}\alpha^2\right)\lambda^3\\ &+\left(\tfrac{62}{2835}-\tfrac{254}{4725}\alpha-\tfrac{4882}{155\,925}\alpha^2\right)\lambda^4+\left(\tfrac{1382}{155\,925}-\tfrac{176}{6075}\alpha-\tfrac{19\,802}{1216\,215}\alpha^2\right)\lambda^5\\ &+\left(\tfrac{21\,844}{6081\,075}-\tfrac{47\,981}{3274\,425}\alpha-\tfrac{5142\,932}{638\,512\,875}\alpha^2\right)\lambda^6+\cdots\Big]\Big\},\end{aligned} \tag{A6}$$

$$\begin{aligned}&\tfrac{1}{2}r_0 a^2 k_F^3(k_F a)\\ &= x^4\Big\{\tfrac{1}{3}-\alpha\Big[\tfrac{4}{9}-\tfrac{1}{15}\alpha^2\lambda+\left(\tfrac{8}{45}-\tfrac{2}{9}\alpha-\tfrac{26}{315}\alpha^2+\tfrac{1}{45}\alpha^3\right)\lambda^2+\left(\tfrac{68}{945}-\tfrac{8}{45}\alpha-\tfrac{52}{2835}\alpha^2+\tfrac{172}{4725}\alpha^3\right)\lambda^3\\ &\qquad+\left(\tfrac{248}{8505}-\tfrac{508}{4725}\alpha+\tfrac{2048}{155\,925}\alpha^2+\tfrac{1406}{42\,525}\alpha^3\right)\lambda^4\\ &\qquad+\left(\tfrac{5528}{467\,775}-\tfrac{352}{6075}\alpha+\tfrac{118\,493}{6081\,075}\alpha^2+\tfrac{25\,904}{1091\,475}\alpha^3\right)\lambda^5\\ &\qquad+\left(\tfrac{87\,376}{18\,243\,225}-\tfrac{95\,962}{3274\,425}\alpha+\tfrac{2042\,882}{127\,702\,575}\alpha^2+\tfrac{9569\,123}{638\,512\,875}\alpha^3\right)\lambda^6+\cdots\Big]\Big\},\end{aligned} \tag{A7}$$

$$\begin{aligned}A_0''(0)k_F^3(k_F a) = x^4\Big[&-\tfrac{1}{3}+\left(\tfrac{1}{6}+\tfrac{4}{9}\alpha+\tfrac{1}{4}\alpha^2+\tfrac{1}{15}\alpha^3\right)\lambda+\left(\tfrac{5}{72}+\tfrac{11}{90}\alpha-\tfrac{1}{72}\alpha^2-\tfrac{73}{1260}\alpha^3-\tfrac{1}{45}\alpha^4\right)\lambda^2\\ &+\left(\tfrac{61}{2160}+\tfrac{67}{2520}\alpha-\tfrac{143}{2880}\alpha^2-\tfrac{1259}{22\,680}\alpha^3-\tfrac{82}{4725}\alpha^4\right)\lambda^3\\ &+\left(\tfrac{277}{24\,192}+\tfrac{29}{19\,440}\alpha-\tfrac{7633}{201\,600}\alpha^2-\tfrac{50\,453}{1425\,600}\alpha^3-\tfrac{442}{42\,525}\alpha^4\right)\lambda^4\\ &+\left(\tfrac{50\,521}{10\,886\,400}-\tfrac{189\,037}{59\,875\,200}\alpha-\tfrac{98\,107}{4354\,560}\alpha^2-\tfrac{609\,157}{31\,135\,104}\alpha^3-\tfrac{730}{130\,977}\alpha^4\right)\lambda^5\\ &+\Big(\tfrac{540\,553}{287\,400\,960}-\tfrac{4365\,313}{1556\,755\,200}\alpha-\tfrac{40\,379\,113}{3353\,011\,200}\alpha^2\\ &\quad-\tfrac{3284\,054\,677}{326\,918\,592\,000}\alpha^3-\tfrac{1794\,487}{638\,512\,875}\alpha^4\Big)\lambda^6+\cdots\Big],\end{aligned} \tag{A8}$$

$$\begin{aligned}&A_1(0)k_F^3\\ &= x^3\Big\{\tfrac{1}{3}-\alpha\Big[\left(\tfrac{1}{3}+\tfrac{1}{3}\alpha+\tfrac{2}{15}\alpha^2+\tfrac{1}{45}\alpha^3\right)(\lambda/1+\alpha)+\left(\tfrac{2}{15}+\tfrac{7}{45}\alpha+\tfrac{8}{105}\alpha^2+\tfrac{4}{210}\alpha^3+\tfrac{2}{945}\alpha^4\right)(\lambda/1+\alpha)^2\\ &\qquad+\left(\tfrac{17}{315}+\tfrac{23}{315}\alpha+\tfrac{122}{2835}\alpha^2+\tfrac{22}{1575}\alpha^3+\tfrac{4}{1575}\alpha^4+\tfrac{1}{4725}\alpha^5\right)(\lambda/1+\alpha)^3\\ &\qquad+\left(\tfrac{62}{2835}+\tfrac{478}{14\,175}\alpha+\tfrac{518}{22\,275}\alpha^2+\tfrac{617}{66\,825}\alpha^3+\tfrac{2}{891}\alpha^4+\tfrac{2}{6237}\alpha^5+\tfrac{2}{93\,555}\alpha^6\right)(\lambda/1+\alpha)^4\\ &\qquad+\Big(\tfrac{1382}{155\,925}+\tfrac{7178}{467\,775}\alpha+\tfrac{2231}{184\,275}\alpha^2+\tfrac{80\,326}{14\,189\,175}\alpha^3+\tfrac{362\,254}{212\,837\,625}\alpha^4+\tfrac{23\,494}{70\,945\,875}\alpha^5\\ &\qquad+\tfrac{2764}{70\,945\,875}\alpha^6+\tfrac{1382}{638\,512\,875}\alpha^7\Big)(\lambda/1+\alpha)^5\\ &\qquad+\Big(\tfrac{21\,844}{6081\,075}+\tfrac{58\,739}{8513\,505}\alpha+\tfrac{3909\,736}{638\,512\,875}\alpha^2+\tfrac{699\,376}{212\,837\,625}\alpha^3+\tfrac{248\,896}{212\,837\,625}\alpha^4\\ &\qquad\quad+\tfrac{543\,398}{1915\,538\,625}\alpha^5+\tfrac{8}{173\,745}\alpha^6+\tfrac{4}{868\,725}\alpha^7+\tfrac{4}{18\,243\,225}\alpha^8\Big)(\lambda/1+\alpha)^6+\cdots\Big]\Big\}.\end{aligned} \tag{A9}$$

APPENDIX B: GROUND-STATE ENERGY

In this appendix we list the expansions (complete and in the ladder approximation) for the shift in the ground-state energy (relative to the ground-state energy of a system of noninteracting fermions) caused by the interaction (6) and specified in Appendix A. The expansions are in powers of x and λ for the multiplicities $\nu=2$ and 4.

For $\nu=2$ the complete series is

$$
\begin{aligned}
\frac{\Delta E m}{N\hbar^2 k_F^2} = {} & (1.0610\times10^{-1}x + 5.5661\times10^{-2}x^2 + 1.1524\times10^{-1}x^3 - 7.410\times10^{-3}x^4 + \cdots) \\
& + \lambda \Bigg[x(-3.5368\times10^{-2}\alpha) + x^2(-3.7107\times10^{-2}\alpha) \\
& \quad + x^3 \bigg[2.1221\times10^{-3}\alpha^3 - 1.975\times10^{-2}\alpha - \frac{\alpha}{(1+\alpha)}(6.3662\times10^{-3}\alpha^3 + 3.8197\times10^{-2}\alpha^2 \\
& \qquad + 9.5493\times10^{-2}\alpha + 9.5493\times10^{-2}) \bigg] \\
& \quad + x^4(4.3295\times10^{-3}\alpha^3 + 3.9200\times10^{-3}\alpha^2 + 9.880\times10^{-3}\alpha + 2.6133\times10^{-3}) + \cdots \Bigg] \\
& + \lambda^2 \Bigg[x(-1.4147\times10^{-2}\alpha) + x^2(6.1846\times10^{-3}\alpha^2 - 1.4843\times10^{-2}\alpha) \\
& \quad + x^3 \bigg[2.6273\times10^{-3}\alpha^3 + 6.583\times10^{-3}\alpha^2 - 7.900\times10^{-3}\alpha \\
& \qquad - \frac{\alpha}{(1+\alpha)^2}(6.0630\times10^{-4}\alpha^4 + 5.4567\times10^{-3}\alpha^3 + 2.1827\times10^{-2}\alpha^2 \\
& \qquad + 4.4563\times10^{-2}\alpha + 3.8197\times10^{-2}) \bigg] \\
& \quad + x^4(-1.4432\times10^{-3}\alpha^4 + 3.1576\times10^{-3}\alpha^3 - 1.6733\times10^{-3}\alpha^2 + 3.0809\times10^{-3}\alpha \\
& \qquad + 1.0889\times10^{-3}) + \cdots \Bigg] \\
& + \lambda^3 \Bigg[x(-5.7262\times10^{-3}\alpha) + x^2(4.9476\times10^{-3}\alpha^2 - 6.0079\times10^{-3}\alpha) \\
& \quad + x^3 \bigg[1.424\times10^{-3}\alpha^3 + 5.267\times10^{-3}\alpha^2 - 3.198\times10^{-3}\alpha \\
& \qquad - \frac{\alpha}{(1+\alpha)^3}(6.0630\times10^{-5}\alpha^5 + 7.2757\times10^{-4}\alpha^4 + 4.0016\times10^{-3}\alpha^3 \\
& \qquad + 1.2328\times10^{-2}\alpha^2 + 2.0918\times10^{-2}\alpha + 1.5461\times10^{-2}) \bigg] \\
& \quad + x^4(-2\ 0654\times10^{-3}\alpha^4 + 2.7893\times10^{-3}\alpha^3 - 1.9430\times10^{-3}\alpha^2 \\
& \qquad + 8.8821\times10^{-4}\alpha + 4.4282\times10^{-4}) + \cdots \Bigg] \\
& + \lambda^4 \Big[x(-2.3204\times10^{-3}\alpha) + x^2(2.9921\times10^{-3}\alpha^2 - 2.4346\times10^{-3}\alpha)
\end{aligned}
$$

$$+x^3\Big[5.904\times10^{-4}\alpha^3+3.185\times10^{-3}\alpha^2-1.296\times10^{-3}\alpha$$

$$-\frac{\alpha}{(1+\alpha)^4}(6.1243\times10^{-6}\alpha^6+9.1864\times10^{-5}\alpha^5+6.4305\times10^{-4}\alpha^4$$

$$+2.6451\times10^{-3}\alpha^3+6.6620\times10^{-3}\alpha^2+9.6604\times10^{-3}\alpha$$

$$+6.2651\times10^{-3})\Big]$$

$$+x^4(-2.0214\times10^{-3}\alpha^4+2.1054\times10^{-3}\alpha^3-1.2979\times10^{-3}\alpha^2+2.1438\times10^{-4}\alpha$$

$$+1.7954\times10^{-4})+\cdots\Big]+\cdots\,. \tag{B1}$$

For $\nu=4$ the complete series is

$$\frac{\Delta Em}{N\hbar^2k_F^2}=(3.1831\times10^{-1}x+1.6698\times10^{-1}x^2+3.9018\times10^{-1}x^3+4.2258\times10^{-1}x^4\ln x+\cdots)$$

$$+\lambda\Big[x(-1.0610\times10^{-1}\alpha)+x^2(-1.1132\times10^{-1}\alpha)$$

$$+x^3\Big[6.3662\times10^{-3}\alpha^3-2.3103\times10^{-1}\alpha$$

$$-\frac{\alpha}{(1+\alpha)}(1.0610\times10^{-2}\alpha^2+6.3662\times10^{-2}\alpha^2+1.5915\times10^{-1}\alpha+1.5915\times10^{-1})\Big]$$

$$+x^4\ln x(-5.6344\times10^{-1}\alpha)+\cdots\Big]$$

$$+\lambda^2\Big[x(-4.2441\times10^{-2}\alpha)+x^2(1.8553\times10^{-2}\alpha^2-4.4528\times10^{-2}\alpha)$$

$$+x^3\Big[7.8819\times10^{-3}\alpha^3+7.7010\times10^{-2}\alpha^2-9.2412\times10^{-2}\alpha$$

$$-\frac{\alpha}{(1+\alpha)^2}(1.0105\times10^{-3}\alpha^4+9.0946\times10^{-3}\alpha^3+3.6378\times10^{-2}\alpha^2$$

$$+7.4272\times10^{-2}\alpha+6.3662\times10^{-2})\Big]$$

$$+x^4\ln x(2.8172\times10^{-1}\alpha^2-2.2538\times10^{-1}\alpha)+\cdots\Big]$$

$$+\lambda^3\Big[x(-1.7179\times10^{-2}\alpha)+x^2(1.4843\times10^{-2}\alpha^2-1.8023\times10^{-2}\alpha)$$

$$+x^3\Big[-2.0894\times10^{-3}\alpha^3+6.1608\times10^{-2}\alpha^2-3.7404\times10^{-2}\alpha$$

$$-\frac{\alpha}{(1+\alpha)^3}(1.0105\times10^{-4}\alpha^5+1.2126\times10^{-3}\alpha^4+6.6694\times10^{-3}\alpha^3$$

$$+ 2.0547\times10^{-2}\alpha^2+3.4863\times10^{-2}\alpha+2.5768\times10^{-2})\Big]$$

$$+ x^4\ln x(-6.2604\times10^{-2}\alpha^3+2.2538\times10^{-1}\alpha^2-9.1224\times10^{-2}\alpha)+\cdots\Big]$$

$$+\lambda^4\Big[x(-6.9613\times10^{-3}\alpha)+x^2(8.9763\times10^{-3}\alpha^2-7.3035\times10^{-3}\alpha)$$

$$+x^3\Big[-5.8635\times10^{-3}\alpha^3+3.7257\times10^{-2}\alpha^2-1.5158\times10^{-2}\alpha$$

$$-\frac{\alpha}{(1+\alpha)^4}(1.0207\times10^{-5}\alpha^6+1.5311\times10^{-4}\alpha^5+1.0718\times10^{-3}\alpha^4$$

$$+4.408\times10^{-3}\alpha^3+1.1103\times10^{-2}\alpha^2+1.6101\times10^{-2}\alpha+1.0442\times10^{-2})\Big]$$

$$+ x^4\ln x(5.2170\times10^{-3}\alpha^4-7.5125\times10^{-2}\alpha^3+1.3630\times10^{-1}\alpha^2-3.6966\times10^{-2}\alpha)+\cdots\Big]$$

$$+\cdots\,. \qquad \text{(B2)}$$

In the ladder approximation for $\nu=2$

$$\frac{\Delta E_L m}{N\hbar^2 k_F^2}=(1.0610\times10^{-1}x+5.5661\times10^{-2}x^2+1.3813\times10^{-1}x^3+1.5339\times10^{-2}x^4+\cdots)$$

$$+\lambda\Big[x(-3.5368\times10^{-2}\alpha)+x^2(-3.7107\times10^{-2}\alpha)$$

$$+x^3\Big[2.1221\times10^{-3}\alpha^3-4.2641\times10^{-2}\alpha$$

$$-\frac{\alpha}{(1+\alpha)}(6.3662\times10^{-3}\alpha^3+3.8197\times10^{-2}\alpha^2+9.5493\times10^{-2}\alpha+9.5493\times10^{-2})\Big]$$

$$+ x^4(4.3295\times10^{-3}\alpha^3+9.5490\times10^{-3}\alpha^2-2.0452\times10^{-2}\alpha+6.3660\times10^{-3})+\cdots\Big]$$

$$+\lambda^2\Big[x(-1.4147\times10^{-2}\alpha)+x^2(6.1846\times10^{-3}\alpha^2-1.4843\times10^{-2}\alpha)$$

$$+x^3\Big[2.6273\times10^{-3}\alpha^3+1.4214\times10^{-2}\alpha^2-1.7057\times10^{-2}\alpha$$

$$-\frac{\alpha}{(1+\alpha)^2}(6.0630\times10^{-4}\alpha^4+5.4567\times10^{-3}\alpha^3+2.1827\times10^{-2}\alpha^2$$

$$+ 4.4563\times10^{-2}\alpha+3.8197\times10^{-2})\Big]$$

$$+x^4(-1.4432\times10^{-3}\alpha^4-5.34\times10^{-6}\alpha^3+1.8184\times10^{-2}\alpha^2-1.0303\times10^{-2}\alpha$$

$$+2.6525\times10^{-3})+\cdots\Big]$$

$$+\lambda^3\Big[x(-5.7262\times10^{-3}\alpha)+x^2(4.9476\times10^{-3}\alpha^2-6.0079\times10^{-3}\alpha)$$

$$+x^3\Big[5.7644\times10^{-4}\alpha^3+1.1371\times10^{-2}\alpha^2-6.9038\times10^{-3}\alpha$$

$$-\frac{\alpha}{(1+\alpha)^3}(6.0630\times10^{-5}\alpha^5+7.2757\times10^{-4}\alpha^4+4.0016\times10^{-3}\alpha^3$$

$$+1.2328\times10^{-2}\alpha^2+2.0918\times10^{-2}\alpha+1.5461\times10^{-2})\Big]$$

$$+x^4(-1.6365\times10^{-3}\alpha^4-4.4677\times10^{-3}\alpha^3+1.3075\times10^{-2}\alpha^2-5.0443\times10^{-3}\alpha$$

$$+1.0787\times10^{-3})+\cdots\Big]$$

$$+\lambda^4\Big[x(-2.3204\times10^{-3}\alpha)+x^2(2.9921\times10^{-3}\alpha^2-2.4346\times10^{-3}\alpha)$$

$$+x^3\Big[-4.2697\times10^{-4}\alpha^3+6.8768\times10^{-3}\alpha^2-2.7976\times10^{-3}\alpha$$

$$-\frac{\alpha}{(1+\alpha)^4}(6.1243\times10^{-6}\alpha^6+9.1864\times10^{-5}\alpha^5+6.4305\times10^{-4}\alpha^4$$

$$+2.6451\times10^{-3}\alpha^3+6.6620\times10^{-3}\alpha^2+9.6604\times10^{-3}\alpha$$

$$+6.2651\times10^{-3})\Big]$$

$$+x^4(-1.0448\times10^{-3}\alpha^4-5.1086\times10^{-3}\alpha^3+7.6080\times10^{-3}\alpha^2-2.3986\times10^{-3}\alpha$$

$$+4.3735\times10^{-4})+\cdots\Big]+\cdots\,. \tag{B3}$$

For $\nu=4$ in the ladder approximation

$$\frac{\Delta E_L m}{N\hbar^2k_F^2}=(3.1831\times10^{-1}x+1.6698\times10^{-1}x^2+2.8708\times10^{-1}x^3+4.6017\times10^{-2}x^4+\cdots)$$

$$+\lambda\Big[x(-1.0610\times10^{-1}\alpha)+x^2(-1.1132\times10^{-2}\alpha)$$

$$+x^3\Big[6.3662\times10^{-3}\alpha^3-1.2792\times10^{-1}\alpha$$

$$-\frac{\alpha}{(1+\alpha)}(1.0610\times10^{-2}\alpha^3+6.3662\times10^{-2}\alpha^2+1.5915\times10^{-1}\alpha+1.5915\times10^{-1})\Big]$$

$$+x^4(1.2988\times10^{-2}\alpha^3+2.8647\times10^{-2}\alpha^2-6.1356\times10^{-2}\alpha+1.9098\times10^{-2})+\cdots\Big]$$

$$
\begin{aligned}
&+\lambda^2 \Bigg[x(-4.2441\times 10^{-2}\alpha)+x^2(1.8554\times 10^{-2}\alpha^2-4.4529\times 10^{2}\alpha) \\
&\quad +x^3 \Bigg[7.8819\times 10^{-3}\alpha^3+4.2642\times 10^{-2}\alpha^2-5.1171\times 10^{-2}\alpha \\
&\qquad -\frac{\alpha}{(1+\alpha)^2}(1.0105\times 10^{-3}\alpha^4+9.0946\times 10^{-3}\alpha^3+3.6378\times 10^{-2}\alpha^2 \\
&\qquad\qquad + 7.4272\times 10^{-2}\alpha+6.3662\times 10^{-2}) \Bigg] \\
&\quad +x^4(-4.3295\times 10^{-3}\alpha^4-1.601\times 10^{-5}\alpha^3+5.4551\times 10^{-2}\alpha^2 \\
&\qquad - 3.0909\times 10^{-2}\alpha+7.9575\times 10^{-3})+\cdots \Bigg] \\
&+\lambda^3 \Bigg[x(-1.7179\times 10^{-2}\alpha)+x^2(1.4843\times 10^{-2}\alpha^2-1.8024\times 10^{-2}\alpha) \\
&\quad +x^3 \Bigg[1.7293\times 10^{-3}\alpha^3+3.4113\times 10^{-2}\alpha^2-2.0711\times 10^{-2}\alpha \\
&\qquad -\frac{\alpha}{(1+\alpha)^3}(1.0105\times 10^{-4}\alpha^5+1.2126\times 10^{-3}\alpha^4+6.6694\times 10^{-3}\alpha^3 \\
&\qquad\qquad + 2.0547\times 10^{-2}\alpha^2+3.4863\times 10^{-2}\alpha+2.5768\times 10^{-2}) \Bigg] \\
&\quad +x^4(-4.9094\times 10^{-3}\alpha^4-1.3403\times 10^{-2}\alpha^3+3.9224\times 10^{-2}\alpha^2 \\
&\qquad - 1.5133\times 10^{-2}\alpha+3.2360\times 10^{-3})+\cdots \Bigg] \\
&+\lambda^4 \Bigg[x(-6.9613\times 10^{-3}\alpha)+x^2(8.9764\times 10^{-3}\alpha^2-7.3037\times 10^{-3}\alpha) \\
&\quad +x^3 \Bigg[-1.2809\times 10^{-3}\alpha^3+2.0630\times 10^{-2}\alpha^2-8.3928\times 10^{-3}\alpha \\
&\qquad -\frac{\alpha}{(1+\alpha)^4}(1.0207\times 10^{-5}\alpha^6+1.5311\times 10^{-4}\alpha^5+1.0718\times 10^{-3}\alpha^4 \\
&\qquad\qquad +4.4085\times 10^{-3}\alpha^3+1.1103\times 10^{-2}\alpha^2+1.6101\times 10^{-2}\alpha + 1.0442\times 10^{-2}) \Bigg] \\
&\quad +x^4(-3.1344\times 10^{-3}\alpha^4-1.5326\times 10^{-2}\alpha^3+2.2824\times 10^{-2}\alpha^2 \\
&\qquad - 7.1959\times 10^{-3}\alpha+1.3120\times 10^{-3})+\cdots \Bigg]+\cdots \, . \qquad \text{(B4)}
\end{aligned}
$$

*On leave from St. Louis University, St. Louis, MO.
†Universidad Nacional, La Plata, Argentina.
‡Universidad Nacional, La Plata, Argentina.

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