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Fluctuation theory in quantum-optical systems

L. A. Lugiato, F. Casagrande, and L. Pizzuto
Istituto di Fisica dell'Università, Via Celoria, 16, 20133 Milano, Italy
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We show that the most complete and accurate description of fluctuations in quantum-optical systems is that obtained using the generalized Wigner distribution for the macroscopic observables of the system. The smallness of the inverse of the saturation photon number entitles us to neglect the terms with derivatives of order higher than second order in the time-evolution equation for the quasiprobability distribution. This treatment allows us to also describe correctly nonclassical effects such as photon antibunching or "squeezing," which are maltreated or even destroyed if one neglects the atom-atom correlations. We derive and analyze the Fokker-Planck equation for the Wigner function in the case of the usual laser and compare the results with those following from other approaches. In the region very high above threshold we find a small antibunching effect, which was not discovered in previous treatments due to the neglect of atom-atom correlations.

I. INTRODUCTION

The study of fluctuations in quantum-optical systems has been an object of continuous interest in the last two decades. Starting from the fundamental works of Haken and co-workers,[1] Lamb and Scully,[2] Risken,[3] Lax and Louisell,[4] a considerable amount of literature on this subject has cumulated over the years. Most of these papers derive and analyze a Fokker-Planck equation for the Glauber-Sudarshan[5,6] quasiprobability distribution, or a suitable master equation for the probability distribution in the photon-number representation. A large variety of equations of this type have been obtained by exploiting different approaches and approximations. These equations lead to results that agree only in part. Hence it is quite natural that we feel a need to put some order in this situation, by trying to choose the approach that can be considered as the most complete and correct. Another reason that moves us in this direction is the fact that in recent years a remarkable attention has been focused on nonclassical effects as photon antibunching[7] and "two-photon coherence" or "squeezing."[8] In fact, the analysis of these effects requires using the most refined techniques available, since the approximations can easily alter or destroy them.

The starting point for most approaches is a suitable operator master equation which describes the coupled dynamics of the electric field and the atoms.[1,4] For the sake of simplicity in this paper we consider only the single-mode description of the electric field in the cavity. The operator master equation is too complicated to be solved, even at steady state. Hence one introduces suitable approximations. The most commonly used is the adiabatic elimination of the atomic variables, which holds when the atomic damping constants are much larger than the field damping constant. This elimination can be performed in different ways in the different approaches. As a result, one obtains a closed time-evolution equation for the Glauber-Sudarshan P function or for the photon-number distribution, that most often can be solved analytically at steady state and numerically in the transient.

We divide the works on this subject into two groups: works that do not contain the one-atom approximation (see, e.g., Refs. 9–18) and works that do (see, e.g., Refs. 19–25). The one-atom approximation consists of neglecting those processes in which the field interacts with two or more atoms simultaneously. The total variation of the field in time is obtained by simply summing up the contributions from the single atoms. In the first group, let us consider, in particular, the work by Haken, Risken, and Weidlich.[1,10] By using suitable operator techniques, these authors translate the operator master equation into a classical-looking partial differential equation for the quasiprobability distribution of the macroscopic observables of the system. This distribution generalizes the Glauber-Sudarshan P function to include also the atomic variables. On the basis of a scaling argument tailormade for the laser in the threshold region, they approximate this equation by a Fokker-Planck equation. Finally, by adiabatic elimination of the atomic variables they recover the previously known Risken equation.[3]

In the second group, let us consider, in particular, the works by Scully and Lamb.[19] These authors do not start from the operator master equation, but analyze in detail the change in time of the density matrix of the electric field induced by the interaction with a single atom. On the basis of suitable assumptions, they derive a well-known master equation which governs the time evolution of the photon-number distribution. In the threshold region, this equation agrees perfectly with the Risken's equation.

Another group of papers which uses the one-atom approximation[21–24] exploits the method to treat open systems, formulated in Ref. 26. This procedure, which will be considered in some detail in Sec. III, derives from the operator master equation a hierarchy of equations for the reduced density matrix of the electric field and for suitable atom-field correlation operators. By eliminating the latter quantities, one obtains a closed time-evolution equation for the electric field, which is readily translated into c-number form using the Glauber-Sudarshan representation.

A work that is, in a sense, intermediate between the two groups is Ref. 12, which does not introduce the one-atom approximation but neglects the atom-atom correlations.

A comparison between all these papers is not easy, because the works which do not use the one-atom approximation introduce other kinds of approximations. The subject of laser fluctuations, which was the focus of the attention in the sixties, is not suitable to make a comparison, even taking into account the abundant experimental data available in this field.[27] In fact, most data concern the threshold region, and in this domain all theories agree almost perfectly with one another. A more proper ground to perform a comparison is offered by the so-called optical bistability, which raised the attention of the quantum-optical community especially since the mid seventies.[28] In fact, fluctuations in optical bistability have been studied both in the framework of the one-atom approach[23] and without the one-atom approximation.[13–17,29] The remarkable fact is that the two approaches lead sometimes to quite different results. For example, the second approach predicts antibunching[16,17] and squeezing[30] in the low-transmission branch of the hysteresis cycle of transmitted versus incident field, whereas these effects are completely absent in the one-atom treatment. It has been shown[13,15,16] that in order to describe these effects it is necessary to use the generalized Wigner function[31,32] instead of the Glauber-Sudarshan representation, because otherwise the diffusion matrix of the Fokker-Planck equation would not be positive definite.

Hence optical bistability is an ideal framework to make the comparison. In Ref. 16 it is suggested that the approach which avoids using the one-atom approximation is more accurate and complete. In this paper we prove this claim explicitly. Furthermore, we derive and analyze the Fokker-Planck equation for the Wigner function in the case of the laser, similarly to what was already done in the case of optical bistability.[15,16] We compare the results obtained from the steady-state solution of this equation with the corresponding results following from other approaches. As expected, we find that in the threshold and below threshold regions the new results differ in a negligible way from the well-known ones. On the contrary, in the region very high above threshold there is a surprise, namely, we find a small antibunching effect. This was not discovered in previous treatments of the laser because they neglected the atom-atom correlations.

In Sec. II we recall the operator master equation which is the starting point of our treatment and translate it into a partial differential equation, both using the generalized Glauber-Sudarshan representation and the generalized Wigner function. The adiabatic elimination of the atomic variables is discussed in Sec. III. In Sec. IV we prove that the one-atom approach is less accurate than the other one. Section V is devoted to the discussion of the Fokker-Planck equation for the Wigner function, in the case of the laser. Section VI contains the final discussion, with particular emphasis on the relative advantages of the Wigner and of the Glauber-Sudarshan representations.

II. FOKKER-PLANCK EQUATIONS FOR THE SYSTEM ATOMS + RESONANT MODE WITH AN INJECTED SIGNAL

Let us consider a homogeneously broadened system of N two-level atoms interacting with a radiation field mode perfectly tuned to the atomic transition frequency ω. The atoms are placed within a resonant cavity of length L and volume V with mirrors of transmittivity T. The ith atom is associated to the raising (lowering) operator $r_i^+(r_i^-)$ and to the inversion operator

$$r_{3i}=\tfrac{1}{2}(r_i^+r_i^- - r_i^-r_i^+)\ ,$$

which obey the angular momentum commutation relations

$$[r_i^+ r_j^-]=2r_{3i}\delta_{ij}, \quad [r_{3i}, r_j^{\pm}]=\pm r_i^{\pm}\delta_{ij} \ . \tag{1}$$

The cavity mode is described by the annihilation (creation) operator A ($A^\dagger$) obeying the boson commutation relation

$$[A, A^\dagger]=1 \ . \tag{2}$$

The atoms and the field are separately coupled to suitable reservoirs in order to describe incoherent pump and decay processes. The dynamics of the full system is ruled by the one-mode laser master equation formulated in the sixties by Haken and co-workers.[1,33] This model had been generalized a few years ago by Bonifacio and Lugiato[34] in order to take into account the possible presence of an external CW coherent field of frequency ω which is injected into the cavity. This allows for treating both the optical bistability (OB) and the laser with injected signal, depending on the value of the pump parameter. In the interaction picture, the statistical operator W of the system resonant mode + atoms with an injected signal evolves in time according to the following master equation (ME):

$$\frac{dW}{dt}=(-iL_{\rm AF}+\Lambda_F+\Lambda_A)W \ , \tag{3}$$

where (i) the first term on the right-hand-side (rhs) describes the atom-field interaction in the dipole and rotating-wave approximations:

$$L_{\rm AF}W=\frac{1}{\hbar}[H_{\rm AF}, W] \ , \tag{4a}$$

$$H_{\rm AF}=i\hbar\bar{g}\sum_{i=1}^{N}(e^{-i\vec{k}\cdot\vec{x}_i}A^+r_i^- - \text{H.c.}) \ , \tag{4b}$$

with $\bar{g}$ being the coupling constant

$$\bar{g}=(2\pi\omega/\hbar V)^{1/2}\mu \ , \tag{4c}$$

μ the modulus of the atomic dipole moment, $\vec{k}$ the wave vector of the radiation mode, and $\vec{x}_i$ the position of the ith atom.

(ii) $\Lambda_F W$ rules the dynamics of the cavity mode including both the cavity losses and the external source field:

$$\Lambda_F W=K\{[A-\alpha, W(A-\alpha)^\dagger] + [(A-\alpha)W, (A-\alpha)^\dagger]\} \ , \tag{5a}$$

where K is the cavity damping constant

$$K=cT/L \tag{5b}$$

and α, which is taken real and positive for definiteness, is a c number proportional to the amplitude of the coherent field injected into the cavity (i.e., α^2 is the photon number of the incident field).

(iii) $\Lambda_A W$ described the atomic decay and pump processes:

$$\Lambda_A W=\tfrac{1}{2}\sum_{i=1}^{N}\{\gamma_\uparrow([r_i^+, Wr_i^-]+[r_i^+W, r_i^-])+\gamma_\downarrow([r_i^-, Wr_i^+]+[r_i^-W, r^+])+\eta([r_{3i}, Wr_{3i}]+[r_3W, r_{3i}])\} \ , \tag{6}$$

where $\gamma_\uparrow(\gamma_\downarrow)$ is the upward (downward) transition rate between the lower (upper) and the upper (lower) level, η a dephasing rate due to elastic collisions. In terms of the latter quantities one defines the transverse and the parallel relaxation rates

$$\gamma_\perp=\frac{\gamma_\uparrow+\gamma_\downarrow+\eta}{2}, \quad \gamma_\parallel=\gamma_\uparrow+\gamma_\downarrow \ , \tag{7a}$$

and the pump parameter

$$\sigma=\frac{\gamma_\uparrow-\gamma_\downarrow}{\gamma_\uparrow+\gamma_\downarrow} \ . \tag{7b}$$

We introduce also the ratio

$$f=\frac{\gamma_\parallel}{2\gamma_\perp} \ ; \tag{7c}$$

for a purely radiative decay $\eta=0$ so that $f=1$.

Equations (3)–(7) describe OB if $\gamma_\uparrow=0$ (i.e., $\sigma=-1$: no pump process), the laser with injected signal if $\gamma_\uparrow>\gamma_\downarrow$ (i.e., $\sigma>0$: positive population inversion), the usual laser if $\gamma_\uparrow>\gamma_\downarrow$ and $\alpha=0$. The generalization of the ME (3) to many modes, including atomic detuning and cavity mistuning as well, has been formulated recently by one of us (L.A.L.).[35] The interaction Hamiltonian (4b) may be written in a more compact form, namely,

$$H_{\rm AF}=i\hbar\bar{g}(A^\dagger R^- - AR^+) \ , \tag{4b$'$}$$

if we introduce the atomic collective operators (macroscopic polarization operators $R^{\pm}$, total population inversion R_3)

$$R^{\pm}=\sum_{i=1}^{N}r_i^{\pm}e^{\pm i\vec{k}\cdot\vec{x}_i}, \quad R_3=\sum_{i=1}^{N}r_{3i} \ , \tag{8}$$

which obey the angular momentum commutation relations

$$[R^+, R^-]=2R_3, \quad [R_3, R^{\pm}]=\pm R^{\pm} \ . \tag{1$'$}$$

We need both the macroscopic and the microscopic descriptions of the atomic system for our purposes.

A first step towards the solution of the ME (3) at least under suitable approximations, is the translation of this operator equation into a c-number partial differential equation, which can be performed via the characteristic function technique.[1,36] More precisely, and referring to the collective description (8) of the atomic system, one introduces a characteristic function by which five c-number quantities are associated to the operators $R^{\pm}$, R_3, A, and $A^{\dagger}$. The Fourier transform of the characteristic function is a quasiprobability distribution function in five c number variables. Its evolution is ruled by a classical-looking partial differential equation which is derived from Eq. (3) by suitable operator techniques. By means of this distribution, one can calculate expectation values of products of operators in suitable order as classical mean values for the corresponding c-number variables. This procedure, first devised by Wigner[31] and then extensively developed by Moyal[32] and Haken and co-workers,[1] creates a bridge between the density operator and a classical distribution. A crucial point concerns the ordering prescription for the operators in the characteristic function. Actually, different choices such as normal, antinormal, or symmetrical ordering, lead to different classical distributions which, in turn, give normal-, antinormal-, or symmetrical-ordered expectation values, respectively. The distributions themselves obey different equations according to the chosen ordering prescription.

Haken, Risken, and Weidlich (HRW) (Refs. 1 and 10) used for the system atoms + field a normal-ordered characteristic function

$$C^{(N)}(\xi,\xi^*,\eta,\zeta,\zeta^*,t)=\mathrm{Tr}[e^{i\xi^* R^+}e^{i\eta R_3}e^{i\xi R^-}e^{i\zeta^* A^\dagger}e^{i\zeta A}W(t)] \tag{9}$$

whose Fourier transform $P(v,v^*,m,\beta,\beta^*,t)$ is a generalized Glauber quasiprobability distribution function, since it extends to the atomic variables the well-known Glauber's P representation of the field density operator.[5] Distribution P allows for calculating normal-ordered expectation values, e.g.,

$$\langle A^\dagger A\rangle(t)=\int d_2v\,dm\,d_2\beta\,P(v,v^*,m,\beta,\beta^*,t)\beta^*\beta,\quad d_2\beta=d\,\mathrm{Re}\beta\,d\,\mathrm{Im}\beta\ , \tag{10}$$

etc. It satisfies a partial differential equation containing derivatives of all orders with respect to $\bar{m}$. However, it can be approximated by a Fokker-Planck equation (FPE), i.e., an equation containing only first- and second-order derivatives (the Fokker-Planck approximation is discussed at the end of this section). The discussion of the steady-state solutions of the semiclassical equations[34] suggests to introduce the following scaled quantities $\bar{v},\bar{m},x,y$:

$$\begin{aligned}&v=-\frac{N}{2}\left[\frac{\gamma_{\parallel}}{\gamma_{\perp}}\right]^{1/2}\bar{v},\quad m=\frac{N}{2}\bar{m}\ ,\\ &\beta=\sqrt{N_s}x,\quad \alpha=\sqrt{N_s}y\ ,\end{aligned} \tag{11}$$

where N_s is the saturation photon number

$$N_s=\frac{\gamma_{\perp}\gamma_{\parallel}}{4\bar{g}^2}\ . \tag{12a}$$

We introduce also the basic cooperation parameter[34]

$$C=\frac{\bar{g}^2N}{2K\gamma_{\perp}}\ . \tag{12b}$$

Then the FPE of HRW (generalized to include an injected signal) reads

$$\begin{aligned}&\frac{\partial P(\bar{v},\bar{v}^*,\bar{m},x,x^*,t)}{\partial t}=\Lambda P(\bar{v},\bar{v}^*,\bar{m},x,x^*,t),\\ &\Lambda=\left[-\frac{\partial}{\partial\bar{v}}[-\gamma_{\perp}(\bar{v}+\bar{m}x)]-\frac{\partial}{\partial x}[-K(x-y+2C\bar{v})]+\text{c.c.}\right]-\frac{\partial}{\partial\bar{m}}\{-\gamma_{\parallel}[\bar{m}-\sigma-\tfrac{1}{2}(\bar{v}^*x+\text{c.c.})]\}\\ &\quad+\frac{\gamma_{\perp}}{K}\frac{1}{4CN_s}\left[\frac{\partial^2}{\partial\bar{v}^*\partial\bar{v}}[\eta(\bar{m}+1)+\gamma_{\parallel}(\sigma+1)]+\frac{\gamma_{\parallel}}{2}\left[\frac{\partial^2}{\partial\bar{v}^2}x\bar{v}+\text{c.c.}\right]\right.\\ &\quad\quad\left.+(\sigma+1)f\gamma_{\parallel}\frac{\partial}{\partial\bar{m}}\left[\frac{\partial}{\partial\bar{v}}\bar{v}+\text{c.c.}\right]+f\gamma_{\parallel}\frac{\partial^2}{\partial\bar{m}^2}[1-\sigma\bar{m}+\tfrac{1}{2}(\bar{v}^*x+\text{c.c.})]\right]\ .\end{aligned} \tag{13}$$

The diffusion matrix of the FPE (13) is not positive definite in general. However, in the case of the usual laser, HRW used a scaling argument, valid only for the laser in the threshold region, by which some terms in the diffusion matrix can be neglected, thus obtaining a well behaved FPE. Then performing the adiabatic elimination of the atomic variables they were able to recover Risken's FPE,[3] which describes the behavior of the laser in the threshold region. However, the FPE (13) is not suitable to describe the laser dynamics far from threshold or OB. Recently Gronchi and Lugiato[13] adopted the symmetrical-ordered characteristic function

$$C^{(s)}(\xi,\xi^*,\eta,\zeta,\zeta^*,t)=\mathrm{Tr}[e^{i(\xi^* R^+ +\eta R_3+\xi R^- +\zeta^* A^\dagger+\zeta A)}W(t)] , \tag{14}$$

whose Fourier transform $P_w(v,v^*,m,\beta,\beta^*,t)$ is a generalization to include the atomic variables of the Wigner quasiprobability distribution function.[31] Symmetrized expectation values are given by the moments of P_w, e.g.,

$$\langle A^\dagger A\rangle_s(t)\equiv\frac{\langle A^\dagger A\rangle(t)+\langle AA^\dagger\rangle(t)}{2}=\int d_2v\, dm\, d_2\beta\, P_w(v,v^*,m,\beta,\beta^* t)\beta^*\beta . \tag{15}$$

In Ref. 13 by a constructive procedure and neglecting the derivatives of an order higher than the second as discussed later on, a FPE is derived for the generalized Wigner distribution P_w which in the notations (11) reads[37]

$$\frac{\partial P_w(\bar{v},\bar{v}^*,\bar{m},x,x^*,t)}{\partial t}=\Gamma P_w(\bar{v},\bar{v}^*,\bar{m},x,x^*,t) ,$$

$$\Gamma=\left[-\frac{\partial}{\partial\bar{v}}[-\gamma_\perp(\bar{v}+\bar{m}x)]-\frac{\partial}{\partial x}[-K(x-y+2C\bar{v})]+\text{c.c.}\right]-\frac{\partial}{\partial\bar{m}}\{-\gamma_\parallel[\bar{m}-\sigma-\tfrac{1}{2}(\bar{v}^*x+\text{c.c.})]\}$$
$$+\frac{\gamma_\perp^2}{K}\frac{1}{2CN_s}\left[\frac{\partial^2}{\partial\bar{v}^*\partial\bar{v}}-\sigma f^2\frac{\partial}{\partial\bar{m}}\left[\frac{\partial}{\partial\bar{v}}\bar{v}+\text{c.c.}\right]+f^2\frac{\partial^2}{\partial\bar{m}^2}(1-\sigma\bar{m})\right]+\frac{k}{N_s}\frac{\partial^2}{\partial x^*\partial x} . \tag{16}$$

As discussed in Ref. 13, Eq. (16) in the case of the laser coincides with the FPE derived by Risken, Schmid, and Weidlich (RSW).[9] These authors followed a semiclassical procedure, starting from the Hamiltonian of the full system without resorting to a ME, and then generalizing to the quantum case the classical definitions of drift and diffusion coefficients. The connection between the equation of RSW and the symmetrical-ordering prescription (14) was not recognized at that time. Not only was this true, but the FPE of RSW was criticized on the basis of arguments that vanish on the light of the new procedure of Ref. 13.

The crucial point with respect to Eq. (16) is now the following. While the drift coefficients of (13) and (16) coincide since they do not depend on any ordering prescription, the diffusion matrix in the Wigner-type FPE (16) is positive definite contrary to the one in the Glauber-type FPE (13), at least in the physically relevant range $|\bar{m}|\leq 1$. For this reason Eq. (16) for $\sigma=-1$ was adopted as the Fokker-Planck equation for OB.[15]

As the last point in this section, let us discuss the problem of the Fokker-Planck approximation. In both Eqs. (13) and (16), the second-order derivative terms are proportional to N_s^{-1}. The fact that N_s is a very large number ensures that the fluctuations are small with respect to the mean values, except in the laser threshold region where the mean value has the same order of magnitude of the variance. The magnitude of N_s guarantees also the validity of the Fokker-Planck approximation. In fact, the terms with derivatives of order higher than the second are proportional to increasing powers of $N_s^{-1/2}$. Hence in our problem we have a smallness parameter $\epsilon=N_s^{-1/2}$ which allows us to truncate the equation at second order. It is true that by a well-known argment of Van Kampen (see, e.g., Ref. 38) one cannot expect that Eq. (16) can, in general, describe correctly the moments higher then the second. However, Eq. (16) is absolutely trustworthy for the calculation of mean values and mean-square fluctuations. Not only is this true, but in most cases the probability distribution is very well approximated by a Gaussian, or by a few Gaussian peaks in the case of multistable systems. Equation (16) is quite suitable to determine these Gaussian functions.

In connection with the problem of the Fokker-Planck approximation, see also Sec. III B after Eq. (23′).

III. FOKKER-PLANCK EQUATIONS FOR THE FIELD QUASIPROBABILITY DISTRIBUTION

A. Wigner and Glauber field distributions

Equations (13) and (16) are two Fokker-Planck equations for the dynamics of the full system atoms + field + injected signal. Actually, we are mainly interested in the dynamics of the field variables in order to treat the laser or OB; hence, both equations contain more information than we need. However, in the situation

$$K \ll \gamma_\perp, \gamma_{||} , \tag{17}$$

in which the atoms have relaxation rates much shorter than the field, the atomic variables can be eliminated adiabatically. This procedure, that is quite familiar in laser physics,[1,2,39] leads from (13) and (16) to two Fokker-Planck equations for quasiprobability distribution functions in the field variables only. This adiabatic elimination can be performed by following the procedures illustrated in Refs. 40 and 11.

It turns out that the FPE (13) reduces to the following FPE for the Glauber distribution $P(x,x^*,t)$ in the normalized field variables (11):

$$\begin{aligned} K^{-1}\frac{\partial P(x,x^*,t)}{\partial t} = \Bigg\{ & -\frac{\partial}{\partial x}\left[-x\left[1-\sigma\frac{2C}{1+|x|^2}\right]+y\right]+\text{c.c.} \\ & -\frac{C}{2N_s}\frac{\partial^2}{\partial x^2}x^2\frac{(1+|x|^2)(1+|x|^2+2\sigma)+\sigma^2(1+2f)}{(1+|x|^2)^3}+\text{c.c.} \\ & +\frac{C}{N_s}\frac{\partial^2}{\partial x^*\partial x}\frac{(1+|x|^2)[(1+|x|^2)(2+|x|^2)+2\sigma]-\sigma^2(1+2f)|x|^2}{(1+|x|^2)^3}\Bigg\}P(x,x^*,t) . \end{aligned} \tag{18}$$

In the case of OB ($\sigma=-1$), Eq. (18) coincides with a FPE derived by Drummond and Walls.[17]

On the other hand, the FPE (16) produces a FPE for the Wigner field distribution $P_w(x,x^*,t)$ which reads

$$\begin{aligned} K^{-1}\frac{\partial P_w(x,x^*,t)}{\partial t} = \Bigg\{ & -\frac{\partial}{\partial x}\left[-x\left[1-\sigma\frac{2C}{1+|x|^2}\right]+y\right]+\text{c.c.}-\frac{C}{2N_s}\frac{\partial^2}{\partial x^2}x^2\frac{(1+|x|^2)^2+\sigma^2(1+2f)}{(1+|x|^2)^3} \\ & +\text{c.c.}+\frac{1}{N_s}\frac{\partial^2}{\partial x^*\partial x}\left[1+C\frac{(1+|x|^2)^2(2+|x|^2)-\sigma^2(1+2f)|x|^2}{(1+|x|^2)^3}\right]\Bigg\}P_w(x,x^*,t) . \end{aligned} \tag{19}$$

The differences in the diffusion coefficients of Eq. (18) and (19) are crucial in order to discuss the feasibility of either equation as FPE for the field dynamics. This point can be best appreciated by rewriting them in polar coordinates. Let us put

$$x = re^{i\varphi} . \tag{20}$$

Equation (18) becomes[41]

$$\begin{aligned} K^{-1}\frac{\partial P(r,\varphi,t)}{\partial t} = \frac{1}{r}\Bigg\{ & \frac{\partial}{\partial r}r\left[r\left[1-\sigma\frac{2C}{1+r^2}\right]-y\cos\varphi\right]+\frac{\partial}{\partial\varphi}y\sin\varphi \\ & +\frac{C}{2N_s}\left[\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\frac{(1+r^2)[1+r^2+\sigma(1-r^2)]-\sigma^2(1+2f)r^2}{(1+r^2)^3}\right. \\ & \left.+\frac{1}{r}\frac{\partial^2}{\partial\varphi^2}\frac{1+\sigma+r^2}{1+r^2}\right]\Bigg\}P(r,\varphi,t) . \end{aligned} \tag{18'}$$

It is easy to verify that the diffusion matrix in (18′) is not positive definite; e.g., for $f=1$ (purely radiative case) its radial coefficient is negative in the case of OB ($\sigma=-1$) when the system is in the low-transmission branch of the hysteresis cycle ($r^2 \ll 1$). Thus Eq. (18) is not directly useful since P does not always exist.

On the other hand, in polar coordinates Eq. (19) assumes the following form:

$$K^{-1}\frac{\partial P_w(r,\varphi,t)}{\partial t}=\frac{1}{r}\left\{\frac{\partial}{\partial r}r\left[r\left[1-\sigma\frac{2C}{1+r^2}\right]-y\cos\varphi\right]+\frac{\partial}{\partial\varphi}y\sin\varphi + \frac{1}{4N_s}\left[\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\left[1+2C\frac{r^4+[2-\sigma^2(1+2f)]r^2+1}{(1+r^2)^3}\right]+(1+2C)\frac{1}{r}\frac{\partial^2}{\partial\varphi^2}\right]\right\}P_w(r,\varphi,t)\,. \tag{19′}$$

The diffusion matrix in Eq. (19′) is manifestly positive definite since $|\sigma|\le 1, f\le 1$. Hence, we can conclude that the FPE (19) for the Wigner distribution $P_w(x,x^*,t)$ is suitable to describe the field dynamics.

Actually, the FPE (19) has been applied in the framework of a quantum statistical treatment of absorptive OB to obtain the statistics of the transmitted light, both as regards the spectrum[15] and the intensity fluctuations.[16] In particular, the spectrum of transmitted light has been carefully described in the limit (17) via linearization around a steady state of this equation (with $\sigma=-1$).[42] Let $x',x^{*\prime}$ represent the deviations of x,x^* from a stationary value x_{st}, respectively:

$$x'=x-x_{st},\quad x^{*\prime}=x^*-x_{st}\,. \tag{20′}$$

The linearized version of (19), i.e., the FPE which rules the dynamics of the system when it is slightly displaced from steady state, is

$$K^{-1}\frac{\partial P_w(x',x^{*\prime},t)}{\partial t}=\left\{-\frac{\partial}{\partial x'}\left[-\left[1-\sigma\frac{2C}{(1+x_{st}^2)^2}\right]x'-\sigma\frac{2Cx_{st}^2}{(1+x_{st}^2)^2}x^{*\prime}\right]+\text{c.c.} - \frac{C}{2N_s}\frac{x_{st}^2}{(1+x_{st}^2)^3}[(1+x_{st}^2)^2+\sigma^2(1+2f)]\frac{\partial^2}{\partial x'^2}+\text{c.c.} + \frac{1}{N_s}\left[1+C\frac{(1+x_{st}^2)(2+x_{st}^2)-\sigma^2(1+2f)x_{st}^2}{(1+x_{st}^2)^3}\right]\frac{\partial^2}{\partial x^{*\prime}\partial x'}\right\}P_w(x',x^{*\prime},t)\,. \tag{21}$$

The full hysteresis cycle in the spectrum of transmitted light has been described on the basis of Eq. (21) specialized to $\sigma=-1$.[15] In particular, a double-peaked structure in the incoherent part of the spectrum of transmitted light was found for $f=1$ in the low-transmission branch of the hysteresis cycle, which is a cooperative and purely quantum effect. Starting from the same equation, the intensity fluctuations have been investigated,[16] and under the same conditions photon antibunching was found.

B. Fokker-Planck equations for field distribution: An alternative approach

The method to treat open systems formulated by one of us[26] was applied to the laser,[21,22,43–45] OB (Refs. 23 and 46) and the laser with a saturable absorber.[44,47,48] This approach leads directly from the ME (3) to a generalized FPE for the Glauber quasiprobability distribution of the field alone $P(x,x^*,t)$. The application to both sides of the ME of the operation of partial trace over the atomic variables Tr_A generates a hierarchy of equations for the reduced statistical operator of the field mode $\rho(t)=\mathrm{Tr}_A W(t)$ and for suitably defined field-atom correlation operators. An approximation scheme must be used in order to truncate this hierarchy and get, at each step, a closed system of equations which allows for deriving the dynamics of $\rho(t)$ that is the one of interest. The model (3) has been extensively analyzed in the "one-atom approximation" in which the field is assumed to interact only with single atoms at a time whereas interaction processes among the field and two or more atoms are neglected. In this approximation one obtains a closed system of time-evolution equations for the operators $\rho(t)$, $W^{(\pm)}(t)=\mathrm{Tr}_A[r_1^{\mp}W(t)]$ and $W^{(+-)}(t)=\mathrm{Tr}_A[r_1^{+}r_1^{-}W(t)]$, where the index 1 indicates any of the N identical atoms. Then in the limit (17) one can eliminate adiabatically $W^{(\pm)}$ and $W^{(+-)}$, thus obtaining a closed time-evolution equation for $\rho(t)$. When translated into the Glauber representation, this equation becomes a c-number differential equation which reads[21]

$$K^{-1}\frac{\partial P(x,x^*,t)}{\partial t}=\Bigg\{\frac{\partial}{\partial x}[x(1+2C)-y]+\text{c.c.} -2C\left[\frac{\partial}{\partial x}x+\text{c.c.}-\frac{1}{N_s}\frac{\partial^2}{\partial x^*\partial x}\right]\left[1+|x|^2-\frac{1}{4N_s}\left[x\frac{\partial}{\partial x}+\text{c.c.}\right]\right]^{-1} \times(1+\sigma+|x|^2)\Bigg\}P(x,x^*,t)\,. \tag{22}$$

It contains derivatives of all orders due to the presence of the inverse operator; in spite of this, its stationary solution (for $y=0$) was derived and discussed.[22]

In order to treat the transient, the Fokker-Planck approximation of (22) was considered in Ref. 44 for the usual laser and the laser with a saturable absorber and in Ref. 23 for OB. It reads

$$K^{-1}\frac{\partial P(x,x^*,t)}{\partial t}=\Bigg\{-\frac{\partial}{\partial x}\left[-x\left[1-\sigma\frac{2C}{1+|x|^2}\right]+y\right]+\text{c.c.} +\frac{C}{N_s}\Bigg[\frac{\partial^2}{\partial x^*\partial x}\frac{(1+\sigma+|x|^2)(2+|x|^2)}{(1+|x|^2)^2} -\frac{1}{2}\left[\frac{\partial^2}{\partial x^2}x^2+\text{c.c.}\right]\frac{1+\sigma+|x|^2}{(1+|x|^2)^2}\Bigg]\Bigg\}P(x,x^*,t)\,. \tag{23}$$

The passage from (22) to (23) follows by expanding the inverse operator of (22) into a Neumann series and then neglecting all terms proportional to N_s^{-n} with $n>1$ since $N_s^{-1}\ll 1$.

In the polar coordinates (20) the FPE (23) becomes

$$K^{-1}\frac{\partial P(r,\varphi,t)}{\partial t}=\frac{1}{r}\Bigg\{\frac{\partial}{\partial r}r\left[r\left[1-\sigma\frac{2C}{1+r^2}\right]-y\cos\varphi\right]+\frac{\partial}{\partial\varphi}y\sin\varphi +\frac{C}{2N_s}\left[\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\frac{1+\sigma+r^2}{(1+r^2)^2}+\frac{1}{r}\frac{\partial^2}{\partial\varphi^2}\frac{1+\sigma+r^2}{1+r^2}\right]\Bigg\}P(r,\varphi,t)\,. \tag{23$'$}$$

This further FPE for the Glauber quasiprobability distribution has a diffusion matrix which is positive definite. We note also that it does not depend on the parameter f, contrary to Eqs. (18) and (19).

In Ref. 44 a comparison was made in the case of the laser between the stationary solution of the full Eq. (22) and of the FPE (23). It turns out that the two stationary distributions have exactly the same Gaussian approximation, which is a very accurate approximation for the laser above threshold. The validity of the Fokker-Planck approximation of Eq. (22) (or generalizations thereof) was later substantiated by other works, in the case of the usual laser,[24] of the laser with saturable absorber,[24,48] and of OB.[23] Hence it works both for first- and for second-order-like phase transitions.

In order to make a comparison with Eq. (19) and its linearized version (21), we quote also the linearized version of Eq. (23) in the variables (20′):

$$K^{-1}\frac{\partial P(x',x^{*\prime},t)}{\partial t}=\Bigg\{-\frac{\partial}{\partial x'}\left[-\left[1-\sigma\frac{2C}{(1+x_{st}^2)^2}\right]x'-\sigma\frac{2Cx_{st}^2}{(1+x_{st}^2)^2}x^{*\prime}\right]+\text{c.c.}$$
$$+\frac{C}{N_s}\left[\frac{(1+\sigma+x_{st}^2)(2+x_{st}^2)}{(1+x_{st}^2)^2}\frac{\partial^2}{\partial x^{*\prime}\partial x'}\right.$$
$$\left.\left.-\frac{1}{2}\frac{x_{st}^2(1+\sigma+x_{st}^2)}{(1+x_{st}^2)^2}\left[\frac{\partial^2}{\partial x'^2}+\text{c.c.}\right]\right]\right\}P(x',x^{*\prime},t)\ . \tag{24}$$

At this point we have two Fokker-Planck equations for field quasiprobability distribution functions, namely, Eq. (19) for a Wigner distribution and Eq. (23) for a Glauber distribution, which are *a priori* valid candidates to describe the field dynamics. Thus we end this section with the same question made in connection with OB in Ref. 44. The answer is given in Sec. IV.

IV. CHOOSING THE BEST APPROACH

At first, a comparison between the two FPE (19) and (23) seems difficult, because they are derived from the one-mode model of Sec. II via quite different methods. The key point to make a comparison between the two approaches outlined in Sec. III is the derivation from the ME (3) of equations having the same physical content of the linearized Fokker-Planck equations [(21) and (24)]. Actually, this procedure allows for establishing a direct connection between the two sequences of steps to be made to obtain either FPE from the model of Sec. II. For definiteness we consider the case of OB; hence we put $\sigma=-1$ throughout this section.

A. Wigner function approach

Let us consider the normalized operators

$$\hat{x}=\frac{A}{\sqrt{N_s}},\quad \hat{x}^{\dagger}=\frac{A^{\dagger}}{\sqrt{N_s}}\ ,$$
$$P^{\pm}=\left[-\frac{N}{2}\left[\frac{\gamma_{\parallel}}{\gamma_{\perp}}\right]^{1/2}\right]^{-1}R^{\pm},\quad P_3=\left[\frac{N}{2}\right]^{-1}R_3\ , \tag{25}$$

which correspond to the c number variables x,x^*,v,v^*,m, respectively [see (11)]. From the ME (3), using the commutation rules (2) and (1′), we obtain the following time-evolution equations for the mean values of the operators (25):

$$\frac{d}{dt}\langle x\rangle=-K(\langle\hat{x}\rangle-y+2C\langle P^{-}\rangle)\ , \tag{26a}$$

$$\frac{d}{dt}\langle P^{-}\rangle=-\gamma_{\perp}(\langle P^{-}\rangle+\langle\hat{x}P_3\rangle)\ , \tag{26b}$$

$$\frac{d}{dt}\langle P_3\rangle=-\gamma_{\parallel}[\langle P_3\rangle+1-\tfrac{1}{2}(\langle\hat{x}P^{+}\rangle+\langle\hat{x}^{\dagger}P^{-}\rangle)]\ , \tag{26c}$$

plus the complex conjugates of Eqs. (26a) and (26b). System (26) is not closed since it contains mean values of operator products and in our quantum statistical treatment no factorization ansatz is made. Hence we derive from the ME (3) the time-evolution equations for the second moments. In doing that, we must choose a definite ordering prescription [this is immaterial in Eqs. (26) because $\hat{x}$ commutes with $P^{\pm}$ and P_3]. In order to ensure consistency with the FPE (19) for the Wigner distribution, we choose the symmetrical-ordering prescription. Namely, we indicate the mean values as follows:

$$\langle \hat{x}^\dagger \hat{x} \rangle_s = \tfrac{1}{2}(\langle \hat{x}^\dagger \hat{x} \rangle + \langle \hat{x}\hat{x}^\dagger \rangle) ,$$
$$\langle P^+P^- \rangle_s = \tfrac{1}{2}(\langle P^+P^- \rangle + \langle P^-P^+ \rangle) ,$$

etc., whereas we drop the index s when the two operators commute. From Eq. (3) we obtain the following set of equations for the symmetrized second moments:

$$\frac{d}{dt}\langle \hat{x}^\dagger \hat{x} \rangle_s = -K \left[2\langle \hat{x}^\dagger \hat{x} \rangle_s - y(\langle \hat{x}^\dagger \rangle_s - y(\langle \hat{x}^\dagger \rangle + \langle \hat{x} \rangle) + 2C(\langle \hat{x}^\dagger P^- \rangle + \langle \hat{x} P^+ \rangle) - \frac{1}{N_s} \right] , \tag{27a}$$

$$\frac{d\langle \hat{x}^2 \rangle}{dt} = -2K(\langle \hat{x}^2 \rangle - y\langle \hat{x} \rangle + 2C\langle \hat{x} P^- \rangle) , \tag{27b}$$

$$\frac{d\langle \hat{x} P^- \rangle}{dt} = -\gamma_\perp \left[\langle \hat{x} P^- \rangle + \langle \hat{x}^2 P_3 \rangle_s + \frac{K}{\gamma_\perp}[\langle \hat{x} P^- \rangle - y\langle P^- \rangle + 2C\langle (P^-)^2 \rangle] \right] , \tag{27c}$$

$$\frac{d\langle \hat{x}^\dagger P^- \rangle}{dt} = -\gamma_\perp \left[\langle \hat{x}^\dagger P^- \rangle + \langle \hat{x}^\dagger \hat{x} P_3 \rangle_s + \frac{K}{\gamma_\perp}(\langle \hat{x}^\dagger P^- \rangle - y\langle P^- \rangle + 2C\langle \hat{P}^+ \hat{P}^- \rangle_s) \right] , \tag{27d}$$

$$\frac{d\langle \hat{x} P_3 \rangle}{dt} = -\gamma_\parallel \left[\langle \hat{x} P_3 \rangle + \langle \hat{x} \rangle - \tfrac{1}{2}(\langle \hat{x}^2 P^+ \rangle + \langle \hat{x}^\dagger \hat{x} P^- \rangle_s) + \frac{K}{\gamma_\parallel}(\langle \hat{x} P_3 \rangle - y\langle P_3 \rangle + 2C\langle P_3 P^- \rangle_s) \right] , \tag{27e}$$

$$\frac{d}{dt}\langle P^+P^- \rangle_s = -\gamma_\perp \left[2\langle P^+P^- \rangle_s + \langle \hat{x} P^+ P_3 \rangle_s + \langle \hat{x}^\dagger P_3 P^- \rangle_s - \frac{\gamma_\perp}{K}\frac{1}{2CN_s} \right] , \tag{27f}$$

$$\frac{d}{dt}\langle (P^-)^2 \rangle = -2\gamma_\perp[\langle (P^-)^2 \rangle + \langle \hat{x} P_3 P^- \rangle_s] , \tag{27g}$$

$$\frac{d}{dt}\langle P_3 P^- \rangle_s = -\gamma_\perp \left[\langle P_3 P^- \rangle_s + \langle \hat{x} P_3^2 \rangle + \frac{\gamma_\parallel}{\gamma_\perp}\{\langle P_3 P^- \rangle_s + \langle P^- \rangle - \tfrac{1}{2}[\langle \hat{x} P^+ P^- \rangle_s + \langle \hat{x}^\dagger (P^-)^2 \rangle]\} - \frac{\gamma_\perp}{K}\frac{f^2}{2CN_s}\langle P^- \rangle \right] , \tag{27h}$$

$$\frac{d\langle P_3^2 \rangle}{dt} = -2\gamma_\parallel \left[\langle P_3^2 \rangle + \langle P_3 \rangle - \tfrac{1}{2}(\langle \hat{x} P^+ P_3 \rangle_s + \langle \hat{x}^\dagger P_3 P^- \rangle_s) - \frac{\gamma_\perp}{K}\frac{f}{8CN_s}(\langle P_3 \rangle + 1) \right] , \tag{27i}$$

plus the complex conjugates of Eqs. (27b)—(27e), (27g), and (27h).

Next we linearize Eqs. (27) around the mean values. This is a well-known procedure. To do that, we put

$$\hat{x} = \langle \hat{x} \rangle + \delta\hat{x}, \quad \hat{x}^\dagger = \langle \hat{x} \rangle + \delta\hat{x}^\dagger ,$$
$$P^\pm = \langle P \rangle + \delta P^\pm, \quad P_3 = \langle P_3 \rangle + \delta P_3 . \tag{28}$$

By inserting (28) into Eqs. (26) and (27), taking into account that $\langle \delta x \rangle = \langle \delta P^\pm \rangle = \langle \delta P_3 \rangle = 0$, and by suitably combining these equations, one obtains a system of time-evolution equations for the variances $\langle \delta x^\dagger \delta x \rangle$, etc. This system is not closed because there appear also mean values of products of three deviations δx, $\delta x^\dagger$, $\delta P^\pm$, δP_3. However, these terms can be dropped on the basis of the following scaling argument.[13] In fact, the quantities in play scale as follows:

$$\langle \hat{x} \rangle, \langle \hat{x}^\dagger \rangle, \langle P^\pm \rangle, \langle P_3 \rangle = O(N_s^0) ,$$
$$\delta\hat{x}, \delta\hat{x}^\dagger, \delta P^\pm, \delta P_3 = O(N_s^{-1/2}) . \tag{29}$$

Using (29) and neglecting the terms of order $N_s^{-3/2}$, one obtains a closed linear system of equations for the variances, in which the mean values $\langle \hat{x} \rangle, \langle \hat{x}^\dagger \rangle, \langle P^\pm \rangle, \langle P_3 \rangle$ appear as known quantities. Since we are interested in the fluctuations around steady state, we insert the stationary mean values. Namely, we put $\langle \hat{x} \rangle = \langle \hat{x}^\dagger \rangle = x_{st}$, $\langle P^+ \rangle = \langle P^- \rangle = P_{st}$, $\langle P_3 \rangle = P_{3,st}$, where x_{st} is a solution of the equation

$$y = x_{st}\left[1 + \frac{2C}{1+x_{st}^2}\right], \tag{30a}$$

while P_{st} and $P_{3,st}$ are given by

$$P_{st} = \frac{x_{st}}{1+x_{st}^2}, \quad P_{3,st} = -\frac{1}{1+x_{st}^2}. \tag{30b}$$

In other words, x_{st}, P_{st}, and $P_{3,st}$ give a stationary solution of Eqs. (26) with all atom-field correlations neglected [i.e., the semiclassical equations for OB (Ref. 34)].

In conclusion, one obtains the following closed system of equations which rules the dynamics of fluctuations around steady state:

$$\frac{d}{dt}\langle \delta\hat{x}^\dagger \delta\hat{x} \rangle_s = -K\left[2\langle \delta\hat{x}^\dagger \delta\hat{x} \rangle_s + 2C(\langle \delta x^\dagger \delta P^- \rangle + \langle \delta\hat{x}\delta P^+ \rangle) - \frac{1}{N_s}\right], \tag{31a}$$

$$\frac{d}{dt}\langle (\delta\hat{x})^2 \rangle = -2K[\langle (\delta\hat{x})^2 \rangle + 2C\langle \delta\hat{x}\delta P^- \rangle], \tag{31b}$$

$$\frac{d}{dt}\langle \delta\hat{x}\delta P^- \rangle = -\gamma_\perp\left[\langle \delta\hat{x}\delta P^- \rangle - \frac{1}{1+x_{st}^2}\langle (\delta\hat{x})^2 \rangle + x_{st}\langle \delta\hat{x}\delta P_3 \rangle + \frac{K}{\gamma_\perp}[\langle \delta\hat{x}\delta P^- \rangle + 2C\langle (\delta P^-)^2 \rangle]\right], \tag{32a}$$

$$\frac{d}{dt}\langle \delta\hat{x}^\dagger \delta P^- \rangle = -\gamma_\perp\Bigg[\langle \delta\hat{x}^\dagger \delta P^- \rangle - \frac{1}{1+x_{st}^2}\langle \delta\hat{x}^\dagger \delta\hat{x} \rangle_s + x_{st}\langle \delta\hat{x}^\dagger \delta P_3 \rangle + \frac{K}{\gamma_\perp}(\langle \delta\hat{x}^\dagger \delta P^- \rangle + 2C\langle \delta P^+ \delta P^- \rangle_s)\Bigg], \tag{32b}$$

$$\frac{d}{dt}\langle \delta\hat{x}\delta P_3 \rangle = -\gamma_\parallel\Bigg[\langle \delta\hat{x}\delta P_3 \rangle - \frac{x_{st}}{2}(\langle \delta\hat{x}\delta P^- \rangle + \langle \delta\hat{x}\delta P^+ \rangle) - \frac{x_{st}}{2(1+x_{st}^2)}[\langle \delta\hat{x}^\dagger \delta\hat{x} \rangle_s + \langle (\delta\hat{x})^2 \rangle] + \frac{K}{\gamma_\parallel}(\langle \delta\hat{x}\delta P_3 \rangle + 2C\langle \delta P_3 \delta P^- \rangle_s)\Bigg], \tag{32c}$$

$$\frac{d}{dt}\langle \delta P^\dagger \delta P^- \rangle_s = -\gamma_\perp\Bigg[2\langle \delta P^+ \delta P^- \rangle_s \frac{1}{1+x_{st}^2}(\langle \delta\hat{x}\delta P^+ \rangle + \langle \delta\hat{x}^\dagger \delta P^- \rangle) + x_{st}(\langle \delta P^+ \delta P_3 \rangle_s + \langle \delta P_3 \delta P^- \rangle_s) - \frac{\gamma_\perp}{K}\frac{1}{2CN_s}\Bigg], \tag{33a}$$

$$\frac{d}{dt}\langle (\delta P^-)^2 \rangle = -2\gamma_\perp\left[\langle (\delta P^-)^2 \rangle - \frac{1}{1+x_{st}^2}\langle \delta\hat{x}\delta P^- \rangle + x_{st}\langle \delta P_3 \delta P^- \rangle_s\right], \tag{33b}$$

$$\frac{d}{dt}\langle \delta P_3 \delta P^- \rangle_s = -\gamma_\perp\Bigg[\langle \delta P_3 \delta P^- \rangle_s - \frac{1}{1+x_{st}^2}\langle \delta\hat{x}\delta P_3 \rangle + x_{st}\langle (\delta P_3)^2 \rangle + 2f\Bigg[\langle \delta P_3 \delta P^- \rangle_s - \frac{x_{st}}{2}[\langle \delta P^+ \delta P^- \rangle_s + \langle (\delta P^-)^2 \rangle] - \frac{x_{st}}{2(1+x_{st}^2)}(\langle \delta\hat{x}\delta P^- \rangle + \langle \delta\hat{x}^\dagger \delta P^- \rangle)\Bigg] - \frac{\gamma_\perp}{K}\frac{f^2}{2CN_s}\frac{x_{st}}{1+x_{st}^2}\Bigg], \tag{33c}$$

$$\frac{d}{dt}\langle(\delta P_3)^2\rangle=-2\gamma_{\parallel}\left[\langle(\delta P_3)^2\rangle-\frac{x_{st}}{2}(\langle\delta P^+\delta P_3\rangle_s+\langle\delta P_3\delta P^-\rangle_s)\right.$$
$$\left.-\frac{x_{st}}{2(1+x_{st}^2)}(\langle\delta\hat{x}\delta P_3\rangle+\langle\delta\hat{x}^\dagger\delta P_3\rangle)-\frac{\gamma_\perp}{K}\frac{f}{4CN_s}\frac{x_{st}^2}{1+x_{st}^2}\right], \quad (33d)$$

plus the complex conjugates of Eqs. (31b), (32a)–(32c) and (33b) and (33c).

Equations (31) rule the dynamics of field fluctuations, Eqs. (32) of atom-field correlations, and Eqs. (33) of atomic fluctuations. Once again, we put ourselves in the limit (17) and perform the adiabatic elimination of the atomic variables from the full system (31)–(33). To this end, first of all we set equal to zero all derivatives of mean values including atomic deviations, namely,

$$\frac{d}{dt}\langle\delta\hat{x}\delta P^-\rangle=\frac{d}{dt}\langle\delta\hat{x}^\dagger\delta P^-\rangle=\frac{d}{dt}\langle\delta\hat{x}\delta P_3\rangle=\frac{d}{dt}\langle\delta P^+\delta P^-\rangle_s$$
$$=\frac{d}{dt}\langle(\delta P^-)^2\rangle=\frac{d}{dt}\langle\delta P_3\delta P^-\rangle_s=\frac{d}{dt}\langle(\delta P_3)^2\rangle=0 \quad (34)$$

together with their complex-conjugate quantities.

Next, we solve the system (33) thereby expressing the atomic fluctuations in terms of atom-field correlations. Then we substitute these expressions into system (32) for the atom-field correlations, obtaining

$$0=\langle\delta\hat{x}\delta P^-\rangle-\frac{1}{1+x_{st}^2}\langle(\delta\hat{x})^2\rangle+x_{st}\langle\delta\hat{x}\delta P_3\rangle-\frac{f}{2N_s}\frac{x_{st}^2}{(1+x_{st}^2)^2}+\frac{K}{\gamma_\perp}\phi_1, \quad (35a)$$

$$0=\langle\delta\hat{x}^\dagger\delta P^-\rangle-\frac{1}{1+x_{st}^2}\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle_s+x_{st}\langle\delta\hat{x}^\dagger\delta P_3\rangle+\frac{1}{2N_s}\left[1-\frac{fx_{st}^2}{(1+x_{st}^2)^2}\right]+\frac{K}{\gamma_\perp}\phi_2, \quad (35b)$$

$$0=\langle\delta\hat{x}\delta P_3\rangle-\frac{x_{st}}{2}(\langle\delta\hat{x}\delta P^-\rangle+\langle\delta\hat{x}\delta P^+\rangle)-\frac{x_{st}}{2(1+x_{st}^2)}(\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle_s+\langle(\delta\hat{x})^2\rangle)+\frac{1}{4N_s}\frac{x_{st}}{(1+x_{st}^2)^2}+\frac{K}{\gamma_{\parallel}}\phi_3, \quad (35c)$$

where ϕ_1, ϕ_2, and ϕ_3 are suitable linear combinations of the atom-field correlations, whose explicit expression is given in the Appendix.

In the limit (17), the terms with ϕ_1, ϕ_2, ϕ_3 can be dropped. Thus, by solving Eqs. (35) one finds the expression of the atom-field correlations as a function of the field fluctuations. If these expressions are substituted into the rhs of Eqs. (31a) and (31b), one obtains a closed system of equations for the field fluctuations, which reads

$$K^{-1}\frac{d}{dt}\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle_s=-2\left[1+\frac{2C}{(1+x_{st}^2)^2}\right]\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle_s+\frac{2Cx_{st}^2}{(1+x_{st}^2)^2}[\langle(\delta\hat{x})^2\rangle+\langle(\delta x^\dagger)^2\rangle]$$
$$+\frac{1}{N_s}\left[1+C\frac{(1+x_{st}^2)^2(2+x_{st}^2)-(1+2f)x_{st}^2}{(1+x_{st}^2)^3}\right], \quad (36a)$$

$$K^{-1}\frac{d}{dt}\langle(\delta\hat{x})^2\rangle=-2\left[1+\frac{2C}{(1+x_{st}^2)^2}\right]\langle(\delta\hat{x})^2\rangle+\frac{4Cx_{st}^2}{(1+x_{st}^2)^2}\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle_s-\frac{C}{N_s}x_{st}^2\frac{(1+x_{st}^2)^2+1+2f}{(1+x_{st}^2)^3}. \quad (36b)$$

As one easily verifies, Eqs. (36) coincide with the equations for the symmetrized field fluctuations that one derives directly from the linearized FPE (21) for the Wigner distribution $P_w(x',x^{*\prime},t)$ of the field variables, with $\sigma=-1$.

B. One-atom approach

Now we go back to the ME (3) and derive the equations of motion for first and second moments of atomic and field operators in the one-atom approximation consistently with the approach of Ref. 26. This means neglecting all terms involving two different atoms in the derivation of such equations. Let us define the single-atom operators

$$\rho^{\pm}=\left[-\frac{1}{2}\left[\frac{\gamma_{\parallel}}{\gamma_{\perp}}\right]^{1/2}\right]^{-1} r^{\pm}, \quad s=r^{+}r^{-}=r_3+\tfrac{1}{2}\,, \tag{37}$$

where we have dropped the index i because all the atoms are identical. Using Eqs. (1), (2), and (4b), from the ME (3) one derives easily the system of equations for first and second moments:

$$\frac{d}{dt}\langle\hat{x}\rangle=-K(\langle\hat{x}\rangle-y+2C\langle\rho^{-}\rangle)\,, \tag{38a}$$

$$\frac{d}{dt}\langle\rho^{-}\rangle=-\gamma_{\perp}[\langle\rho^{-}\rangle+(2\langle\hat{x}s\rangle-\langle\hat{x}\rangle)]\,, \tag{38b}$$

$$\frac{d}{dt}\langle s\rangle=-\gamma_{\parallel}[\langle s\rangle-\tfrac{1}{4}(\langle\hat{x}\rho^{+}\rangle+\langle\hat{x}^{\dagger}\rho^{-}\rangle)]\,, \tag{38c}$$

$$\frac{d}{dt}\langle\hat{x}^{\dagger}\hat{x}\rangle=-K[2\langle\hat{x}^{\dagger}\hat{x}\rangle-y(\langle\hat{x}^{\dagger}\rangle+\langle\hat{x}\rangle)+2C(\langle\hat{x}^{\dagger}\rho^{-}\rangle+\langle\hat{x}\rho^{+}\rangle)]\,, \tag{39a}$$

$$\frac{d}{dt}\langle(\hat{x})^2\rangle=-2K(\langle\hat{x}^2\rangle-y\langle\hat{x}\rangle+2C\langle\hat{x}\rho^{-}\rangle)\,, \tag{39b}$$

$$\frac{d}{dt}\langle\hat{x}\rho^{-}\rangle=-\gamma_{\perp}\left[\langle\hat{x}\rho^{-}\rangle+2\langle\hat{x}^2 s\rangle-\langle\hat{x}^2\rangle+\frac{K}{\gamma_{\perp}}(\langle\hat{x}\rho^{-}\rangle-y\langle\rho^{-}\rangle)\right], \tag{39c}$$

$$\frac{d}{dt}\langle\hat{x}^{\dagger}\rho^{-}\rangle=-\gamma_{\perp}\left[\langle\hat{x}^{\dagger}\rho^{-}\rangle+2\langle\hat{x}^{\dagger}\hat{x}s\rangle-\langle\hat{x}^{\dagger}\hat{x}\rangle+\frac{\langle s\rangle}{N_s}+\frac{K}{\gamma_{\perp}}(\langle\hat{x}^{\dagger}\rho^{-}\rangle-y\langle\rho^{-}\rangle)\right], \tag{39d}$$

$$\frac{d}{dt}\langle\hat{x}s\rangle=-\gamma_{\parallel}\left[\langle\hat{x}s\rangle-\tfrac{1}{4}(\langle\hat{x}^{\dagger}\hat{x}\rho^{-}\rangle+\langle\hat{x}^2\rho^{+}\rangle)+\frac{K}{\gamma_{\perp}}(\langle\hat{x}s\rangle-y\langle s\rangle)\right], \tag{39e}$$

together with the complex conjugates of (38a) and (38b) and (39b)—(39e). In order to ensure consistency with the FPE (23) for the Glauber distribution, we have used now the normal-ordering prescription for the operators in play. Note that according to the single-atom approximation the atomic second moments are absent; actually, $(r^{+})^2=(r^{-})^2=0$, $r^{+}r^{-}=r_3+\frac{1}{2}$, $r^{+}r_3=-r^{+}/2$, and so on. Next, we follow the same procedure of the previous case. As a first step, we linearize system (39) around the semiclassical stationary state

$$\begin{aligned}&\langle\hat{x}\rangle=\langle\hat{x}^{\dagger}\rangle=x_{\rm st}\,,\\ &\langle\rho^{-}\rangle=\langle\rho^{+}\rangle=\frac{x_{\rm st}}{1+x_{\rm st}^2}\,,\\ &\langle s\rangle=\frac{x_{\rm st}^2}{2(1+x_{\rm st}^2)}\,,\end{aligned} \tag{30}$$

thus obtaining the closed system of equations for the field fluctuations and the atom-field correlations:

$$\frac{d}{dt}\langle\delta\hat{x}^{\dagger}\delta\hat{x}\rangle=-K[2\langle\delta\hat{x}^{\dagger}\delta\hat{x}\rangle+2C(\langle\delta\hat{x}^{\dagger}\delta\rho^{-}\rangle+\langle\delta\hat{x}\delta\rho^{\dagger}\rangle)]\,, \tag{40a}$$

$$\frac{d}{dt}\langle(\delta\hat{x})^2\rangle=-2K[\langle(\delta\hat{x})^2\rangle+2C\langle\delta\hat{x}\delta\rho^{-}\rangle]\,, \tag{40b}$$

$$\frac{d}{dt}\langle \delta\hat{x}\delta\rho^-\rangle = -\gamma_\perp \left[\langle \delta\hat{x}\delta\rho^-\rangle - \frac{1}{1+x_{st}^2}\langle(\delta\hat{x})^2\rangle + 2x_{st}\langle\delta\hat{x}\delta s\rangle + \frac{K}{\gamma_\perp}\left[\langle\delta\hat{x}\delta\rho^-\rangle - 2C\frac{x_{st}^2}{(1+x_{st}^2)^2}\right]\right], \quad (41a)$$

$$\frac{d}{dt}\langle \delta\hat{x}^\dagger\delta\rho^-\rangle = -\gamma_\perp \left[\langle \delta x^\dagger\delta\rho^-\rangle - \frac{1}{1+x_{st}^2}\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle + 2x_{st}\langle\delta\hat{x}^\dagger\delta s\rangle + \frac{1}{2N_s}\frac{x_{st}^2}{1+x_{st}^2} + \frac{K}{\gamma_\perp}\left[\langle\delta\hat{x}^\dagger\delta\rho^-\rangle + 2C\frac{x_{st}^2}{(1+x_{st}^2)^2}\right]\right], \quad (41b)$$

$$\frac{d}{dt}\langle \delta\hat{x}\delta s\rangle = -\gamma_\parallel \left[\langle \delta\hat{x}\delta s\rangle - \frac{x_{st}}{4}(\langle\delta\hat{x}\delta\rho^-\rangle + \langle\delta\hat{x}\delta\rho^+\rangle) - \frac{x_{st}}{1+x_{st}^2}[\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle + \langle(\delta\hat{x})^2\rangle] + \frac{K}{\gamma_\parallel}\left[\langle\delta\hat{x}\delta s\rangle - C\frac{x_{st}^3}{(1+x_{st}^2)^2}\right]\right], \quad (41c)$$

plus the complex conjugates of (40b) and (41a)—(41c).

In the limit (17) we set

$$\frac{d}{dt}\langle\delta\hat{x}\delta\rho^-\rangle = \frac{d}{dt}\langle\delta\hat{x}^\dagger\delta\rho^-\rangle = \frac{d}{dt}\langle\delta\hat{x}\delta s\rangle = 0$$

and solve the system (41) neglecting all terms proportional to $K/\gamma_\perp, K/\gamma_\parallel$. Substituting the expressions of $\langle\delta\hat{x}\delta\rho^-\rangle$, $\langle\delta\hat{x}^\dagger\delta\rho^-\rangle$, and $\langle\delta\hat{x}\delta s\rangle$ as functions of $\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle$ and $\langle(\delta\hat{x})^2\rangle$ into (40a) and (40b) we obtain

$$K^{-1}\frac{d}{dt}\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle = \frac{1}{(1+x_{st})^2}\left[-2[(1+x_{st}^2)^2+2C]\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle + 2Cx_{st}^2[\langle(\delta\hat{x}^\dagger)^2\rangle + \langle(\delta\hat{x})^2\rangle] + \frac{C}{N_s}x_{st}^2(x_{st}^2+2)\right], \quad (42a)$$

$$K^{-1}\frac{d}{dt}\langle(\delta\hat{x})^2\rangle = \frac{1}{(1+x_{st}^2)^2}\left[-2[(1+x_{st}^2)^2+2C]\langle(\delta\hat{x})^2\rangle + 4Cx_{st}^2\langle\delta\hat{x}^\dagger\delta\hat{x}\rangle - \frac{C}{N_s}x_{st}^4\right]. \quad (42b)$$

As one immediately verifies, Eqs. (42) coincide with the equations for the normal-ordered field fluctuations directly obtained from the linearized FPE (24) for the Glauber distribution $P(x',x^{*\prime},t)$ with $\sigma=-1$.

After the above analysis we can answer the question raised at the end of the previous section and conclude that the approach by means of the generalized Wigner distribution[13,15] is more satisfactory than that by the one-atom approximation in the framework of the method of Ref. 26. In fact, Eqs. (42) [which have essentially the same physical content of the linearized version of Eq. (23)] have been derived from the ME (3) *via the same procedure that leads to* Eqs. (36) [which have essentially the same physical content of the linearized version of Eq. (19)], except for the fact that *it incorporates one approximation more* (the one-atom approximation). *Therefore one concludes that the FPE (19) for the Wigner distribution gives a more complete description of the fluctuations.*

Up to Eqs. (33), the analysis of part A of this section is substantially equivalent to that given in Ref. 49. The differences are the following. (1) We consider the case of OB instead of the laser. This makes it easier, because in OB the output field has a well definite phase at steady state, whereas in the laser the phase is arbitrary. Hence, we avoid to introduce an *ad hoc* value for the phase. (2) We introduce from the very beginning scaled variables, so that the order of magnitude of the various terms is immediately evident. (3) We use symmetrical ordering instead of normal ordering. (4) In Ref. 49 the correlations involving only atomic variables are subdivided into a part containing the correlations between different atoms and a part containing the correlations of each atom with itself. (5) We consider only the case of weak atomic relaxation, using the nomenclature of Ref. 49. After Eqs. (33) our treatment deviates from that of Ref. 49 because we consider the situation (17), whereas Ref. 49 assumes either $\gamma_\perp \gg K, \gamma_\parallel$ or $K \gg \gamma_\perp, \gamma_\parallel$. In the case $\gamma_\perp \gg K, \gamma_\parallel$, which is not completely incompatible with (17), the conclusion of Ref. 49 that the normalized atom-field correlations are inversely proportional to the number of photons agrees with Eqs. (35), which show that these correlations scale as N_s^{-1}. In fact, the photon number at steady state is $N_s x_{st}^2$, and x_{st} is on the order of unity.

V. ONE-MODE LASER PHOTON STATISTICS

After ascertaining that the right FPE is (19) for the Wigner quasiprobability distribution $P_w(x,x^*,t)$, let us analyze in detail this equation in the case of the usual laser.[50] Hence we start from Eq. (19′), that is, the version of the FPE (19) in the polar coordinates (20), where we put $y=0$ (no injected signal) and introduce another relevant laser parameter, namely, the atomic threshold inversion per atom

$$\sigma_T = \frac{k\gamma_\perp}{N\bar{g}^2} = \frac{1}{2C} . \tag{43}$$

At steady state

$$[\partial P_w(r,\varphi,t)/\partial t = \partial P_w(r,\varphi,t)/\partial\varphi = 0]$$

P_w depends only on the variable

$$z = r^2 \tag{44}$$

and obeys the following equation:

$$\frac{d}{dz}[f(z)P_w(z)] + g(z)P_w(z) = 0 , \tag{45}$$

where

$$f(z) = \frac{1}{N_s}\left[1 + \frac{1}{\sigma_T} - \frac{z}{\sigma_T}\frac{(1+z)^2 + \sigma^2(1+2f)}{(1+z)^3}\right] , \tag{46a}$$

$$g(z) = 2\left[1 - \frac{\sigma}{\sigma_T}\frac{1}{1+z}\right] . \tag{46b}$$

The solution is

$$P_w^{st}(z) = \mathcal{N}[f(z)]^{-1}\exp\left[-\int_0^z \frac{g(z')}{f(z')}dz'\right] , \tag{47}$$

where $\mathcal{N}$ is a suitable normalization constant.

From (47) one can calculate the symmetrized moments $\langle z^n \rangle_s$ of z. In particular, the first and second moments of z are linked to the stationary mean value $\langle n \rangle$ and the variance $(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$ of the photon number

$$n = A^\dagger A \tag{48}$$

by the following relations:

$$\begin{aligned} &\langle z \rangle_s = N_s^{-1}(\langle n \rangle + \tfrac{1}{2}) , \\ &\langle z^2 \rangle_s = N_s^{-2}(\langle n^2 \rangle + \langle n \rangle + \tfrac{1}{2}) , \\ &\Longrightarrow (\Delta n)^2 = N_s^2(\langle z^2 \rangle_s - \langle z \rangle_s^2) - \tfrac{1}{4} . \end{aligned} \tag{49}$$

However, we need not consider the exact stationary distribution (47) to discuss the laser photon statistics. Let us distinguish between the case of the laser well below threshold ($\sigma \approx -1$) and that of the laser in the threshold region or well above threshold ($\sigma_T \lesssim \sigma \lesssim 1$).

A. Laser well below threshold

For $\sigma \approx -1$, one has $z \ll 1$ so that Eq. (45) can be approximated by

$$0 = 2\left[1 - \frac{\sigma}{\sigma_T}\right]P_w(z) + \frac{1}{N_s}\left[1 + \frac{1}{\sigma_T}\right]\frac{d}{dz}P_w(z) . \tag{50}$$

The solution of (50) is a blackbody stationary distribution, namely,

$$P_w^{st}(z) = \frac{2(\sigma_T - \sigma)}{1+\sigma_T} e^{-2(\sigma_T - \sigma/1+\sigma_T)z} . \tag{51}$$

Such a distribution is typical of a chaotic field, as it is the laser field below threshold. From Eq. (51) using (49), one can calculate the stationary mean value of the photon number (squared photon number) $\langle n \rangle (\langle n^2 \rangle)$:

$$\begin{aligned} \langle n \rangle &= \frac{1+\sigma}{2(\sigma_T - \sigma)} , \\ \langle n^2 \rangle &= \frac{1}{2}\frac{(1+\sigma)(1+\sigma_T)}{(\sigma_T - \sigma)^2} . \end{aligned} \tag{52}$$

TABLE I. Mean stationary intensity $\langle I \rangle$ vs pump parameter A obtained from (i) the stationary Wigner distribution $P_w^{\rm st}$ [Eq. (54)], (ii) the stationary solution of the generalized Fokker-Planck equation (22), and (iii) the stationary solution of the Fokker-Planck equation (23). The parameters are $\sigma_T = 10^{-2}$, $N_s = 10^4$, $f = 1$.

A	(i)	(ii)	(iii)
0.8	2.1041×10^{-2}	2.1049×10^{-2}	2.1428×10^{-2}
0.9	3.2115×10^{-2}	3.2156×10^{-2}	3.2926×10^{-2}
1.00	5.6650×10^{-2}	5.6690×10^{-2}	5.8426×10^{-2}
1.02	6.4567×10^{-2}	6.4606×10^{-2}	6.6641×10^{-2}
1.04	7.3885×10^{-2}	7.3924×10^{-2}	7.6290×10^{-2}
1.06	8.4753×10^{-2}	8.4792×10^{-2}	8.7509×10^{-2}
1.08	9.7261×10^{-2}	9.7301×10^{-2}	1.0037×10^{-1}
1.1	1.1141×10^{-1}	1.1145×10^{-1}	1.1485×10^{-1}
1.2	2.0050×10^{-1}	2.0054×10^{-1}	2.0466×10^{-1}
1.3	2.9995×10^{-1}	3.0000×10^{-1}	3.0389×10^{-1}
1.4	3.9995×10^{-1}	4.0000×10^{-1}	4.0361×10^{-1}

The results (52) coincide with the corresponding ones derived from Risken's FPE (Ref. 3), which rules the time evolution of the Glauber quasiprobability field distribution in this regime and up to the threshold region.

B. Laser in the threshold region and above threshold

In the case $\sigma_T \lesssim \sigma \lesssim 1$, following a widely used procedure in laser physics, the exact stationary distribution (47) can be very well approximated by a Gaussian. Actually, the function $g(z)$ vanishes at

$$z = \bar{z} = \frac{\sigma}{\sigma_T} - 1 , \tag{53}$$

which is just the scaled semiclassical value of the laser intensity.[1] The factor $f(z)$ has the only effect to produce a very small shift (on the order of N_s^{-1}) in the position of the maximum of $P_w^{\rm st}(z)$ with respect to $\bar{z}$. We neglect this effect by replacing $f(z)$ by $f(\bar{z})$ in the inverse factor in (47). By developing the argument of the exponential in (47) up to second order around $z = \bar{z}$, we find

$$P_w^{\rm st}(z) = \mathcal{N}' e^{-[(z-\bar{z})^2/q_s^2]} , \tag{54}$$

where $\mathcal{N}'$ is the normalization constant and

$$\begin{aligned} q_s^2 &= 2(\langle z^2 \rangle_s - \langle z \rangle_s^2) \\ &= \frac{1}{N_s}\left[\frac{1+\sigma}{\sigma_T} - (\sigma - \sigma_T)(1+2f)\right] . \end{aligned} \tag{55}$$

From Eqs. (54) and (55) we derive both analytical and numerical results on laser photon statistics and compare them with those obtained by a number of authors using quite different approaches.

First of all, we calculate the mean value and the fluctuations of the stationary field intensity versus the pump parameter

$$A = \frac{\sigma}{\sigma_T} \tag{56}$$

and compare them with those computed from the exact stationary solution of the generalized FPE (22) (Ref. 22) and of its Fokker-Planck approximation (23).[44] To this end, we have only to compare our data with those given by P. Mandel,[24] who performed a careful numerical comparison among the photon statistics derived from Eqs. (22) and (23) and that obtained in the framework of his "semiquantum" approximation. Table I compares the scaled mean stationary intensity $\langle I \rangle = N_s^{-1} \langle n \rangle = \langle z \rangle_s - (2N_s)^{-1}$ vs A as obtained from (i) distribution (54), (ii) the stationary solution of the generalized FPE (22), and (iii) the stationary solution of the FPE (23). The parameters are $\sigma_T = 10^{-2}, N_s = 10^4$. It turns out that the values calculated from (54) are extremely close to those given by (22), also for $A < 1$ where the Gaussian approximation becomes questionable. More precisely, the mean stationary intensities obtained from $P_w^{\rm st}$ are very slightly lower (less than 1%) with respect to those calculated from Eq. (22) at steady state. Table II shows a parallel comparison, but concerning the stationary intensity fluctuations

$$\langle I^2 \rangle / \langle I \rangle^2 = (\langle z^2 \rangle_s - \langle z \rangle_s / N_s) / [\langle z \rangle_s - (2N_s)^{-1}]^2 .$$

Again the results given by (54) are very close to those derived from (22), in this case very slightly

TABLE II. Same as in Table I but for the stationary fluctuation of intensity $\langle I^2\rangle/\langle I\rangle^2$.

A	(i)	(ii)	(iii)
0.8	1.8778	1.8713	1.8674
0.9	1.7711	1.7677	1.7600
1.00	1.5718	1.5705	1.5579
1.02	1.5199	1.5190	1.5057
1.04	1.4657	1.4649	1.4514
1.06	1.4104	1.4099	1.3965
1.08	1.3560	1.3555	1.3427
1.1	1.3042	1.3037	1.2919
1.2	1.1231	1.1228	1.1179
1.3	1.0563	1.0561	1.0545
1.4	1.0317	1.0315	1.0309

higher than the latter ones, and become closer and closer as A increases, i.e., well above threshold. These results confirm that in the description of the laser in the threshold region also quite different approaches give substantially the same results.

Now we go back to Eq. (54), to point out some interesting consequences of our treatment of laser fluctuations and to further elucidate the connections with other approaches. From Eq. (54) plus the relations (49) one calculates the variance of the laser photon number, obtaining $(\Delta n)^2 = N_s^2 q_s^2/2 - 1/4$, or (neglecting the last term)

$$(\Delta n)^2 = \frac{N_s}{2}\left[\frac{1+\sigma}{\sigma_T} - (\sigma-\sigma_T)(1+2f)\right]. \quad (57)$$

Many comments are in order with respect to the result (57). First of all, since by (49) and (53)

$$\langle n\rangle \simeq N_s \langle z\rangle_s \simeq N_s\left[\frac{\sigma}{\sigma_T}-1\right], \quad (58)$$

using (57) and (58) one has for $\sigma \gg \sigma_T$

$$\frac{(\Delta n)^2}{\langle n\rangle} = \frac{1}{2}\left[\frac{1+\sigma}{\sigma-\sigma_T} - \sigma_T(1+2f)\right] \simeq \frac{1+\sigma}{2\sigma}. \quad (59)$$

It follows that $(\Delta n)^2/\langle n\rangle \to 1$ for $\sigma \to 1$, i.e., at very high pumping the stationary distribution approaches a Poisson distribution, completing the change of the photon statistics from chaotic ($\sigma \simeq -1$) to coherent ($\sigma \simeq 1$). This is a well-known result, that was already established in the sixties on the basis of different approaches.[1,2,12,19,39]

Now, let us consider Eq. (57) more closely. The main contribution to photon-number fluctuations, namely, $N_s(1+\sigma)/2\sigma_T$, coincides with the variance calculated by Weidlich, Risken, and Haken[12] via a normal-ordering prescription. These authors made a factorization ansatz which amounts to neglecting atom-atom correlations.[51] Thus the remaining contribution to photon-number fluctuations, $N_s(\sigma-\sigma_T)(1+2f)/2$ is the effect of such correlations which have been taken into due account in the present treatment. This term of atomic cooperative origin vanishes at threshold. Expression (57) of the variance perfectly agrees with the expression recently derived by Pokrovski and Khazanov.[18] In Ref. 18, Eq. (57) is, however, not obtained from a FPE but on analyzing the moment equations in the limit $K \ll \gamma_\perp, \gamma_\parallel$.

Let us now consider the behavior of the intensity correlation function

$$g^{(2)}(0) = \frac{\langle A^\dagger A^\dagger A A\rangle}{\langle A^\dagger A\rangle^2} = 1 + \frac{(\Delta n)^2 - \langle n\rangle}{\langle n\rangle^2}. \quad (60)$$

As it is well known, for a coherent field $g^{(2)}(0)=1$. Moreover, one has *bunching* when $g^{(2)}(0)>1$ (e.g., in a thermal field) and *antibunching*[54] when $g^{(2)}(0)<1$. Photon antibunching is a purely quantum effect, first observed in resonance fluorescence from two-level atoms,[55] and can be explained only by a fully quantum treatment of radiation. In our case, from (57) and (58) we obtain

$$(\Delta n)^2 - \langle n\rangle = \frac{N_s}{2}\left[\frac{1-\sigma+2\sigma_T}{\sigma_T} - (\sigma-\sigma_T)(1+2f)\right]. \quad (61)$$

For $\sigma \to 1$ the above expression approaches the value

$$\frac{N_s}{2}[2-(1-\sigma_T)(1+2f)] \simeq \frac{N_s}{2}(1-2f). \quad (62)$$

Hence when $f \gtrsim 1/2$ (e.g., in the purely radiative case $f=1$) $g^{(2)} < 1$, i.e., *we have photon antibunching*. This is in a sense surprising, because this effect occurs in the region very high above threshold ($\sigma \approx 1$), where the photon statistics is closest to Poissonian. However, we stress that this antibunching effect is only a curiosity because

$$[(\Delta n)^2 - \langle n\rangle]/\langle n\rangle^2 \propto N_s^{-1} \ll 1.$$

This phenomenon arises from the atomic correlation term

$$N_s(\sigma-\sigma_T)(1+2f)/2$$

of (57). This explains why it was not predicted in earlier treatments of the laser,[1,2,12,19,22] which systematically neglected atom-atom correlations.

Another remarkable nonclassical effect, which is presently attracting the attention of quantum-optical physicists, is the so-called "squeezing." Let us consider the two components in quadrature of the electric field:

$$x_1=\frac{A+A^\dagger}{2},\quad x_2=\frac{A-A^\dagger}{2i}\ . \tag{63}$$

This effect arises when the variance of either of the two components is smaller than in the usual coherent Glauber states.[8] Namely, we have squeezing when

$$\langle \delta x_1^2\rangle<\tfrac{1}{4}\ \text{ or }\ \langle \delta x_2^2\rangle<\tfrac{1}{4}\ . \tag{64}$$

Recently, it has been demonstrated that this effect arises in degenerate parametric amplifiers and also, to a smaller extent, in optical bistability.[30,56]

VI. CONCLUDING DISCUSSION

We have shown that the presence of a smallness parameter, namely, the inverse of the saturation photon number N_s, allows obtaining a consistent and tractable description of fluctuations in quantum-optical systems.

The conclusion of our analysis is that the most complete and accurate description is that obtained using the quasiprobability distribution for the macroscopic observables of the system *á la* Haken and co-workers,[1,10] with two differences. The first is that, in general, we must not use the scaling argument adopted in Ref. 1, which holds only for the laser in the threshold region. This argument leads to drop, in the time-evolution equation for the quasiprobability distribution, not only the terms with derivative of an order higher than the second but also many of the diffusion terms. Instead, one must keep, in general, all the terms with derivatives of second order. The second difference is that we must use the generalized Wigner function (i.e., the symmetrical-ordering prescription) instead of the generalized Glauber-Sudarshan distribution (i.e., the normal-ordering prescription).

In such a way, we can describe also nonclassical effects as photon antibunching or squeezing. As we have shown, these effects are maltreated or even destroyed if one neglects the atom-atom correlations or one introduces the one-atom approximation.

The condition $N_s \gg 1$ allows us to neglect the terms with derivatives of an order higher than the second. We stress that this condition is not, in general, equivalent to assuming that the fluctuations are normal. In fact, the full (nonlinearized) Fokker-Planck equation holds also when the variance of the electric field is on the same order of its mean value, as in the laser near threshold. When the fluctuations are normal, one can use a simplified description obtained by linearization, as it is made, e.g., in Sec. IV.

Let us now discuss at length the matter of the Wigner and Glauber-Sudarshan representations. The second one is more popular in the quantum-optical community, because most quantities of interest are naturally normal ordered. However, we are forced to use the Wigner distribution by the fact that the diffusion matrix of the Fokker-Planck equation obtained using the Glauber-Sudarshan representation is not, in general, positive definite. In fact, *the lack of positive definiteness is a signal of the presence of nonclassical effects.* In order to illustrate this point, let us consider the following simplifying assumptions:

(a) There is an injected coherent field, so that the stationary mean value of the electric field is different from zero;

(b) the fluctuations are normal, so that we can use a linearized Fokker-Planck equation;

(c) the diffusion matrix of the linearized Fokker-Planck equation in polar variables r,φ is diagonal (this holds when there is perfect resonance between the incident field, the atoms, and the cavity).

Let us consider the linearized FPE in the Glauber-Sudarshan representation which will have the structure

$$\frac{\partial P(r',\varphi',t)}{\partial t}=\left[K_r\frac{\partial}{\partial r'}r'+K_\varphi\frac{\partial}{\partial\varphi'}\varphi'+D_r\frac{\partial^2}{\partial r'^2}+D_\varphi\frac{\partial^2}{\partial\varphi'^2}\right]P(r',\varphi',t)\ , \tag{65}$$

where K_r and K_φ are the drift coefficients and D_r,D_φ the diffusion coefficients. Taking into account the relations

$$\langle\delta x_1^2\rangle=\tfrac{1}{4}+N_s\langle (r')^2\rangle\ ,\qquad \langle\delta x_2^2\rangle=\tfrac{1}{4}+N_s x_{\rm st}^2\langle(\varphi')^2\rangle\ , \tag{66}$$

where x_1 and x_2 have been defined in (63), one has easily from (65) that at steady state

$$\langle\delta x_1^2\rangle=\tfrac{1}{4}+N_s\frac{D_r}{K_r}\ ,\qquad \langle\delta x_2^2\rangle=\tfrac{1}{4}+N_s x_{\rm st}^2\frac{D_\varphi}{K_\varphi}\ . \tag{67}$$

The term $\frac{1}{4}$ comes from the fact that Eq. (65) is associated with the normal-ordering prescription. Now, the stability of the steady state requires that $K_r>0, K_\varphi>0$. Hence whenever $D_r<0$ ($D_\varphi<0$) one has squeezing in the component $x_1(x_2)$ [see Eq. (64)]. Furthermore, when the squeezing concerns the component x_1 there is also photon antibunching. In fact, as it is shown in Ref. 16, when the fluctuations are normal the second-order correlation function $g^{(2)}(0)$ is given by

$$g^{(2)}(0)=1+\frac{4}{N_s x_{\rm st}^2}(\langle\delta x_1^2\rangle-\tfrac{1}{4}) \;. \tag{68}$$

Hence when $\langle\delta x_1^2\rangle<\frac{1}{4}$ one has $g^{(2)}(0)<1$, i.e., there is antibunching. This situation occurs, e.g., in absorptive optical bistability.[16,30]

At this point, one can ask to what extent the nonpositive definiteness of the diffusion matrix in the Glauber-Sudarshan representation forces us to use the Wigner function. First of all, we observe that the two hierarchies of equations for the moments obtained from the two Fokker-Planck equations (with normal-ordering and symmetrical-ordering prescription, respectively) are perfectly equivalent and correct. However, if one wants to obtain the quasiprobability distribution one must use the Wigner function formulation, because the Glauber-Sudarshan P function does not exist, in general, when nonclassical effects are present. One can use the Glauber-Sudarshan function formulation only by adopting the formalism of the complex P representation, devised by Drummond and Gardiner.[57] This method has been proven useful in a number of examples in which it leads to an exact solution to a nonlinear Fokker-Planck equation whenever the detailed balance condition holds.[58] On the other hand, in other situations as the ones considered in this paper, the Wigner function formalism is more suitable because it gives directly the quasiprobability distribution. Using the complex P representation, the distribution itself is complex and one must calculate the moments at the cost of integrations along suitable contours in the complex plane.

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APPENDIX

We quote the expression of the quantities ϕ_i ($i=1,2,3$) appearing on the rhs of Eqs. (35). One finds

$$\phi_1=\langle\delta\hat{x}\delta P^-\rangle+2C(\beta_1-x_{\rm st}\beta_2) \;, \tag{A1a}$$

$$\phi_2=\langle\delta\hat{x}^\dagger\delta P^-\rangle+2C\beta_3 \;, \tag{A1b}$$

$$\phi_3=\langle\delta\hat{x}\delta P_3\rangle+2C\beta_2 \;, \tag{A1c}$$

where

$$\beta_1=\frac{1}{1+x_{\rm st}^2}\langle\delta\hat{x}\delta P^-\rangle \;, \tag{A2a}$$

$$\beta_2=\frac{1}{1+f(2+x_{\rm st}^2)}\{x_{\rm st}[f(\alpha_1+\beta_1)-\alpha_2+(1+f)\alpha_4]+\alpha_3\} \;, \tag{A2b}$$

$$\beta_3=\alpha_1+\alpha_4 \;, \tag{A2c}$$

and

$$\alpha_1=\frac{1}{2(1+x_{\rm st}^2)}(\langle\delta\hat{x}\delta P^+\rangle+\langle\delta\hat{x}^\dagger\delta P^-\rangle) \;, \tag{A3a}$$

$$\alpha_2=\frac{x_{\rm st}}{2(1+x_{\rm st}^2)}(\langle\delta\hat{x}\delta P_3\rangle+\langle\delta\hat{x}^\dagger\delta P_3\rangle) \;, \tag{A3b}$$

$$\alpha_3=\frac{1}{1+x_{\rm st}^2}[\langle\delta\hat{x}\delta P_3\rangle+f x_{\rm st}(\langle\delta\hat{x}\delta P^-\rangle+\langle\delta\hat{x}^\dagger\delta P^-\rangle)] \;, \tag{A3c}$$

$$\alpha_4=\frac{x_{\rm st}}{2(1+2f)(1+x_{\rm st}^2)}[2x_{\rm st}\alpha_2-(\alpha_3+\alpha_3^*)-f x_{\rm st}(2\alpha_1+\beta_1+\beta_1^*)] \;. \tag{A3d}$$

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