Angular momentum of the quantized electromagnetic field with periodic boundary conditions

D. Lenstra and L. Mandel
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 22 February 1982)

When an electromagnetic field is quantized within a box of side L with periodic boundary conditions, the total angular momentum $\hat{\vec{J}}$ is not strictly a constant of the motion even when $L\to\infty$. As a result, the conditions for isotropy of such a field involve subtle differences from the usual conditions. The same constraints apply to the orbital part of $\hat{\vec{J}}$, but not to the spin part. The expectation value of $\hat{\vec{J}}$ for any monochromatic plane wave is shown to vanish. Commutation relations are derived between $\hat{\vec{J}}$ and an arbitrary field vector, which show explicitly how $\hat{\vec{J}}$ generates rotation.

I. INTRODUCTION

The angular momentum of a quantized, free electromagnetic field is an important dynamical variable.[1] It acts as the generator of rotations[2] and, within the framework of optical coherence theory, it provides a simple test for the isotropy of the field.[3] Moreover, the total angular momentum of the free field is normally taken to be a constant of the motion.[4]

However, when the electromagnetic field is quantized within a box of side L with periodic boundary conditions, as is frequently the case, the usual definition leads to certain difficulties. These difficulties stem from the fact that the quantization imposes preferred directions in space, which persist even after the normalization volume is allowed to grow to infinity. As a result, the total angular momentum operator of the free field is no longer a constant of the motion in the strict sense, and various transformation properties are affected also. As the use of periodic boundary conditions is so common in problems in quantum optics, we thought it might be worthwhile to summarize the special features that the quantized angular momentum acquires under those conditions. For example, we shall show that, even though the angular momentum may not be a constant of the motion, its rate of change is reducible to normally ordered operators at the boundary of the quantization volume, whose expectations will of course be very small for an excitation that is localized well inside the volume. Similar remarks apply to the orbital part of the angular momentum, whereas the spin part is strictly a constant of the motion for a free field. We derive commutation relations between the total angular momentum and an arbitrary field vector, and use it to obtain the condition for isotropy of the field. The constraints that prevent the total angular momentum from being strictly constant do not, however, prevent the field within a cube of side L from being isotropic as $L\to\infty$. We illustrate this by reference to blackbody radiation.

II. DEFINITIONS AND RATE OF CHANGE OF ANGULAR MOMENTUM

We consider a free, quantized electromagnetic field defined within a cube of side L, with periodic boundary conditions. The quantized field vectors[5] $\hat{\vec{E}}(\vec{r},t)$ and $\hat{\vec{B}}(\vec{r},t)$ obey Maxwell's equations

$$\begin{aligned}
&\vec{\nabla}\cdot\hat{\vec{E}}(\vec{r},t)=0\ ,\\
&\vec{\nabla}\cdot\hat{\vec{B}}(\vec{r},t)=0\ ,\\
&\vec{\nabla}\times\hat{\vec{E}}(\vec{r},t)=-\frac{\partial\hat{\vec{B}}(\vec{r},t)}{\partial t}\ ,\\
&\vec{\nabla}\times\hat{\vec{B}}(\vec{r},t)=(1/c^2)\frac{\partial\hat{\vec{E}}(\vec{r},t)}{\partial t}\ ,
\end{aligned}\tag{1}$$

and they can be given expansions in plane-wave modes in the form

$$\hat{\vec{E}}(\vec{r},t)=\frac{1}{L^{3/2}}\sum_{\vec{k},s}\left[\frac{\hbar\omega}{2\epsilon_0}\right]^{1/2}\{i\hat{a}_{\vec{k},s}\vec{\epsilon}_{\vec{k},s}\exp[i(\vec{k}\cdot\vec{r}-\omega t)]+\text{H.c}\}\ , \tag{2}$$

$$\hat{\vec{B}}(\vec{r},t)=\frac{1}{L^{3/2}}\sum_{\vec{k},s}\left[\frac{\hbar}{2\omega\epsilon_0}\right]^{1/2}\{i\hat{a}_{\vec{k},s}(\vec{k}\times\vec{\epsilon}_{\vec{k},s})\exp[i(\vec{k}\cdot\vec{r}-\omega t)]+\text{H.c.}\}\ . \tag{3}$$

Here, $\vec{k}$ is the wave vector with components $2\pi n_1/L$, $2\pi n_2/L$, $2\pi n_3/L$ $(n_1,n_2,n_3=0,\pm1,\pm2,\ldots)$, and $\omega=ck$. $\vec{\epsilon}_{\vec{k},s}$ $(s=1,2)$ describes a pair of orthogonal, transverse, possibly complex, unit-polarization vectors, defined up to a unitary transformation by

$$\begin{aligned}&\vec{k}\cdot\vec{\epsilon}_{\vec{k},s}=0\ ,\\&\vec{\epsilon}^{\,*}_{\vec{k},s}\cdot\vec{\epsilon}_{\vec{k},s'}=\delta_{ss'}\ ,\\&\vec{\epsilon}_{\vec{k},1}\times\vec{\epsilon}_{\vec{k},2}=\vec{k}/k\equiv\vec{\kappa}\ .\end{aligned} \tag{4}$$

$\hat{a}_{\vec{k},s}, \hat{a}^\dagger_{\vec{k},s}$ are photon annihilation and creation operators, obeying the usual commutation rules. The total angular momentum operator $\hat{\vec{J}}(\vec{r}_0,t)$ with respect to point $\vec{r}_0$ is defined as the integral of the angular momentum density, just as in classical electromagnetic theory, except that the operator has to be symmetrized to make it Hermitian. We then have

$$\hat{\vec{J}}(\vec{r}_0,t)=\tfrac{1}{2}\epsilon_0\int_{L^3}d^3x(\vec{r}-\vec{r}_0)\times[\hat{\vec{E}}(\vec{r},t)\times\hat{\vec{B}}(\vec{r},t)-\hat{\vec{B}}(\vec{r},t)\times\hat{\vec{E}}(\vec{r},t)]\ . \tag{5}$$

It follows immediately from the definition that

$$\begin{aligned}\hat{\vec{J}}(\vec{r}_0,t)&=\hat{\vec{J}}(0,t)-\vec{r}_0\times\tfrac{1}{2}\epsilon_0\int_{L^3}d^3x[\hat{\vec{E}}(\vec{r},t)\times\hat{\vec{B}}(\vec{r},t)-\hat{\vec{B}}(\vec{r},t)\times\hat{\vec{E}}(\vec{r},t)]\\&=\hat{\vec{J}}(0,t)-\vec{r}_0\times\hat{\vec{P}}\ ,\end{aligned} \tag{6}$$

where $\hat{\vec{P}}$ is the total momentum operator of the field. From the mode expansions for $\hat{\vec{E}}(\vec{r},t)$ and $\hat{\vec{B}}(\vec{r},t)$ we readily find that $\hat{\vec{P}}$ is expressible in the form

$$\hat{\vec{P}}=\sum_{\vec{k},s}\hbar\vec{k}\hat{n}_{\vec{k},s}\ , \tag{7}$$

where $\hat{n}_{\vec{k}s}=\hat{a}^\dagger_{\vec{k},s}\hat{a}_{\vec{k},s}$ is the photon number operator. This shows explicitly that $\hat{\vec{P}}$ is a constant of the motion, because $\hat{n}_{\vec{k},s}$ is a constant of the motion. $\hat{\vec{J}}(\vec{r}_0,t)$ therefore changes in time only if $\hat{\vec{J}}(0,t)$ changes in time.

From the definition of $\hat{\vec{J}}(0,t)$, and with the help of the third and fourth Maxwell equations (1), we immediately have

$$\begin{aligned}\frac{\partial\hat{\vec{J}}(\vec{r}_0,t)}{\partial t}&=\frac{\partial\hat{\vec{J}}(0,t)}{\partial t}\\&=\tfrac{1}{2}\epsilon_0\int_{L^3}d^3x\,\vec{r}\times\left[\frac{\partial\hat{\vec{E}}(\vec{r},t)}{\partial t}\times\hat{\vec{B}}(\vec{r},t)+\hat{\vec{E}}(\vec{r},t)\times\frac{\partial\hat{\vec{B}}(\vec{r},t)}{\partial t}+\text{H.c.}\right]\\&=-\int_{L^3}d^3x\,\vec{r}\times\{\epsilon_0\hat{\vec{E}}(\vec{r},t)\times[\vec{\nabla}\times\hat{\vec{E}}(\vec{r},t)]+(1/\mu_0)\hat{\vec{B}}(\vec{r},t)\times[\vec{\nabla}\times\hat{\vec{B}}(\vec{r},t)]\}\ .\end{aligned} \tag{8}$$

In writing the last expression we have used the fact that all equal-time commutators of $\hat{\vec{E}}$ with $\hat{\vec{E}}$ and $\hat{\vec{B}}$ with $\hat{\vec{B}}$ vanish. We expand the vector triple products under the integral by writing

$$\hat{\vec{E}}\times(\vec{\nabla}\times\hat{\vec{E}})=\hat{E}_i\vec{\nabla}\hat{E}_i-\hat{\vec{E}}\cdot\vec{\nabla}\hat{\vec{E}}=\tfrac{1}{2}\vec{\nabla}\hat{\vec{E}}^2-\vec{\nabla}\cdot\hat{\vec{E}}\hat{\vec{E}}\ ,$$

since $\vec{\nabla}\cdot\hat{\vec{E}}=0$, so that

$$\vec{r}\times[\hat{\vec{E}}\times(\vec{\nabla}\times\hat{\vec{E}})]=\tfrac{1}{2}(\vec{r}\times\vec{\nabla})\hat{\vec{E}}^2-\vec{r}\times(\vec{\nabla}\cdot\hat{\vec{E}}\hat{\vec{E}})=-\tfrac{1}{2}\vec{\nabla}\times(\vec{r}\hat{\vec{E}}^2)-\vec{\nabla}\cdot(\hat{\vec{E}}\,\vec{r}\times\hat{\vec{E}})\ , \tag{9}$$

and similarly for the term in $\hat{\vec{B}}$. We now insert this under the integral in Eq. (8), and apply the generalized Gauss theorem to convert the volume integral to an integral over the surface of the cube L^3. We then obtain

$$\frac{\partial \hat{\vec{J}}(\vec{r}_0,t)}{\partial t}=\frac{1}{2}\oint_{L^3} \vec{d}S\times\vec{r}\left[\epsilon_0\hat{\vec{E}}^2(\vec{r},t)+\frac{1}{\mu_0}\hat{\vec{B}}^2(\vec{r},t)\right]$$
$$+\oint_{L^3}\vec{d}S\cdot\left[\epsilon_0\hat{\vec{E}}(\vec{r},t)\vec{r}\times\hat{\vec{E}}(\vec{r},t)+\frac{1}{\mu_0}\hat{\vec{B}}(\vec{r},t)\vec{r}\times\hat{\vec{B}}(\vec{r},t)\right]. \tag{10}$$

The first surface integral vanishes, as is obvious if we combine contributions from surface elements at $(\frac{1}{2}L,y,z)$ and $(-\frac{1}{2}L,y,z)$, for example. The vector $\vec{d}S\times\vec{r}$ points in opposite directions at these two surface elements, whereas $\hat{\vec{E}}(\frac{1}{2}L,y,z)=\hat{\vec{E}}(-\frac{1}{2}L,y,z)$, and similarly for $\hat{\vec{B}}$. We are therefore left with the second integral, and the relation can be expressed in component form

$$\frac{\partial \hat{J}_l(\vec{r}_0,t)}{\partial t}=\in_{lmn}\oint_{L^3} dS_p r_m\left[\epsilon_0\hat{E}_n\hat{E}_p+\frac{1}{\mu_0}\hat{B}_n\hat{B}_p\right], \tag{11}$$

with summation over repeated indices understood. The terms with $m\neq p$ also vanish in pairs at the boundary as before, and only those with $m=p$ remain. Because of the presence of the antisymmetric tensor $\in_{lmn}$, the only nonvanishing contributions must therefore come from terms with $n\neq p$.

We may decompose the Hermitian operators $\hat{\vec{E}}$ and $\hat{\vec{B}}$ into their positive and negative frequency parts by using the expansions (2) and (3), so that

$$\hat{\vec{E}}(\vec{r},t)=\hat{\vec{E}}^{(+)}(\vec{r},t)+\hat{\vec{E}}^{(-)}(\vec{r},t),$$

where $\hat{\vec{E}}^{(+)}(\vec{r},t)$ and $\hat{\vec{E}}^{(-)}(\vec{r},t)$ contain only annihilation and creation operators, respectively. Hence,

$$\hat{E}_n\hat{E}_p=\hat{E}_n^{(+)}\hat{E}_p^{(+)}+\hat{E}_n^{(-)}\hat{E}_p^{(-)}+\hat{E}_n^{(-)}\hat{E}_p^{(+)}+\hat{E}_n^{(+)}\hat{E}_p^{(-)}.$$

Of the four terms on the right, the first three are in normal order but the last one is not. However, with the help of the mode expansion (2) we can readily show that $\hat{E}_n^{(+)}$ commutes with $\hat{E}_p^{(-)}$ when $n\neq p$, so that the order of $\hat{E}_n^{(+)}\hat{E}_p^{(-)}$ can be inverted. A similar argument also applies to the magnetic fields, and we have finally

$$\frac{\partial \hat{J}_l(\vec{r}_0,t)}{\partial t}=\in_{lmn}\oint_{L^3} dS_p r_m\left[\epsilon_0(\hat{E}_n^{(+)}\hat{E}_p^{(+)}+\hat{E}_n^{(-)}\hat{E}_p^{(-)}+\hat{E}_n^{(-)}\hat{E}_p^{(+)}+\hat{E}_p^{(-)}\hat{E}_n^{(+)})\right.$$
$$\left.+\frac{1}{\mu_0}(\hat{B}_n^{(+)}\hat{B}_p^{(+)}+\hat{B}_n^{(-)}\hat{B}_p^{(-)}+\hat{B}_n^{(-)}\hat{B}_p^{(+)}+\hat{B}_p^{(-)}\hat{B}_n^{(+)})\right]. \tag{12}$$

We have therefore reduced the rate of change of the total angular momentum to a surface integral over the boundary involving only normally ordered operators.

It is clear from this relation that between any states $|\psi_1\rangle$, $|\psi_2\rangle$ of the field for which

$$\hat{\vec{E}}^{(+)}(\vec{r},t)|\psi_i\rangle=0,\quad \hat{\vec{B}}^{(+)}(\vec{r},t)|\psi_i\rangle=0,\quad i=1,2$$

when $\vec{r}$ lies on the boundary, the matrix element

$$\left\langle\psi_2\left|\frac{\partial \hat{J}_l(\vec{r}_0,t)}{\partial t}\right|\psi_1\right\rangle=0. \tag{13}$$

The states $|\psi_1\rangle$, $|\psi_2\rangle$ have the property that the excitations of the field are localized, at least in an approximate sense, far from the boundary and they correspond roughly to a classical field that vanishes at the boundary. It is in this limited sense that the total angular momentum $\hat{\vec{J}}(\vec{r}_0)$ of the field may be regarded as a constant of the motion.

III. DECOMPOSITION OF $\hat{\vec{J}}(\vec{r}_0)$ INTO SPIN AND ORBITAL PARTS

It is well known from classical electromagnetic theory that the total angular momentum $\vec{J}(\vec{r}_0)$ may be decomposed into the sum of two parts, one of which depends on $\vec{r}_0$, whereas the other one does not. We have, in general,[1,2]

$$\epsilon_0 \int_{L^3} d^3x(\vec{r}-\vec{r}_0)\times[\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)]=\epsilon_0\int_{L^3} d^3x\, E_i(\vec{r},t)[(\vec{r}-\vec{r}_0)\times\vec{\nabla}]A_i(\vec{r},t)$$
$$-\epsilon_0\oint_{L^3} d\vec{S}\cdot\vec{E}(\vec{r},t)(\vec{r}-\vec{r}_0)\times\vec{A}(\vec{r},t)$$
$$+\epsilon_0\int_{L^3} d^3x\,\vec{E}(\vec{r},t)\times\vec{A}(\vec{r},t)\,, \tag{14}$$

in which $\vec{A}(\vec{r},t)$ is the vector potential. The first part, which depends on $\vec{r}_0$, is interpreted as the orbital part $\vec{L}$ of the total angular momentum, and the second which does not contain $\vec{r}_0$, as the intrinsic or spin part $\vec{S}$. When the field is quantized the same relation holds, and we can define Hermitian operators $\hat{\vec{L}}$ and $\hat{\vec{S}}$ by symmetrizing

$$\hat{\vec{L}}=\tfrac{1}{2}\epsilon_0\int_{L^3} d^3x\{\hat{E}_i(\vec{r},t)[(\vec{r}-\vec{r}_0)\times\vec{\nabla}]\hat{A}_i(\vec{r},t)+[(\vec{r}-\vec{r}_0)\times\vec{\nabla}\hat{A}_i(\vec{r},t)]\hat{E}_i(\vec{r},t)\}$$
$$-\tfrac{1}{2}\epsilon_0\oint_{L^3}[d\vec{S}\cdot\hat{\vec{E}}(\vec{r},t)(\vec{r}-\vec{r}_0)\times\hat{\vec{A}}(\vec{r},t)+(\vec{r}-\vec{r}_0)\times\hat{\vec{A}}(\vec{r},t)\hat{\vec{E}}(\vec{r},t)\cdot d\vec{S}]\,, \tag{15}$$

$$\hat{\vec{S}}=\tfrac{1}{2}\epsilon_0\int_{L^3} d^3x[\hat{\vec{E}}(\vec{r},t)\times\hat{\vec{A}}(\vec{r},t)-\hat{\vec{A}}(\vec{r},t)\times\hat{\vec{E}}(\vec{r},t)]\,, \tag{16}$$

with

$$\hat{\vec{J}}=\hat{\vec{L}}+\hat{\vec{S}}\,. \tag{17}$$

The contribution of the surface integral over the boundary in Eq. (15) can be written in normal order, as before.

Let us examine $\hat{\vec{S}}$ in a little more detail. By making use of the mode expansions[6] for $\hat{\vec{E}}(\vec{r},t)$ and $\hat{\vec{A}}(\vec{r},t)$ we readily obtain from Eq. (16), after integrating over the volume L^3,

$$\hat{\vec{S}}=i\hbar\sum_{\vec{k}}\sum_{s,s'}(\hat{a}^\dagger_{\vec{k},s'}\hat{a}_{\vec{k},s}+\tfrac{1}{2}\delta_{ss'})(\vec{\epsilon}_{\vec{k},s}\times\vec{\epsilon}^{\,*}_{\vec{k},s'})\,. \tag{18}$$

This is a general expression, in which the unit-polarization vectors $\vec{\epsilon}_{\vec{k},s}$ may be chosen arbitrarily, subject to the conditions (4). If we choose $\vec{\epsilon}_{\vec{k},1}$ and $\vec{\epsilon}_{\vec{k},-1}$ to represent orthogonal states of right and left circular polarization, then

$$\vec{\epsilon}_{\vec{k},s}\times\vec{\epsilon}^{\,*}_{\vec{k},s'}=-is\vec{\kappa}\delta_{ss'}\quad(s,s'=\pm1) \tag{19}$$

where $\vec{\kappa}\equiv\vec{k}/k$. This choice of $\vec{\epsilon}_{\vec{k},s}$ immediately allows Eq. (18) to be reduced to the simpler form

$$\hat{\vec{S}}=\sum_{\vec{k}}\hbar\vec{\kappa}(\hat{n}_{\vec{k},1}-\hat{n}_{\vec{k},-1})\,, \tag{20}$$

which is diagonal in photon number states. The spin angular momentum is therefore expressible in terms of the difference between the number of right and left circularly polarized photons, and it is strictly a constant of the motion since $\hat{n}_{\vec{k},s}$ is also. As

$$\hat{\vec{L}}=\hat{\vec{J}}-\hat{\vec{S}}\,,$$

it follows that the orbital angular momentum $\hat{\vec{L}}$ is a constant of the motion only in the weaker sense implied by Eq. (13).

IV. TOTAL ANGULAR MOMENTUM OF A MONOCHROMATIC PLANE WAVE

We consider a state in which the only occupied modes are of the type $\vec{k},s=1$ and $\vec{k},s=2$. This corresponds to a plane wave with wave vector $\vec{k}$, but of arbitrary polarization. If the state is pure, it must be expressible as a linear superposition of Fock states of the form

$$|n_{\vec{k},1},n_{\vec{k},2},\{0\}\rangle\,,$$

where the $\{0\}$ signifies that all other modes are empty. If the modes $\vec{k},1$ and $\vec{k},2$ happen to correspond to right and left circular polarizations, it follows from Eq. (20) that the expectation value of $\hat{\vec{S}}$ in the state $|n_{\vec{k},1},n_{\vec{k},2},\{0\}\rangle$ is given by

$$\langle\{0\},n_{\vec{k},2},n_{\vec{k},1}|\hat{\vec{S}}|n_{\vec{k},1},n_{\vec{k},2},\{0\}\rangle$$
$$=\hbar\vec{\kappa}(n_{\vec{k},1}-n_{\vec{k},2})\,, \tag{21}$$

and this points in the direction of $\vec{k}$ and is nonzero if $n_{\vec{k},1} \neq n_{\vec{k},2}$. However, the expectation of the $\vec{k}$ component of the total angular momentum $\hat{\vec{J}}$ in the same state is always zero. Indeed, it vanishes in any plane-wave state that is expressible as a linear superposition of states of the type $|n_{\vec{k},1}, n_{\vec{k},2}, \{0\}\rangle$. In order to show this we now calculate the matrix element of $\hat{\vec{J}}(\vec{r}_0,t)$ between two such states.

With the help of Eq. (6) and the mode expansions (2) and (3) we have

$$
\begin{aligned}
&\langle \{0\}, n'_{\vec{k},2}, n'_{\vec{k},1} | \hat{J}_l(\vec{r}_0,t) | n_{\vec{k},1}, n_{\vec{k},2}, \{0\}\rangle \\
&\quad = -\vec{r}_0 \times \langle \{0\}, n'_{\vec{k},2}, n'_{\vec{k},1} | \hat{\vec{P}} | n_{\vec{k},1}, n_{\vec{k},2}, \{0\}\rangle \\
&\quad + \frac{1}{L^3} \sum_{\vec{k}',s'} \sum_{\vec{k}'',s''} \left[\frac{\hbar\omega'}{2}\right]^{1/2} \left[\frac{\hbar}{2\omega''}\right]^{1/2} \\
&\in_{lmn} \Big[\langle \{0\}, n'_{\vec{k},2}, n'_{\vec{k},1} | \hat{a}_{\vec{k}',s'} \hat{a}_{\vec{k}'',s''} | n_{\vec{k},1}, n_{\vec{k},2}, \{0\}\rangle \\
&\qquad \times [\vec{\epsilon}_{\vec{k}',s'} \times (\vec{k}'' \times \vec{\epsilon}_{\vec{k}'',s''})]_n \int_{L^3} d^3x\, r_m \exp\{i[(\vec{k}'+\vec{k}'')\cdot\vec{r} - (\omega'+\omega'')t]\} + \text{c.c.} \\
&\qquad - \tfrac{1}{2} \langle \{0\}, n'_{\vec{k},2}, n'_{\vec{k},1} | \hat{a}_{\vec{k}',s'} \hat{a}^{\dagger}_{\vec{k}'',s''} + \hat{a}^{\dagger}_{\vec{k}'',s''} \hat{a}_{\vec{k}',s'} | n_{\vec{k},1}, n_{\vec{k},2}, \{0\}\rangle\, [\vec{\epsilon}_{\vec{k}',s'} \times (\vec{k}'' \times \vec{\epsilon}_{\vec{k}'',s''})]_n \\
&\qquad \times \int_{L^3} d^3x\, r_m \exp\{i[(\vec{k}'-\vec{k}'')\cdot\vec{r} - (\omega'-\omega'')t]\} + \text{c.c.} \Big] .
\end{aligned}
$$

The expression (7) for $\hat{\vec{P}}$ allows us to evaluate the first term immediately. In the remaining terms we arrange the operators in normal order, whereupon it is apparent that the only nonvanishing contributions to the double sums come from terms with $\vec{k}' = \vec{k}'' = \vec{k}$. We also observe that the last integral vanishes by symmetry when $\vec{k}' = \vec{k}''$, whereas the first integral reduces to the form

$$\int_{L^3} d^3x (\vec{r} \times \vec{k}) \exp[2i(\vec{k}\cdot\vec{r} - \omega t)] ,$$

and this also vanishes by symmetry. We finally obtain

$$
\begin{aligned}
&\langle \{0\}, n'_{\vec{k},2}, n'_{\vec{k},1} | \hat{\vec{J}}(\vec{r}_0,t) | n_{\vec{k},1}, n_{\vec{k},2}, \{0\}\rangle \\
&\quad = -(\vec{r}_0 \times \hbar\vec{k})(n_{\vec{k},1} + n_{\vec{k},2}) \delta_{n_{\vec{k},1} n'_{\vec{k},1}} \delta_{n_{\vec{k},2} n'_{\vec{k},2}} ,
\end{aligned}
\tag{22}
$$

and it follows from this that the corresponding matrix element for the $\vec{k}$ component of $\hat{\vec{J}}(\vec{r}_0,t)$ is always zero. The expectation value of $\vec{k}\cdot\hat{\vec{J}}(\vec{r}_0,t)$ in any monochromatic plane-wave state, therefore, must vanish also.

This result may seem surprising at first because of the possible nonzero expectation of $\vec{k}\cdot\hat{\vec{S}}$ given by Eq. (21), but too much physical significance should not be attached to it. Any measurement of the angular momentum with a detector of finite size necessarily disturbs the field in such a way that angular momentum can be absorbed from the field and transferred to the detector. This point has been discussed in some detail in Ref. 1.

V. COMMUTATION RELATIONS BETWEEN $\hat{\vec{J}}(\vec{r}_0,t)$ AND FIELD VECTORS

In a free field, any one of the important electromagnetic fields, such as $\hat{\vec{E}}(\vec{r},t)$, $\hat{\vec{B}}(\vec{r},t)$, $\hat{\vec{A}}(\vec{r},t)$, can be given a mode expansion of the general form

$$\hat{\vec{F}}(\vec{r},t) = \frac{1}{L^{3/2}} \sum_{\vec{k},s} f(\omega) \hat{a}_{\vec{k},s} \vec{\epsilon}_{\vec{k},s} \exp[i(\vec{k}\cdot\vec{r} - \omega t)] + \text{H.c.} \equiv \hat{\vec{V}}(\vec{r},t) + \hat{\vec{V}}^{\dagger}(\vec{r},t) . \tag{23}$$

Here $\hat{\vec{F}}$ stands for any one of the vectors, $f(\omega)$ is some slowly varying function of $\omega = ck$, and $\hat{\vec{V}}(\vec{r},t)$ is the positive frequency part of $\hat{\vec{F}}(\vec{r},t)$. We shall now establish the commutation relation between $\hat{\vec{J}}(\vec{r}_0,t_0)$ and $\hat{\vec{V}}(\vec{r},t)$.

We start by evaluating certain c-number commutators that are needed below. From the mode expansions and the commutation relations between $\hat{a}_{\vec{k},s}$ and $\hat{a}^{\dagger}_{\vec{k}',s'}$ we have immediately

$$\begin{aligned}
[\hat{V}_m(\vec{r},t),\hat{E}_n(\vec{r}',t_0)]
&=\frac{-i}{L^3}\sum_{\vec{k},s}\sum_{\vec{k}',s'} f(\omega)\left[\frac{\hbar\omega'}{2\epsilon_0}\right]^{1/2}[\hat{a}_{\vec{k},s},\hat{a}^{\dagger}_{\vec{k}',s'}](\vec{\epsilon}_{\vec{k},s})_m(\vec{\epsilon}^{\,*}_{\vec{k}',s'})_n\exp[i(\vec{k}\cdot\vec{r}-\vec{k}'\cdot\vec{r}'-\omega t+\omega' t_0)]\\
&=\frac{-i}{L^3}\sum_{\vec{k},s} f(\omega)\left[\frac{\hbar\omega}{2\epsilon_0}\right]^{1/2}(\vec{\epsilon}_{k,s})_m(\vec{\epsilon}^{\,*}_{\vec{k},s})_n\exp\{i[\vec{k}\cdot(\vec{r}-\vec{r}')-\omega(t-t_0)]\}\\
&=\frac{-i}{L^3}\sum_{\vec{k}} f(\omega)\left[\frac{\hbar\omega}{2\epsilon_0}\right]^{1/2}(\delta_{mn}-\kappa_m\kappa_n)\exp\{i[\vec{k}\cdot(\vec{r}-\vec{r}')-\omega(t-t_0)]\}\ ,
\end{aligned}\tag{24}$$

after making use of the tensor relation

$$\sum_s(\vec{\epsilon}_{k,s})_m(\vec{\epsilon}^{\,*}_{\vec{k},s})_n=\delta_{mn}-\kappa_m\kappa_n\ .\tag{25}$$

Similarly we may show that

$$[\hat{V}_m(\vec{r},t),\hat{B}_n(\vec{r}',t_0)]=\frac{-i}{L^3c}\sum_{\vec{k}} f(\omega)\left[\frac{\hbar\omega}{2\epsilon_0}\right]^{1/2}\in_{mnp}\kappa_p\exp\{i[\vec{k}\cdot(\vec{r}-\vec{r}')-\omega(t-t_0)]\}\ .\tag{26}$$

We now apply these results to the calculation of the commutator of $\hat{\vec{V}}(\vec{r},t)$ with $\hat{\vec{J}}(\vec{r}_0,t_0)$. By making use of the commutation relations between $\hat{\vec{E}}$ and $\hat{\vec{B}}$ vectors, we first express $\hat{\vec{J}}(\vec{r}_0,t_0)$ given by Eq. (5) in the form

$$\hat{J}_n(\vec{r}_0,t_0)=\epsilon_0\int_{L^3}d^3r'(r_l'-r_{0l})[\hat{E}_n(\vec{r}',t_0)\hat{B}_l(\vec{r}',t_0)-\hat{E}_l(\vec{r}',t_0)\hat{B}_n(\vec{r}',t_0)]+C\ ,\tag{27}$$

where the symbol C represents a c number. We then have from Eqs. (24), (26), and (27), with the help of the mode expansions (2) and (3),

$$\begin{aligned}
[\hat{V}_m(\vec{r},t),\hat{J}_n(\vec{r}_0,t_0)]
&=\frac{-i\hbar}{2cL^{5/2}}\sum_{\vec{k}}\sum_{\vec{k}',s'} f(\omega)(\omega\omega')^{1/2}\int_{L^3}d^3r'(r_l'-r_{0l})\exp\{i[\vec{k}\cdot(\vec{r}-\vec{r}')-\omega(t-t_0)]\}\\
&\qquad\times\{\hat{a}_{\vec{k}'s'}\exp[i(\vec{k}'\cdot\vec{r}'-\omega' t_0)]+\hat{a}^{\dagger}_{\vec{k}'s'}\exp[-i(\vec{k}'\cdot\vec{r}'-\omega' t_0)]\}\\
&\qquad\times[(\delta_{mn}-\kappa_m\kappa_n)(\vec{\kappa}'\times\vec{\epsilon}_{\vec{k}',s'})_l+\in_{mlp}\kappa_p(\vec{\epsilon}_{\vec{k}',s'})_n\\
&\qquad-(\delta_{ml}-\kappa_m\kappa_l)(\vec{\kappa}'\times\vec{\epsilon}_{\vec{k}',s'})_n-\in_{mnp}\kappa_p(\vec{\epsilon}_{\vec{k}',s'})_l]\ .
\end{aligned}\tag{28}$$

For simplicity we have taken $\vec{\epsilon}_{\vec{k}',s'}$ to be real in this expansion.

The summations over $\vec{k}$ can be simplified. Each of the four volume integrals in Eq. (28) has the structure

$$\int_{L^3}d^3r'\sum_{\vec{k}}g(\vec{k})(r_l'-r_{0l})\exp\{i[\vec{k}\cdot(\vec{r}-\vec{r}')+\vec{k}'\cdot\vec{r}']\}\ ,$$

where $g(\vec{k})$ is of the form

$$g(\vec{k})=\sqrt{\omega}f(\omega)\exp[-i\omega(t-t_0)]F\ ,\tag{29}$$

where the factor F represents 1 or κ_p or $\kappa_m\kappa_n$ or $\kappa_m\kappa_l$. It can be shown (see Appendix A) that in all four cases

$$\int_{L^3}d^3r'\sum_{\vec{k}}g(\vec{k})(r_l'-r_{0l})\exp\{i[\vec{k}\cdot(\vec{r}-\vec{r}')+\vec{k}'\cdot\vec{r}']\}=L^3\exp(i\vec{k}'\cdot\vec{r})\left[-i\frac{\partial}{\partial k_l'}g(\vec{k}')+g(\vec{k}')(r_l-r_{0l})\right]\tag{30}$$

provided that $\vec{r}$ does not lie on the boundary of L^3. This allows us to simplify Eq. (28). After some rather lengthy, but essentially straightforward, manipulation and rearrangement of the remaining terms, and evaluation of the sum over $\vec{k}'$, s', we arrive at the result

$$[\hat{V}_m(\vec{r},t),\hat{J}_n(\vec{r}_0,t_0)]=i\hbar\left[\in_{mnp}\hat{V}_p(\vec{r},t)-\in_{nlp}(r_l-r_{0l})\frac{\partial}{\partial r_p}\hat{V}_m(\vec{r},t)\right] \tag{31}$$

provided that $\vec{r}$ does not lie on the boundary of the normalization volume. There is no corresponding restriction on $\vec{r}_0$. We shall now apply this result to a rotational transformation of $\hat{\vec{V}}(\vec{r},t)$.

VI. TOTAL ANGULAR MOMENTUM AND ISOTROPY OF THE FIELD

We consider the following infinitesimal unitary transformation of the vector operator $\hat{\vec{V}}(\vec{r},t)$ to some new operator:

$$\hat{\vec{V}}'=\exp[i\hat{\vec{J}}(\vec{r}_0,t_0)\cdot\delta\vec{\theta}/\hbar]\hat{\vec{V}}(\vec{r},t)\exp[-i\hat{\vec{J}}(\vec{r}_0,t_0)\cdot\delta\vec{\theta}/\hbar] , \tag{32}$$

where $\delta\vec{\theta}$ is a small vector angle. By expanding to the first order in $\delta\vec{\theta}$, we obtain

$$\hat{V}'_m=\hat{V}_m(\vec{r},t)+\frac{i}{\hbar}[\hat{J}_n(\vec{r}_0,t_0),\hat{V}_m(\vec{r},t)]\delta\theta_n ,$$

and with the help of Eq. (31), after making a cyclic permutation of the indices in the last term, we can write

$$\hat{V}'_m=\hat{V}_m(\vec{r},t)+\in_{mnp}\delta\theta_n\hat{V}_p(\vec{r},t)-\in_{pnl}\delta\theta_n(r_l-r_{0l})\frac{\partial}{\partial r_p}\hat{V}_m(\vec{r},t) .$$

Hence, to the first order in $\delta\vec{\theta}$, we have the vector relation

$$\hat{\vec{V}}'=\hat{\vec{V}}(\vec{r},t)+\delta\vec{\theta}\times\hat{\vec{V}}(\vec{r},t)-[\delta\vec{\theta}\times(\vec{r}-\vec{r}_0)]\cdot\vec{\nabla}\hat{\vec{V}}(\vec{r},t)=(1+\delta\vec{\theta}\times)\hat{\vec{V}}(\vec{r}-\delta\hat{\theta}\times(\vec{r}-\vec{r}_0),t) . \tag{33}$$

$\vec{V}'$ is a new vector operator that is obtained from the old one by rotating the position vector $\vec{r}$ that labels the operator backwards through $\delta\theta$ about the point $\vec{r}_0$ and then rotating the vector operator forward through $\delta\vec{\theta}$. This relation shows explicitly that the total angular momentum about $\vec{r}_0$ is the generator of rotation about $\vec{r}_0$.[2,3] Since the electric field $\hat{\vec{E}}$, the magnetic field $\hat{\vec{B}}$, the vector potential $\hat{\vec{A}}$, etc., are all expressible as linear combinations of an operator of the type $\hat{\vec{V}}$ with $\hat{\vec{V}}^\dagger$, it follows that a similar transformation law must hold for any of the field vectors. Equation (33) can also be rewritten in the symbolic form[3]

$$\hat{\vec{V}}'=R\hat{\vec{V}}(R^{-1}\vec{r},t) , \tag{34}$$

where R is the infinitesimal rotation matrix corresponding to the rotation $\delta\vec{\theta}$ about $\vec{r}_0$.

In the statistical description of fluctuating electromagnetic fields one often encounters correlation functions among $\hat{\vec{V}}$ and $\hat{\vec{V}}^\dagger$ operators, such as

$$\Gamma^{(1,1)}\equiv\langle\hat{V}_i^\dagger(\vec{r}_1,t_1)\hat{V}_j(\vec{r}_2,t_2)\rangle ,$$

$$\Gamma^{(2,2)}\equiv\langle\hat{V}_i^\dagger(\vec{r}_1,t_1)\hat{V}_j^\dagger(\vec{r}_2,t_2)\times\hat{V}_j(\vec{r}_2,t_2)\hat{V}_i(\vec{r}_1,t_1)\rangle ,$$

etc., which can be used to identify certain statistical symmetry properties of the field. For example, if all correlations are invariant under time translation, then the field is stationary in time, and if all correlations are invariant under space translation, then the field is spatially homogeneous. In a similar way, we may identify an electromagnetic field as being statistically isotropic about $\vec{r}_0$ if all correlations are invariant under rotation about $\vec{r}_0$, or, more explicitly, if

$$R_{i_1j_1}\cdots R_{i_mj_m}\langle\hat{V}_{j_1}^\dagger(R^{-1}\vec{r}_1,t_1)\cdots\hat{V}_{j_m}(R^{-1}\vec{r}_m,t_m)\rangle=\langle\hat{V}_{i_1}^\dagger(\vec{r}_1,t_1)\cdots\hat{V}_{i_m}(\vec{r}_m,t_m)\rangle\equiv\Gamma . \tag{35}$$

With the help of Eqs. (32) and (34) we can reexpress this isotropy condition for any infinitesimal rotation $\delta\vec{\theta}$

in the alternate form

$$\Gamma=\mathrm{Tr}\{\hat{\rho}\exp[i\hat{\vec{J}}(\vec{r}_0,t_0)\cdot\delta\vec{\theta}/\hbar]\hat{V}_{i_1}^{\dagger}(\vec{r}_1,t_1)$$
$$\times\exp[-i\hat{\vec{J}}(\vec{r}_0,t_0)\cdot\delta\vec{\theta}/\hbar]\cdots\exp[i\hat{\vec{J}}(\vec{r}_0,t_0)\cdot\delta\vec{\theta}/\hbar]\hat{V}_{i_m}(\vec{r}_m,t_m)\exp[-i\hat{\vec{J}}(\vec{r}_0,t_0)\cdot\delta\vec{\theta}/\hbar]\}\ ,$$

where $\hat{\rho}$ is the density operator of the free field. By making a cyclic permutation of the last unitary operator we obtain the condition

$$\Gamma=\mathrm{Tr}\{\exp[-i\hat{\vec{J}}(\vec{r}_0,t_0)\cdot\delta\vec{\theta}/\hbar]\hat{\rho}\exp[i\hat{\vec{J}}(\vec{r}_0,t_0)\cdot\delta\vec{\theta}/\hbar]\hat{V}_{i_1}^{\dagger}(\vec{r}_1,t_1)\cdots\hat{V}_{i_m}(\vec{r}_m,t_m)\}\tag{36}$$

and, if this relation is to be independent of $\delta\vec{\theta}$, we must have

$$[\hat{\rho},\hat{\vec{J}}(\vec{r}_0,t_0)]=0\tag{37}$$

for isotropy. The density operator, therefore, has to commute with the total angular momentum about $\vec{r}_0$ if the field is to be isotropic about $\vec{r}_0$.

In the special case in which the field is also spatially homogeneous, it must be isotropic about every point if it is isotropic about any one point. The reason is that

$$[\hat{\rho},\hat{\vec{P}}]=0\tag{38}$$

for a homogeneous field, and according to Eq. (6) the total angular momentum about one point $\vec{r}_1$ differs from the total angular momentum about another point $\vec{r}_2$ by $(\vec{r}_1-\vec{r}_2)\times\hat{\vec{P}}$. It follows that if $\hat{\rho}$ commutes with $\hat{\vec{P}}$ and $\hat{\vec{J}}(\vec{r}_1,t)$, it must also commute with $\hat{\vec{J}}(\vec{r}_2,t)$, etc. However, isotropy cannot be expected to hold in the strict sense when the field lies within a cube and obeys periodic boundary conditions.

An example of this difficulty is provided by blackbody radiation for which the density operator takes the form

$$\hat{\rho}=\exp(-\hat{H}/KT)/\mathrm{Tr}[\exp(-\hat{H}/KT)]$$
$$=\prod_{\vec{k},s}[1-\exp(-\hbar\omega/KT)]\exp(-\hat{n}_{\vec{k},s}\hbar\omega/KT)\ ,\tag{39}$$

where $\hat{H}$ is the total energy, K is Boltzmann's constant, and T is the absolute temperature. In this case it is clear by inspection that

$$[\hat{\rho},\hat{H}]=0=[\hat{\rho},\hat{\vec{P}}]\ ,\tag{40}$$

so that the field is both stationary and homogeneous. If $\hat{\vec{J}}$ were strictly a constant of the motion, so that

$$[\hat{\vec{J}}(\vec{r}_0),\hat{H}]=0\ ,$$

then from Eq. (39) we would have

$$[\hat{\rho},\hat{\vec{J}}(\vec{r}_0)]=0\ ,\tag{41}$$

and the radiation field would also be strictly isotropic about every point. However, as we have seen, $\hat{\vec{J}}$ is not a constant of the motion in the strict sense so that one would not expect the field to be strictly isotropic either. However, once the discrete set of modes is replaced by a continuum and sums over modes become integrals, the field can exhibit isotropy in the strict sense.

As an example, we consider the second rank correlation tensors of blackbody radiation.[7] From the mode expansions (2) and (3) and the form (39) of the density operator, one can readily show that

$$\langle\hat{E}_i^{(-)}(\vec{r}_1,t_1)\hat{E}_j^{(+)}(\vec{r}_2,t_2)\rangle=\frac{\hbar}{2\epsilon_0}\frac{1}{L^3}\sum_{\vec{k}}\frac{\omega(\delta_{ij}-k_ik_j/k^2)}{[\exp(\hbar\omega/KT)-1]}\exp\{i[\vec{k}\cdot(\vec{r}_2-\vec{r}_1)-\omega(t_2-t_1)]\}\ .\tag{42}$$

The stationarity and homogeneity of the field are exhibited by the fact that the correlation tensor depends only on the differences t_2-t_1 and $\vec{r}_2-\vec{r}_1$, respectively. However, the sum over $\vec{k}$ does not strictly satisfy Eq. (35), and the field is therefore not isotropic in the strict sense. But if we allow L to become infinite, and replace the sum by an integral, we obtain

$$\langle \hat{E}_i^{(-)}(\vec{r}_1,t_1)\hat{E}_j^{(+)}(\vec{r}_2,t_2)\rangle = \left[\frac{\hbar}{2\epsilon_0}\right]\frac{1}{(2\pi)^3}\int d^3k \frac{\omega(\delta_{ij}-k_ik_j/k^2)}{[\exp(\hbar\omega/KT)-1]}\exp\{i[\vec{k}\cdot(\vec{r}_2-\vec{r}_1)-\omega(t_2-t_1)]\} , \tag{43}$$

which does satisfy Eq. (35) exactly. The field is therefore isotropic, despite the fact that the density operator $\hat{\rho}$ does not commute with the total angular momentum in the strict sense. This illustrates the somewhat subtle incongruities that can result from the use of box quantization with periodic boundary conditions.

ACKNOWLEDGMENT

This work was supported by the National Science Foundation and by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

APPENDIX: PROOF OF AN INTEGRAL RELATION

We consider the functions encountered in Eq. (28),

$$G \equiv \sum_{\vec{k}} \int_{L^3} d^3r' g(\vec{k})(r_l'-r_{0l})\exp(i\vec{k}\cdot\vec{r})\exp[i(\vec{k}'-\vec{k})\cdot\vec{r}'] ,$$
$$= \sum_{\vec{k}} \int_{L^3} d^3r' g(\vec{k}) r_l' \exp\{i[\vec{k}\cdot\vec{r}+(\vec{k}'-\vec{k})\cdot\vec{r}']\} - L^3 r_{0l} g(\vec{k}')\exp(i\vec{k}'\cdot\vec{r}) , \tag{A1}$$

where $g(\vec{k})$ has the form

$$g(\vec{k}) = \sqrt{\omega} f(\omega)\exp[-i\omega(t-t_0)]F , \tag{A2}$$

where F represents 1 or κ_p or $\kappa_m\kappa_l$ or $\kappa_m\kappa_n$. For notational simplicity, and without loss of generality, we assume that the index l corresponds to the x component, and we integrate with respect to y' and z'. Then

$$G + L^3 x_0 g(\vec{k}')\exp(i\vec{k}'\cdot\vec{r}) = L^2 \sum_{k_x} g(k_x,k_y',k_z')\exp[i(k_y'y+k_z'z)] \times \int_{-(1/2)L}^{(1/2)L} dx' \exp(ik_x x)\exp[i(k_x'-k_x)x']x' . \tag{A3}$$

Before performing the integration, we expand $g(k_x,k_y',k_z')$ as a power series in k_x about the origin in the form

$$g(k_x,k_y',k_z') = \sum_{s=0}^{\infty} \frac{1}{s!} g^{(s)}(0,k_y',k_z')k_x^s . \tag{A4}$$

Then Eq. (A3) can be written

$$G + L^3 x_0 g(\vec{k}')\exp(i\vec{k}'\cdot\vec{r})$$
$$= L^2 \sum_{s=0}^{\infty} \frac{1}{s!} g^{(s)}(0,k_y',k_z')\exp[i(k_y'y+k_z'z)]\frac{1}{i^s}\left[\frac{d}{dx}\right]^s \int_{-(1/2)L}^{(1/2)L} dx' x' \exp(ik_x'x') \sum_{k_x}\exp[ik_x(x-x')]$$
$$= L^3 \sum_{s=0}^{\infty} \frac{1}{s!} g^{(s)}(0,k_y',k_z')\exp[i(k_y'y+k_z'z)]\frac{1}{i^s}\left[\frac{d}{dx}\right]^s \int_{-(1/2)L}^{(1/2)L} dx' x' \exp(ik_x'x') \sum_{N=-\infty}^{\infty} \delta(x-x'-NL)$$
$$= L^3 \sum_{s=0}^{\infty} \frac{1}{s!} g^{(s)}(0,k_y',k_z')\exp[i(k_y'y+k_z'z)]\frac{1}{i^s}\left[\frac{d}{dx}\right]^s x\exp(ik_x'x)$$
$$= L^3 \sum_{s=0}^{\infty} \frac{1}{s!} g^{(s)}(0,k_y',k_z')\exp[i(k_y'y+k_z'z)](xk_x'^{\,s} - isk_x'^{\,(s-1)})\exp(ik_x'x) . \tag{A5}$$

We have made use of the δ-function representation of the sum over k_x in the second line[8] and, subsequently, we have assumed that $x \neq \frac{1}{2}L$ in order to ensure that only the term $N=0$ makes a contribution. With the help of Eq. (A4) we now perform the summation over s, and obtain

$$G+L^3 x_0 g(\vec{k}')\exp(i\vec{k}'\cdot\vec{r})=L^3 x g(\vec{k}')\exp(i\vec{k}'\cdot\vec{r})-iL^3\frac{\partial g(\vec{k}')}{\partial k_x'}\exp(i\vec{k}'\cdot\vec{r})$$

or, in the original notation,

$$G=L^3\left[-i\frac{\partial g(\vec{k}')}{\partial k_l'}+(r_l-r_{0l})g(\vec{k})\right]\exp(i\vec{k}'\cdot\vec{r}) . \tag{A6}$$

This is the relation that was called Eq. (30) in the main text.

---

[1]For a detailed introductory discussion of angular momentum see, for example, J. W. Simmons and M. J. Guttmann, *States, Waves and Photons* (Addison-Wesley, Reading, Mass., 1970), Chap. 9 and Appendix VI.

[2]K. Gottfried, *Quantum Mechanics* (Benjamin, New York, 1966), Vol. I.

[3]K. Dialetis and C. L. Mehta, Nuovo Cimento 56, 89 (1968).

[4]F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965), Chap. 4.

[5]We identify Hilbert space operators of the quantized field by the caret.

[6]Since $\hat{\vec{E}}=-\partial\hat{\vec{A}}/\partial t$, the expansion for $\hat{\vec{A}}$ looks just like the expansion (2) for $\hat{\vec{E}}$, but with the coefficient $i(\hbar\omega/2\epsilon_0)^{1/2}$ replaced by $(\hbar/2\omega\epsilon_0)^{1/2}$.

[7]C. L. Mehta and E. Wolf, Phys. Rev. 134, A1149 (1964).

[8]See, for example, M. J. Lighthill, *Fourier Analysis and Generalized Functions* (Cambridge University Press, Cambridge, 1962), Sec. 5.4.