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Photon-number statistics in resonance fluorescence

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The theory of photon-number statistics in resonance fluorescence is treated, starting with the general formula for the emission probability of n photons during a given time interval T. The results fully confirm formerly obtained results by Cook that were based on the theory of atomic motion in a traveling wave. General expressions for the factorial moments are derived and explicit results for the mean and the variance are given. It is explicitly shown that the distribution function tends to a Gaussian when T becomes much larger than the natural lifetime of the excited atom. The speed of convergence towards the Gaussian is found to be typically slow, that is, the third normalized central moment (or the skewness) is proportional to $T^{-1/2}$. However, numerical results illustrate that the overall features of the distribution function are already well represented by a Gaussian when T is larger than a few natural lifetimes only, at least if the intensity of the exciting field is not too small and its detuning is not too large.

I. INTRODUCTION

Let $p(n,T)$ be the probability that n photons ($n=0,1,2,\ldots$) are emitted in a given time (the counting time T, $T>0$) by a two-level atom when the atom is driven by a coherent monochromatic electromagnetic field. The first attempt to calculate $p(n,T)$ was made by Mandel.[1] By starting from the general formula for $p(n,T)$, explicit expressions for the first few moments of the distribution function were derived, which confirmed that, due to the antibunching in time of the fluorescent photons,[2] sub-Poissonian statistics are to be expected. Furthermore, explicit expressions for the distribution function in case of resonant excitation were obtained in the limiting cases of very small and very large counting time (as compared to the natural lifetime of the excited atom). In the latter case the result is expressed in a rather cumbersome infinite series.

An important step forward towards the complete calculation of $p(n,T)$ was made by Cook,[3] who obtained a simple analytical expression for the Laplace transform (with respect to T) of the generating function

$$G(z,T)=\sum_{n=0}^{\infty} z^n p(n,T)\ , \tag{1}$$

which enabled him to derive explicit, though generally rather cumbersome, exact expressions for $p(n,T)$. However, Cook's derivation is based on the theory of atomic motion in a traveling wave[3–5] and it is therefore of interest to investigate whether the same results can be derived by starting with the general formula for the probability that n photons are emitted, without introducing atomic motion. Indeed, we will present here such a derivation which has the advantage of being more directly related to the theory of resonance fluorescence, rather than to the theory of atomic motion in a traveling wave.

We will derive general expressions for the factorial moments and, from these, obtain more explicit results for the mean and the variance, especially for the case in which the counting time T is a few times larger than the lifetime of the excited atom. Furthermore, by studying the behavior of the generating function for large T, we will show that the distribution function $p(n,T)$ becomes more and more Gaussian with increasing T, in agreement with the central limit theorem. The speed of convergence towards the Gaussian will be shown to depend strongly on the frequency detuning and the intensity of the exciting field. Some numerical examples will be given which indicate that, as long as the intensity of the exciting field is not too small and the detuning is not too large, the values of $p(n,T)$ for $n=0,1,2,\ldots$ are already quite accurately given by $(2\pi\sigma^2)^{-1/2}\exp[-(n-\langle n\rangle)^2/2\sigma^2]$, when T is only a few times larger than the natural lifetime and where the mean $\langle n\rangle$ and the variance σ^2 are given by simple expressions.

II. CALCULATION OF THE GENERATING FUNCTION

Following Mandel,[1] our starting point is the formula for the probability that n photons are emitted during the time interval from $t=0$ to $t=T$,

$$p(n,T)=\left\langle \mathscr{T}{:}\frac{1}{n!}\left[\int_0^T dt\,\hat{I}(t)\right]^n \exp\left[-\int_0^T dt\,\hat{I}(t)\right]{:}\right\rangle , \qquad (2)$$

where $\mathscr{T}$ is the time-ordering symbol; :: stands for normal ordering; $\langle\ \rangle$ means the expectation value for the total state of atom plus field; and $\hat{I}(t)$ is the operator for the total flux expressed in units of photons per second (the caret will be used to indicate symbols that represent operators rather than c numbers). According to (1), the generating function can be expressed as

$$G(z,T)=\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}\int_0^T dt_1\int_0^T dt_2\cdots\int_0^T dt_n\langle\mathscr{T}{:}\hat{I}(t_1)\hat{I}(t_2)\cdots\hat{I}(t_n){:}\rangle , \qquad (3)$$

where it is to be understood that the term with $n=0$ is equal to 1. Since

$$\int_0^T dt_1\int_0^T dt_2\cdots\int_0^T dt_n\langle\mathscr{T}{:}\hat{I}(t_1)\hat{I}(t_2)\cdots\hat{I}(t_n){:}\rangle$$
$$=n!\int_0^T dt_n\int_0^{t_n} dt_{n-1}\cdots\int_0^{t_2} dt_1\langle\mathscr{T}{:}\hat{I}(t_n)\hat{I}(t_{n-1})\cdots\hat{I}(t_1){:}\rangle , \qquad (4)$$

we can rewrite (3) as

$$G(z,T)=\sum_{n=0}^{\infty}(z-1)^n\int_0^T dt_n\int_0^{t_n} dt_{n-1}\cdots\int_0^{t_2} dt_1\langle\mathscr{T}{:}\hat{I}(t_n)\hat{I}(t_{n-1})\cdots\hat{I}(t_1){:}\rangle . \qquad (5)$$

On the right-hand side of (5) the quantity $\langle\mathscr{T}{:}\hat{I}(t_n)\hat{I}(t_{n-1})\cdots\hat{I}(t_1){:}\rangle$ can be interpreted as the joint probability density of photon emissions at the successive times $t_1,t_2,\ldots,t_n$.[6] Therefore, this quantity is equal to the probability density for a photon emission at time t_1 multiplied by the conditional probability density for an emission at time t_2, given that there has been one at t_1, multiplied by the conditional probability density for an emission at t_3, given that there has been one at t_2, etc. The differential probability density for an emission at t_1 is equal to $\langle\hat{I}(t_1)\rangle$, and this depends on the state of the atom at $t=0$. As the atom returns to the ground state immediately after each emission, the conditional probability density for a photon emission at t_2, given that there has been one at t_1, is given by $\langle\hat{I}(t_2)\rangle_{t_1;G}$, where $\langle\ \rangle_{t_1;G}$ means that the expectation value must be evaluated for the case that the atom is in the ground state at t_1. For a monochromatic coherent field we have

$$\langle\hat{I}(t_2)\rangle_{t_1;G}=\langle\hat{I}(t_2-t_1)\rangle_{0;G} . \qquad (6)$$

If we introduce the dimensionless functions $(t\geq 0)$,

$$f(t)\equiv\langle\hat{I}(t)\rangle/2\beta ;$$
$$f_0(t)\equiv\langle\hat{I}(t)\rangle_{0;G}/2\beta \qquad (7)$$

(β is half the Einstein coefficient for the transition), then we can write (5), in view of the above-given remarks, as

$$G(z,T)=1+2\beta(z-1)\int_0^T dt\,f(t)+\sum_{n=2}(2\beta)^n(z-1)^n$$
$$\times\int_0^T dt_n\int_0^{t_n} dt_{n-1}\cdots\int_0^{t_2} dt_1 f_0(t_n-t_{n-1})f_0(t_{n-1}-t_{n-2})\cdots f_0(t_2-t_1)f(t_1) . \qquad (8)$$

A powerful method in dealing with the multiple integrations on the right-hand side (rhs) of (8) is the Laplace transformation technique. For any function $h(t)$ we denote its Laplace transform with respect to t by

$$\tilde{h}(s)=\int_0^\infty dt\,h(t)e^{-st} . \qquad (9)$$

After taking the Laplace transform with respect to T of both sides of (8) and using some elementary properties of Laplace transforms, we obtain

$$\tilde{G}(z,s)=1/s+2\beta(z-1)\tilde{f}(s)/s$$
$$+\sum_{n=2}^{\infty}[2\beta(z-1)]^n\tilde{f}_0(s)^{n-1}\tilde{f}(s)/s . \qquad (10)$$

The summation on the rhs of (10) can easily be performed and the resulting expression can be written as

$$\tilde{G}(z,s)=\frac{1+2\beta(z-1)[\tilde{f}(s)-\tilde{f}_0(s)]}{s[1-2\beta(z-1)\tilde{f}_0(s)]} . \quad (11)$$

Expressions for $\tilde{f}(s)$ and $\tilde{f}_0(s)$ can be extracted from the general expression for the Laplace transform of $\langle \hat{I}(t)\rangle$ derived by Kimble and Mandel.[7] In fact, we find

$$\tilde{f}(s)=\frac{1}{2s}+\frac{\tilde{y}(s)}{1+\frac{\Omega^2(s+\beta)}{(s+2\beta)[(s+\beta)^2+\Delta^2]}} , \quad (12)$$

where

$$\tilde{y}=-\frac{1}{2s}+\frac{\langle \hat{R}_3(0)\rangle+\frac{1}{2}}{s+2\beta}-\frac{1}{2}\frac{\Omega}{\beta}\left[\frac{\langle \hat{b}_s^\dagger(0)\rangle e^{i\phi}}{1-i\theta}\left[\frac{1}{s+\beta(1+i\theta)}-\frac{1}{s+2\beta}\right]+\frac{\langle \hat{b}_s(0)\rangle e^{-i\phi}}{1+i\theta}\left[\frac{1}{s+\beta(1-i\theta)}-\frac{1}{s+2\beta}\right]\right] . \quad (13)$$

Let us explain the meaning of all the symbols appearing in (12) and (13). Ω is the Rabi frequency, characterizing the strength of the exciting field; Δ is the detuning frequency, i.e., the difference between the field frequency and the atomic resonance frequency; $\hat{R}_3(t)$ is half the population difference operator for the two-level atom; $\hat{b}_s^\dagger(t)$ and $\hat{b}_s(t)$ are the slow time-varying atomic raising and lowering operators; ϕ is the phase of the exciting field at $t=0$ and $\theta=\Delta/\beta$.

The expression for $\tilde{f}_0(s)$ is obtained from (12) by substituting the ground-state expectation value for $\tilde{y}(s)$, i.e., $\tilde{y}(s)=-1/2s$. After substitution of the thus obtained expressions for $\tilde{f}(s)$ and $\tilde{f}_0(s)$ in the rhs of (11), the final result can be written in the simple form

$$\tilde{G}(z,s)=\frac{D(s)+\beta(z-1)B(s)}{sD(s)-\beta\Omega^2(z-1)(s+\beta)} , \quad (14)$$

where

$$\begin{aligned} D(s)&=s^3+4\beta s^2+(5\beta^2+\Omega^2+\Delta^2)s \\ &\quad +\beta(\Omega^2+2\beta^2+2\Delta^2) , \\ B(s)&=(w_0+1)(s+\beta)^2-v_0\Omega(s+\beta) \\ &\quad +u_0\Omega\Delta+(w_0+1)\Delta^2 . \end{aligned} \quad (15)$$

In (15), u_0, v_0, and w_0 describe the initial state of the atom at the beginning of the counting time interval, i.e., at $t=0$,

$$\begin{aligned} u_0&=2\,\mathrm{Im}\,[\langle \hat{b}_s(0)\rangle e^{-i\phi}] , \\ v_0&=2\,\mathrm{Re}\,[\langle \hat{b}_s(0)\rangle e^{-i\phi}] , \\ w_0&=2\langle \hat{R}_3(0)\rangle . \end{aligned} \quad (16)$$

If the atom is in a stationary state already at $t=0$, then we have [see Ref. 7, Eqs. (42) and (43)]

$$\begin{aligned} u_0&=-\frac{2\Omega\Delta}{\Omega^2+2\beta^2+2\Delta^2} , \\ v_0&=-\frac{2\Omega\beta}{\Omega^2+2\beta^2+2\Delta^2} , \\ w_0&=\frac{\Omega^2}{\Omega^2+2\beta^2+2\Delta^2}-1 , \end{aligned} \quad (17)$$

whereas, if the atom is in the ground state at $t=0$,

$$u_0=v_0=0, \quad w_0=-1 . \quad (18)$$

It can easily be verified that (14), although written here in a slightly different form, is completely identical to Cook's results [(i.e., Eq. (21) of Ref. 3)].

III. CALCULATION OF THE MEAN AND THE VARIANCE

The rth factorial moment $\langle n(n-1)\cdots(n-r+1)\rangle\equiv\langle n^{(r)}\rangle$, can be obtained from the generating function by

$$\langle n^{(r)}\rangle=\frac{d^r}{dz^r}G(z,T)\,|_{z=1} . \quad (19)$$

Using (14) we find for the Laplace transform of the rth factorial moment ($r\geq 1$):

$$\widetilde{\langle n^{(r)}\rangle}=\frac{r!\beta B(s)[\beta\Omega^2(s+\beta)]^{r-1}}{s^rD(s)^r}+\frac{r![\beta\Omega^2(s+\beta)]^r}{s^{r+1}D(s)^r} . \quad (20)$$

By inverse Laplace transformation we find

$$\langle n^{(r)}\rangle = r\frac{d^{r-1}}{ds^{r-1}}\left[\frac{\beta B(s)[\beta\Omega^2(s+\beta)]^{r-1}e^{sT}}{D(s)^r}\right]_{s=0} + \frac{d^r}{ds^r}\left[\frac{[\beta\Omega^2(s+\beta)]^r e^{sT}}{D(s)^r}\right]_{s=0}$$
$$+ r\sum_{i=1}^{3}\frac{d^{r-1}}{ds^{r-1}}\left[\left[\frac{\beta B(s)[\beta\Omega^2(s+\beta)]^{r-1}}{s^r D(s)^r} + \frac{[\beta\Omega^2(s+\beta)]^r}{s^{r+1}D(s)^r}\right](s-s_i)^r e^{sT}\right]_{s=s_i}, \tag{21}$$

where s_1, s_2, and s_3 are the three roots of the cubic equation

$$D(s) \equiv s^3 + 4\beta s^2 + (5\beta^2 + \Omega^2 + \Delta^2)s + \beta(\Omega^2 + 2\beta^2 + 2\Delta^2) = 0 . \tag{22}$$

General expressions for s_1, s_2, and s_3, can be found in Kimble and Mandel [Eqs. (55a) and (55b) of Ref. 7]. We recall that one of the roots is real while the two remaining roots are the complex conjugates of each other. Furthermore, it can easily be shown that the real part of any of the roots is always smaller than or at most equal to $-\beta$. It follows from this that if T is one order of magnitude larger than $1/\beta$, the last term in the rhs of (21) is already very small, because of the exponent $\exp(s_i T)$. We conclude that for, say, $T \gtrsim 5/\beta$ the factorial moments are given by the first two terms in the rhs of (21) only.

General expressions for the mean $\langle n\rangle = \langle n^{(1)}\rangle$ and the variance $\sigma^2 = \langle n^2\rangle - \langle n\rangle^2 = \langle n^{(2)}\rangle + \langle n\rangle - \langle n\rangle^2$ can be obtained by using (21). The results are

$$\langle n\rangle = \frac{\beta[B(0)+\Omega^2]}{D(0)} + \beta^2\Omega^2\frac{d}{ds}\left[\frac{e^{sT}}{D(s)}\right]_{s=0}$$
$$+ \sum_{i=1}^{3}\left[\frac{\beta B(s_i)}{s_i(s_i-s_j)(s_i-s_k)} + \frac{\beta\Omega^2(s_i+\beta)}{s_i^2(s_i-s_j)(s_i-s_k)}\right]e^{s_i T}, \tag{23}$$

$$\sigma^2 = \frac{2\beta^2\Omega^2[B(0)+\Omega^2]}{D(0)^2} + 2\beta^3\Omega^2\frac{d}{ds}\left[\frac{B(s)+2\Omega^2}{D(s)^2}e^{sT}\right]_{s=0} + \beta^4\Omega^4\frac{d^2}{ds^2}\left[\frac{e^{sT}}{D(s)^2}\right]_{s=0}$$
$$+ 2\sum_{i=1}^{3}\frac{d}{ds}\left[\left[\frac{\beta^2\Omega^2 B(s)(s+\beta)}{s^2(s-s_j)^2(s-s_k)^2} + \frac{\beta^2\Omega^4(s+\beta)^2}{s^3(s-s_j)^2(s-s_k)^2}\right]e^{sT}\right]_{s=s_i} + \langle n\rangle - \langle n\rangle^2 , \tag{24}$$

where $\{i,j,k\}$ is a permutation of $\{1,2,3\}$. For $T \gtrsim 5/\beta$ we can neglect the terms in (23) and (24) that involve the exponential factors $\exp(s_i T)$. In that case we find after performing the differentiations and substituting for $D(s)$,

$$\langle n\rangle \to \frac{B(0)}{\Omega^2+2\beta^2+2\Delta^2} - \frac{\Omega^2(3\beta^2-\Delta^2)}{(\Omega^2+2\beta^2+2\Delta^2)^2} + \frac{\Omega^2\beta T}{\Omega^2+2\beta^2+2\Delta^2}, \tag{25}$$

$$\sigma^2 \to \frac{B(0)}{\omega^2+2\beta^2+2\Delta^2} - \frac{2B(0)\Omega^2(5\beta^2+\Omega^2+\Delta^2)}{(\Omega^2+2\beta^2+2\Delta^2)^3} + \frac{2\beta\Omega^2 B'(0) - B(0)^2}{(\Omega^2+2\beta^2+2\Delta^2)^2} - \frac{\Omega^2(3\beta^2-\Delta^2)}{(\Omega^2+2\beta^2+2\Delta^2)^2}$$
$$- \frac{\Omega^4(5\Omega^2+44\beta^2+4\Delta^2)}{(\Omega^2+2\beta^2+2\Delta^2)^3} + \frac{5\Omega^4(5\beta^2+\Omega^2+\Delta^2)^2}{(\Omega^2+2\beta^2+2\Delta^2)^4}$$
$$+ \frac{\Omega^2}{\Omega^2+2\beta^2+2\Delta^2}\left[1 - \frac{2\Omega^2(3\beta^2-\Delta^2)}{(\Omega^2+2\beta^2+2\Delta^2)^2}\right]\beta T . \tag{26}$$

As these expressions are valid for $T \gtrsim 5/\beta$ as well as for arbitrary detuning, they extend earlier results obtained by Mandel[1] and by Cook.[8]

For values of $T < 5/\beta$ we must add to the right-hand sides of (25) and (26), respectively, the corresponding terms proportional to the exponentials. These terms, however, appear to be representable by relatively simple expressions in the case of zero detuning ($\Delta = 0$) only, namely, in that case we have

$$s_1 = -\beta;\; s_2 = -3\beta/2 + i\beta\Omega';\; s_3 = -3\beta/2 - i\beta\Omega' , \tag{27}$$

where

$$\Omega' \equiv (\Omega^2/\beta^2 - \tfrac{1}{4})^{1/2} . \tag{28}$$

For the mean we find

$$\langle n \rangle = \frac{B(0)}{\Omega^2+2\beta^2} - \frac{3\Omega^2\beta^2}{(\Omega^2+2\beta^2)^2} + \frac{\Omega^2\beta T}{\Omega^2+2\beta^2}$$
$$+ \left[(w_0+1)\beta - \frac{3}{2}\frac{\beta B(0)}{\Omega^2+2\beta^2} + \frac{\beta\Omega^2(10\beta^2/4-\Omega^2)}{(\Omega^2+2\beta^2)^2} \right] \frac{e^{-3\beta T/2}\sin(\beta\Omega' T)}{\beta\Omega'}$$
$$+ \left[\frac{-B(0)}{\Omega^2+2\beta^2} + \frac{3\Omega^2\beta^2}{(\Omega^2+2\beta^2)^2} \right] e^{-3\beta T/2}\cos(\beta\Omega' T) , \tag{29}$$

where $B(0)$ is here given by [see (15), with $\Delta=0$]

$$B(0) = \beta[(w_0+1)\beta - v_0\Omega] . \tag{30}$$

Note that if the atom was in the steady state at $t=0$, (29) reduces to the simple result

$$\langle n \rangle = \frac{\Omega^2\beta T}{\Omega^2+2\beta^2} . \tag{31}$$

An explicit general expression for the variance σ^2 when $\Delta=0$ can also be derived. However, as already suggested by (26), this expression is very complicated, except when the atom is in the steady state at $t=0$. In this latter case the variance reads

$$\sigma^2 = \frac{2\Omega^4\beta^2(7\beta^2-\Omega^2)}{(\Omega^2+2\beta^2)^4} + \left[1 - \frac{6\Omega^2\beta^2}{(\Omega^2+2\beta^2)^2} \right] \frac{\Omega^2\beta T}{\Omega^2+2\beta^2}$$
$$- \frac{\Omega^4\beta^2}{(\Omega^2+2\beta^2)^4} e^{-3\beta T/2} \left[2(7\beta^2-\Omega^2)\cos(\Omega'\beta T) + 9\frac{\beta^2-\Omega^2}{\Omega'}\sin(\Omega'\beta T) \right] . \tag{32}$$

Our results (31) and (32) fully agree with Mandel.[1]

IV. GAUSSIAN BEHAVIOR OF THE PHOTON NUMBER DISTRIBUTION FOR LARGE COUNTING TIMES

According to (25) and (26) both the mean $\langle n \rangle$ and the variance σ^2 are proportional to T for $T\to\infty$. A powerful method for studying the behavior of the distribution function for large T is then the investigation of the probability distribution function of the normalized variable

$$x_n = \frac{n-\langle n \rangle}{\sigma} \quad (n=0,1,2,\dots) . \tag{33}$$

In particular, we will study the behavior for $T\to\infty$ of the corresponding cumulant generating function

$$K(y,T) = \ln\left[\sum_{n=0}^{\infty} e^{y(n-\langle n \rangle)/\sigma} p(n,T) \right] , \tag{34}$$

which is related to the formerly introduced generating function $G(z,T)$ by

$$K(y,T) = -\langle n \rangle y/\sigma + \ln[G(e^{y/\sigma},T)] . \tag{35}$$

As σ is proportional to $T^{1/2}$ for $T\to\infty$, the behavior of $K(y,T)$ for $T\to\infty$ is clearly related to the behavior of $G(z,T)$ for $z\to 1$ and $T\to\infty$. Therefore, we will first focus our attention on $G(z,T)$ for $z\to 1$ and large T.

We recall that for T larger than $5/\beta$, the behavior of $G(z,T)$ is governed by the pole of $\tilde{G}(z,s)$ which lies closest to $s=0$. From (14) we see that this pole is in $s=0$ when $z=1$, but for $z\neq 1$ the pole will shift away from 0. In fact, we can expand the position s_0 of the pole in increasing powers of $z-1$, i.e.,

$$s_0 = \sum_{m=1}^{\infty} s^{(m)}(z-1)^m . \tag{36}$$

The expansion coefficients $s^{(m)}$ can be found by equating the denominator of (14) with $s=s_0$ to zero. For the first three coefficients we find

$$s^{(1)} = \frac{\beta\Omega^2}{\Omega^2+2\beta^2+2\Delta^2} ; \tag{37a}$$

$$s^{(2)}=\frac{\beta\Omega^2 s^{(1)}-(5\beta^2+\Omega^2+\Delta^2)(s^{(1)})^2}{\beta(\Omega^2+2\beta^2+2\Delta^2)}=\frac{-\beta\Omega^4(3\beta^2-\Delta^2)}{(\Omega^2+2\beta^2+2\Delta^2)^3}\ ; \tag{37b}$$

$$s^{(3)}=\frac{\beta\Omega^2 s^{(2)}-2(5\beta^2+\Omega^2+\Delta^2)s^{(1)}s^{(2)}-4\beta(s^{(1)})^3}{\beta(\Omega^2+2\beta^2+2\Delta^2)}\ . \tag{37c}$$

The expression for $G(z,T)$ when $T \gtrsim 5/\beta$ is given by the residue of $\widetilde{G}(z,s)\exp(sT)$ in $s=s_0$. Hence, using (14), we find

$$G(z,T)=\frac{D(s_0)+\beta B(s_0)(z-1)}{D(s_0)+s_0 D'(s_0)-\beta\Omega^2(z-1)}e^{s_0 T} \quad (T \gtrsim 5/\beta)\ , \tag{38}$$

which can be written as

$$G(z,T)=\left[1+\sum_{k=1}^{\infty}A(k)(z-1)^k\right]\exp\left[\sum_{m=1}^{\infty}s^{(m)}(z-1)^m T\right]\ , \tag{39}$$

where the coefficients $A(k)$ are given by $(k=1,2,3,\ldots)$

$$A(k)=\frac{1}{k!}\frac{d^k}{dz^k}\left[\frac{D(s_0)+\beta B(s_0)(z-1)}{D(s_0)+s_0 D'(s_0)-\beta\Omega^2(z-1)}\right]_{z=1}\ . \tag{40}$$

The first two $A(k)$ are

$$A(1)=\frac{B(0)}{\Omega^2+2\beta^2+2\Delta^2}-\frac{\Omega^2(3\beta^2-\Delta^2)}{(\Omega^2+2\beta^2+2\Delta^2)^2}\ ; \tag{41a}$$

$$A(2)=\frac{\Omega^2[\beta B'(0)+B(0)]}{(\Omega^2+2\beta^2+2\Delta^2)^2}-\frac{2\Omega^2(5\beta^2+\Omega^2+\Delta^2)B(0)}{(\Omega^2+2\beta^2+2\Delta^2)^3}+\frac{\Omega^4(23\beta^4-\Delta^4-2\Omega^2\beta^2-2\Omega^2\Delta^2-26\beta^2\Delta^2)}{(\Omega^2+2\beta^2+2\Delta^2)^4}\ . \tag{41b}$$

By substitution of (39) in (35) we find for the cumulant generating function

$$K(y,T)=-\langle n\rangle y/\sigma+T\sum_{k=1}^{\infty}s^{(k)}(e^{y/\sigma}-1)^k+\ln\left[1+\sum_{k=1}^{\infty}A(k)(e^{y/\sigma}-1)^k\right]\ , \tag{42}$$

and by consistently expanding this in increasing powers of y/σ, we obtain

$$K(y,T)=[s^{(1)}T+A(1)-\langle n\rangle]y/\sigma+\tfrac{1}{2}\{[s^{(1)}+2s^{(2)}]T+A(1)+2A(2)-A(1)^2\}(y/\sigma)^2$$
$$+\tfrac{1}{6}\{[s^{(1)}+6s^{(2)}+6s^{(3)}]T+A(1)+6A(2)+6A(3)-3A(1)^2-6A(1)A(2)+2A(1)^3\}(y/\sigma)^3+\cdots\ . \tag{43}$$

It can easily be verified, by using (37a) (37b), (41a), (41b), (25), and (26), that (43) can be written as

$$K(y,T)=\tfrac{1}{2}y^2+O(T^{-1/2})\ , \tag{44}$$

or,

$$\lim_{T\to\infty}K(y,T)=\tfrac{1}{2}y^2\ . \tag{45}$$

This proves that for large counting times T the normalized quantity $(n-\langle n\rangle)/\sigma$ $(n=0,1,2,\ldots)$ is distributed as a Gaussian with zero mean and unit variance. Therefore, $p(n,T)$ satisfies

$$\lim_{T\to\infty}p(n,T)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(n-\langle n\rangle)^2/2\sigma^2}\ , \tag{46}$$

where $\langle n\rangle$ and σ^2 are given by (25) and (26), respectively. This asymptotic Gaussian behavior of $p(n,T)$ was earlier found by Cook,[5] by using the explicit form of the distribution function, but only for the special case $\Delta=0$, $\Omega=\beta$. We have shown here that it is true for all values of detuning and for all intensities of the exciting field. Moreover, it is interesting to note that the precise nature of the two-photon correlation function was not important in the derivation of the result. In fact, we would like to point out that a probability distribution function of the form (2) will always tend to a Gaussian for times much larger than the typical correlation time, at least if the correlations die out exponentially.

This can best be argued by referring to (11), which can also be written as

$$\widetilde{G}(z,s)=\frac{1+(z-1)[\widetilde{I}(s)-\widetilde{I}_0(s)]}{s-(z-1)s\widetilde{I}_0(s)}\ , \tag{47}$$

where $\widetilde{I}$ and $\widetilde{I}_0$ are the Laplace transforms of $\langle \hat{I}(t)\rangle$ and $\langle \hat{I}(t)\rangle_{0;G}$, respectively. Introducing the correlation function $\lambda(t)$ by writing

$$\langle \hat{I}(t)\rangle_{0;G}=R[1+\lambda(t)]\ , \tag{48}$$

where $R=\langle \hat{I}(\infty)\rangle$ is the steady-state photon-emission rate, we have

$$\widetilde{I}_0(s)=R/s+R\widetilde{\lambda}(s)\ . \tag{49}$$

We will now assume that $\widetilde{\lambda}(s)$ has a Taylor expansion around $s=0$, which is certainly the case if $\lambda(t)$ goes exponentially to zero for $t\to\infty$. In that case we can write

$$\widetilde{I}_0(s)=R/s+R\int_0^\infty dt\,\lambda(t)+R\int_0^\infty dt\,t\lambda(t)s+\cdots\ . \tag{50}$$

The pole of $\widetilde{G}(z,s)$ which governs the large T behavior of $G(z,t)$ can now be expressed as

$$s_0=R(z-1)+R^2(z-1)^2\int_0^\infty dt\,\lambda(t)+\cdots\ . \tag{51}$$

Therefore, for large T we find

$$G(z,T)\sim\exp\left[\left[R(z-1)+R^2(z-1)^2\int_0^\infty dt\,\lambda(t)+\cdots\right]T\right]\ , \tag{52}$$

or

$$\ln[G(e^x,T)]=RTx+\tfrac{1}{2}RT\left[1+2R\int_0^\infty dt\,\lambda(t)\right]x^2+\cdots\ . \tag{53}$$

Thus, $p(n,T)$ tends to a Gaussian with mean RT and variance $RT[1+2R\int_0^\infty dt\,\lambda(t)]$ for $T\to\infty$.

As a matter of course, these results are also obtained after application of the central limit theorem. Namely, for large T we can divide the counting interval into N identical intervals of length T_1, where T_1 is still much larger than the correlation time (which is of the order of $1/\beta$ in resonance fluorescence). The photon numbers emitted during each subinterval are then almost identically distributed, with mean $\mu(T_1)$ and variance $\sigma(T_1)^2$, and almost statistically independent. The central limit theorem, if applicable, then tells us that $p(n,T)$ is given by a Gaussian with mean $(T/T_1)\mu(T_1)$ and variance $(T/T_1)\sigma(T_1)^2$. Since both $\mu(T_1)$ and $\sigma(T_1)^2$ were found to be proportional to T_1 for large T_1, this agrees with (46) and (53). However, though the central limit theorem leads to the correct result, it does not give any insight into the speed of convergence of $p(n,T)$ towards the Gaussian, whereas our derivation certainly does. For instance, it follows from (43) that for $T\to\infty$ the kth cumulant is proportional to $(T^{k-2})^{-1/2}$. In particular, the slowest decreasing cumulant is the third, for which we can write according to (43)

$$K_3=\frac{s^{(1)}+6s^{(2)}+6s^{(3)}}{6(s^{(1)}+2s^{(2)})^{3/2}}\frac{1}{\sqrt{T}}+O(1/T)\ . \tag{54}$$

The third cumulant is closely related to the skewness, i.e., the third normalized central moment, namely,

$$\left\langle\left[\frac{n-\langle n\rangle}{\sigma}\right]^3\right\rangle=6K_3\ . \tag{55}$$

Therefore, adopting the skewness as a measure for the deviation of the actual distribution function $p(n,T)$ from its asymptotic Gaussian form, the typical time scale T_3 connected with the speed of convergence is, in view of (54), given by

$$T_3\sim\frac{(s^{(1)}+6s^{(2)}+6s^{(3)})^2}{(s^{(1)}+2s^{(2)})^3}\ , \tag{56}$$

and by substitution of (37a)–(37c)

$$\beta T_3\sim\frac{[(\Omega^2+2\beta^2+2\Delta^2)^4-6\Omega^2(3\beta^2-\Delta^2)(\Omega^2+2\beta^2+2\Delta^2)^2+6\Omega^4(16\beta^4-\Omega^2\beta^2-\Omega^2\Delta^2-16\beta^2\Delta^2)]^2}{\Omega^2(\Omega^2+2\beta^2+2\Delta^2)(\Omega^4+4\beta^4+4\Delta^4-2\Omega^2\beta^2+6\Omega^2\Delta^2+8\beta^2\Delta^2)^3}\ . \tag{57}$$

As long as $\Omega^2\gtrsim\beta^2$ and $\Delta^2\lesssim\beta^2$, βT_3 is of order unity, indicating that the skewness is already smaller than 0.3 for $T>10/\beta$. For $\Omega^2\ll\beta^2$, however, $\beta T_3\simeq 2(\beta^2+\Delta^2)/\Omega^2$, indicating that the convergence will be very slow. Indeed, this corresponds to the case in which, due to the small intensity of the exciting field, the probability that two successive photon emissions are correlated becomes very small, which implies that $p(n,T)$ tends to a Poissonian with mean equal to $\Omega^2\beta T/2(\beta^2+\Delta^2)$. In order that this Poissonian becomes Gaussian, the mean must be much larger than 1, or, $\beta T\gg 2(\beta^2+\Delta^2)/\Omega^2$, in agreement with the aforementioned estimate.

The other known Poissonian limit case[3] occurs

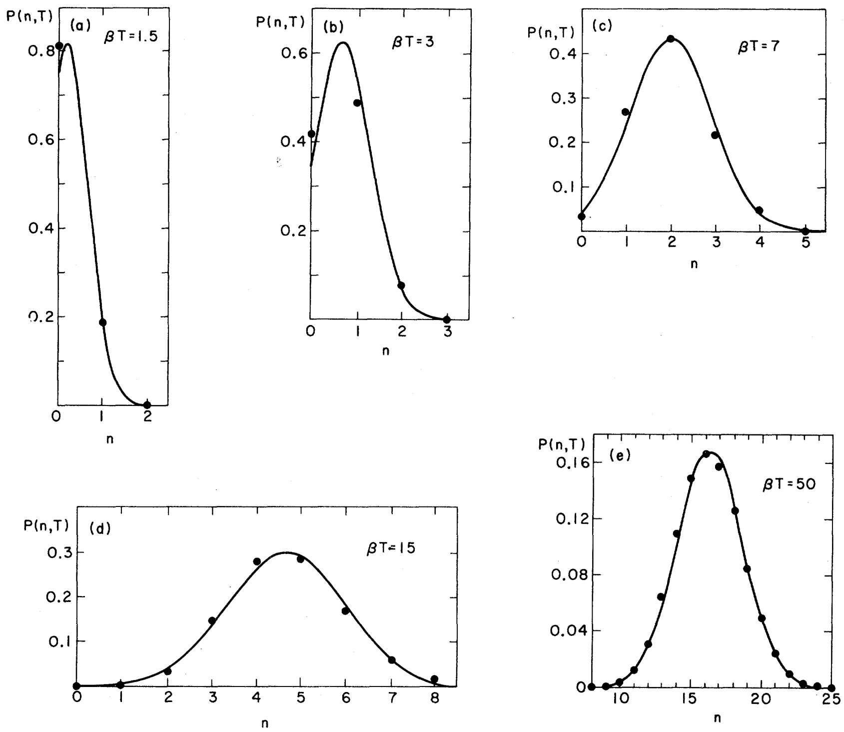

FIG. 1 (a)—(e) Exact distribution function (58) (indicated by dots) and the Gaussian approximation (46) (drawn as a continuous function of n) for several values of βT as indicated.

for very large intensities of the exciting field, i.e., $\Omega^2 \gg \beta^2, \Delta^2$. Since the mean is in this case equal to βT, the Gaussian limit will be reached for $\beta T \gg 1$, in agreement with the value $\beta T_3 = 1$ which follows from (57). The effect of large detunings, i.e., $\Delta^2 \gg \beta^2, \Omega^2$, is to slow down the convergence towards Gaussian. Namely, in this case (57) gives us $\beta T_3 \sim 2\Delta^2/\Omega^2$.

Finally, we will give some numerical results for the special case $\Omega = \beta$; $\Delta = 0$. When the atom is in the ground state at the beginning of the counting interval, we have

$$p(n,T) = \left[1 + \frac{\beta T}{3n+1} + \frac{(\beta T)^2}{(3n+1)(3n+2)}\right] \times \frac{(\beta T)^{3n} e^{-\beta T}}{(3n)!}, \tag{58}$$

which is the exact result derived by Cook.[3] We have compared this function with the Gaussian given by (46), whose mean and variance are taken equal to the values given by (25) and (26) [with $B(0) = B'(0) = 0$; $\Omega = \beta, \Delta = 0$], i.e.,

$$\langle n \rangle = (\beta T - 1)/3, \quad \sigma^2 = (3\beta T + 2)/27 . \tag{59}$$

The results for several values of T are given in Fig. 1. The dots represent the exact result given by (58) and the solid curves represent the Gaussian with mean and variance as in (59). The Gaussians are drawn as continuous functions of n. It is seen that even for T as small as $1.5/\beta$, the Gaussian is not a poor approximation at all. We conclude that the overall features of the photon number distribution function for $T \gtrsim 1/\beta$ are very well represented by a

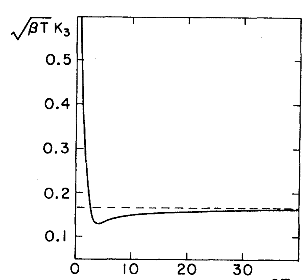

FIG. 2. A plot of $\sqrt{\beta T} K_3$ as a function of βT, illustrating that $K_3(:)T^{-1/2}$ for $\beta T \gg 1$. This numerical result has been obtained using the exact expression (58) for the distribution function.

Gaussian. At a closer look, however, the figures show that the convergence towards the Gaussian is rather slow indeed: even for $\beta T = 50$, the exact distribution function is seen to be slightly skewed to the left, which is directly related to the slow convergence of the third cumulant [e.g., (54)] to zero. This is also illustrated in Fig. 2, where the quantity $\sqrt{\beta T} K_3$ has been plotted versus βT. For large βT this quantity is seen to approach the theoretically predicted value of $\frac{1}{6}$.

V. CONCLUSION

We have theoretically investigated the statistics of photons emitted by a single two-level atom when the atom is driven by a coherent monochromatic electromagnetic field, which is nearly at resonance with the atomic transition. Our approach was based on the general formula for the emission probability of n photons during a given time interval and our results confirm earlier obtained results by Cook, whose treatment, however, is based on the theory of atomic motion in a traveling wave. We have presented general expressions for the factorial moments and from these we have deduced more explicit expressions for the mean and the variance, with emphasis on the case in which the counting time T is a few times larger than the lifetime of the excited atom. Furthermore, we have investigated the behavior of the photon distribution function for T larger than the lifetime of the excited atom and we have explicitly shown that the distribution function tends to a Gaussian for $T \rightarrow \infty$. Although the speed of convergence was found to be typically slow, i.e., the third cumulant (which indicates the skewness) was proportional to $T^{-1/2}$, numerical results illustrate that the overall features of the distribution function are already well represented by a Gaussian for $T \gtrsim 1/\beta$, at least if the intensity of the exciting field is not too small and its detuning frequency is not too large.

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