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Lifetime of oscillatory steady states

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We introduce a method to derive expressions for the distribution ρ of large fluctuations about a stable oscillatory steady state and for the transition rate from that state into another stable state. Our method is based on a WKB-type expansion of the solution of the Fokker-Planck equation. The expression for ρ has a form similar to the Boltzmann distribution with the energy replaced by a function W, which is the solution of a Hamilton-Jacobi—type equation. For the case of small dissipation, a simple analytical approximation to W, in terms of an action increment, is derived. Our results are employed to predict various measurable quantities in physical systems. Specifically we consider the problems of the physical pendulum, the shunted Josephson junction, and the transport of charge-density-wave excitations.

I. INTRODUCTION

The effects of thermal noise on the dynamical behavior of physical systems has been a subject of continuing interest for many years.[1-10] For example, the fluctuations about stable equilibrium states of physical systems are well known to be described by the Boltzmann distribution.[11,12] Furthermore, systems in which multiple stable states can coexist exhibit transitions between those states. The transition rate from a stable equilibrium state, which is inversely proportional to the lifetime of that state, has also been discussed in the literature.[3,4,13-18] That rate has been shown to be of the form $\Omega \exp(-\Delta E/T)$, where ΔE denotes the activation energy, i.e., the height of the potential barrier to be overcome in order to escape the state, and T denotes temperature. The dependence of the attempt frequency Ω on the parameters of the probelm varies with the particular problem under consideration.

Of equal interest are the questions of fluctuations about, and transitions from, stable oscillatory steady states. Indeed considerable effort has gone into calculating the distribution of fluctuations about such states.[19-26] The distribution of fluctuations may be characterized by the contours of constant probability density. In Ref. 25 the damped pendulum driven by constant torque was considered. The contours were approximated by contours of constant "energy" E_p, where E_p was taken to be the periodic part of the energy, though the potential energy for the driven pendulum is not periodic. In our earlier work,[26] the contours were approximated by steady-state trajectories in phase space corresponding to different values of the action. In the present paper we show that at low temperatures, the contours of constant probability density correspond to the family of steady-state phase-space trajectories of the pendulum, determined by varying the dissipation coefficient. Our analysis shows the validity of the results of Refs. 25 and 26 in the limit of small dissipation.

The ability to calculate transition rates is important not only in itself, but also for the determination of the relative stability of different stable states, including oscillatory states. Examples of physical systems of current interest, for which it is important to answer the above questions, include various configurations of Josephson junctions, quantum parametric amplifiers and oscillators in both optical and microwave frequency ranges, various field-induced diffusion processes, and systems exhibiting charge-density-wave transport to name but a few. With the exception of Ref. 26 results on the transition rate from stable nonequilibrium steady states seem to have been limited to numerical simulations.

This is apparently due to the fact that most discussions of the fluctuations about such states were restricted to small fluctuations.

In this paper we present a method for deriving explicit analytical formulas for the distribution of fluctuations about, and transition rates from, stable oscillatory steady states. Our method is not restricted to small fluctuations, but includes large fluctuations as well. To illustrate the method, we consider the problem of a one-dimensional physical pendulum with friction coefficient G, constant external torque I, and a Langevin white-noise term which represents thermal fluctuations. The pendulum is perhaps the simplest example of a system which, in the absence of noise, has both equilibrium states and a nonequilibrium steady state. The latter corresponds to a continuously whirling motion, and exists for all values of G in the range $0<G<G_M(I)$ [$G_M(I)$ is calculated in Sec. II]. In addition, this model of the pendulum is directly applicable to other physical systems, e.g., a Josephson junction[27,28] and a model of charge-density-wave transport.[29,30] Our aim is to calculate the distribution of fluctuations about the whirling state and the transition rate from the whirling state to an equilibrium state.

A white-noise term is appropriate to model the situation when the microscopic interactions that the pendulum is subject to consist of a large number of small random kicks, each of which causes a small change in velocity. Although for ease of presentation we consider the specific example described above, our method is applicable to a wide class of dynamical systems perturbed not only by Langevin white noise, but by state-dependent noise, such as shot noise[31] and to higher-dimensional systems such as the dc SQUID.[32] The probability density for fluctuations in such a system is the solution of a Fokker-Planck equation.

In order to calculate ρ, we derive an asymptotic solution of the Fokker-Planck equation by the WKB method. The leading term in this expansion is of the form $\rho_0\exp(-W/T)$, where the function W satisfies a Hamilton-Jacobi- (eikonal-) type equation and ρ_0 satisfies a transport equation. The constant-W contours are shown to be the family of nonequilibrium steady-state trajectories parametrized by G. Furthermore, an approximation to W, valid for small G/I, is given by $W\cong(\Delta A)^2/2$, where ΔA is the increment in the action between the nonequilibrium steady-state trajectory and the trajectory through the point at which the fluctuation is computed. Introducing W as one of the independent variables, we show that the Fokker-Planck equation is approximated by a Smoluchowski-type equation in W. Solving this equation leads to an explicit expression for ρ, and also for the transition rate $\bar{\tau}_s^{-1}\cong\Omega\exp[-(W_M/T)]\cong\exp[-(\Delta A_M)^2/2T]$ from the nonequilibrium steady state. The quantity ΔA_M represents the increment in the action between the the nonequilibrium steady states corresponding to G and to $G_M(I)$. Finally, we remark that the WKB expansion is valid for $T/W_M\ll 1$.

Some of the results in this paper were anticipated in our previous work,[26] where the Fokker-Planck equation was also approximated by a Smoluchowski-type equation, though in the variable A rather than W. The results of that paper are here shown to be the leading term in an asymptotic expansion of the present results in powers of G/I. Moreover, the results of this article close a gap in the argument developed in Ref. 26, where the size of the fluctuations considered was somewhat limited. In the present paper the limitation is removed.

In Sec. II we review the deterministic dynamics of the forced pendulum with friction. In Sec. III we compute the distribution ρ and describe a comparison of our results with numerical simulations of the Langevin equation, which exhibits remarkable agreement. In Sec. IV, we compute the mean lifetime $\bar{\tau}_s$. In Sec. V we present applications of the theory to the specific examples of a shunted Josephson junction and to a model of charge-density-wave transport. We also describe a comparison of our results for $\bar{\tau}_s$, with numerical simulations of the Langevin equation, which also exhibits excellent agreement. Finally, in Sec. VI, we summarize our results and discuss their relationship with other works on the subject.

II. THE UNDERDAMPED FORCED PENDULUM

In this section we review the dynamics of the forced underdamped pendulum in the absence of thermal fluctuations. In dimensionless variables, the equation of motion can be written as

$$\ddot{\theta}+G\dot{\theta}+U'(\theta)=0\ , \tag{2.1}$$

where θ is the deflection angle of the pendulum, G is the friction coefficient, and the potential $U(\theta)$ is given by

$$U(\theta)=-\cos\theta-I\theta \tag{2.2}$$

with I denoting the external torque. A sketch of the potential is shown in Fig. 1(a). The dynamics of

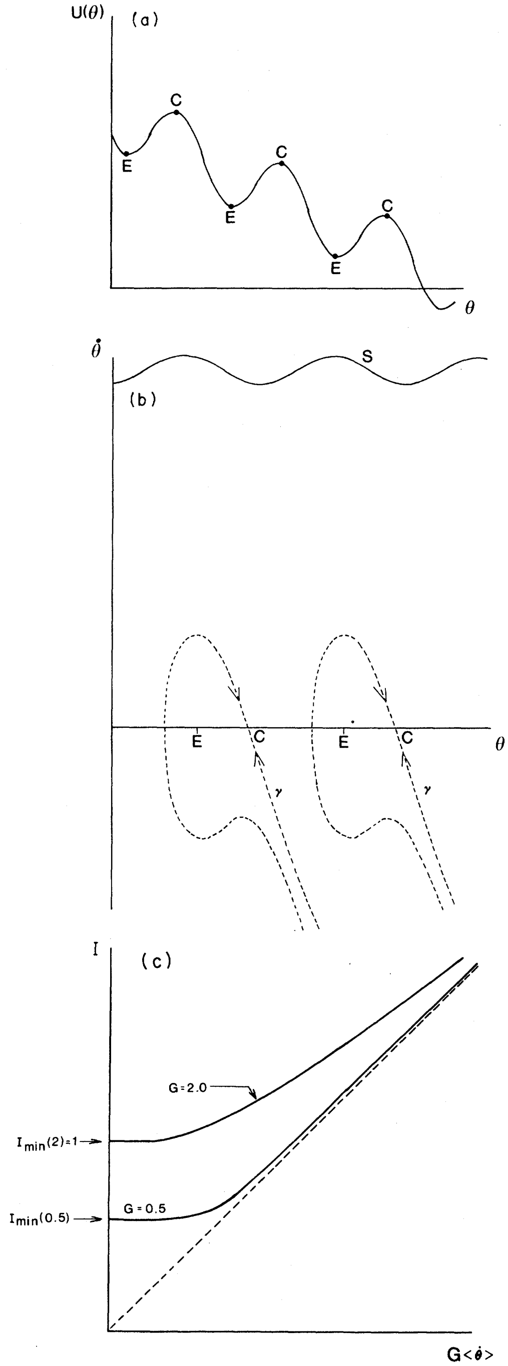

FIG. 1. (a) Sketch of potential $U(\theta)$ of underdamped forced pendulum for $I<1$. (b) Typical phase-space plot of underdamped forced pendulum for $G<G_c$ and $I_{\min}(G)<I<1$. The curve S is the phase-space trajectory of the nonequilibrium steady state, the curves γ are separatrices, and the points E and C are, respectively, stable and unstable equilibria. Curves were calculated for $I=0.5$ and $G=0.1$. Dashed lines extend to infinity. (c) Plot of torque I vs average velocity $\langle\dot{\theta}\rangle$ for various values of G.

the pendulum depend on the values of the coefficients G and I. Thus, for example, for $G>G_c\approx\pi/4$ (see Appendix A), the asymptotically stable solution of (2.1) is an equilibrium solution located at one of the minima of $U(\theta)$ if $I<1$, or a nonequilibrium steady-state solution corresponding to motion down the potential with a periodically varying velocity whose time average $\langle\dot{\theta}\rangle$ is nonzero, if $I>1$.

In contrast, for $G<G_c$, and I in the range $I_{\min}(G)<I<1$ [$I_{\min}(G)$ is calculated in Appendix A], stable equilibrium solutions and the stable nonequilibrium solution can coexist, and the system can exhibit hysteresis. A typical phase-space picture for $G<G_c$ and $I_{\min}(G)<I<1$ is shown in Fig. 1(b). Finally a typical plot of torque versus average velocity is shown in Fig. 1(c). In this parameter regime, phase space is divided into a basin of attraction D_s of the stable nonequilibrium steady state S and a basin of attraction for each of the stable equilibrium states E, denoted generically by D_E. The basins are separated from each other by separatrices, which correspond to solutions of (2.1) that converge asymptotically to the unstable equilibrium points at $\dot{\theta}=0$ and at the local maxima of $U(\theta)$[33] [cf. Fig.

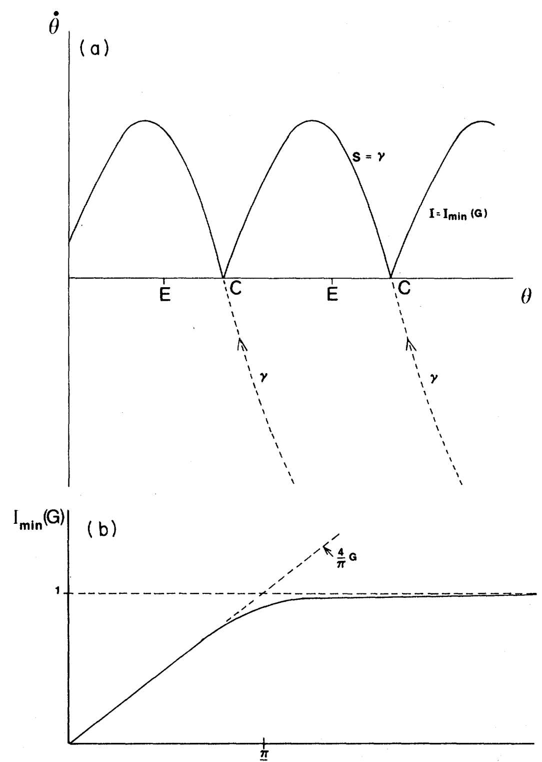

FIG. 2. (a) Phase-space plot of the underdamped forced pendulum when the trajectory S and the separatrices γ coalesce at $I=I_{\min}(G)$. (b) Plot of $I_{\min}(G)$ vs G.

1(a)].

For a given value of $G < G_c$, as $I(<1)$ is decreased towards $I_{\min}$, the separatrices and the nonequilibrium steady state S approach each other. When $I = I_{\min}(G)$, these curves coalesce, leading to the phase-space picture shown in Fig. 2(a). The dependence of $I_{\min}$ on G is shown in Fig. 2(b). For small G, $I_{\min}(G)$ is given approximately[27] by (see Appendix A)

$$I_{\min}(G) \approx \frac{4G}{\pi} . \tag{2.3}$$

For a given $I \leq 1$, there is a corresponding value of G, denoted by $G_M(I)$, for which

$$I_{\min}(G_M) = I \tag{2.4}$$

[note that $G_M(1) = G_c \approx \pi/4$]. Thus for $G < G_M(I)$ there is hysteresis, while for $G = G_M(I)$, the phase-space picture is again as shown in Fig. 2(a). For a given value of $I < 1$, when G increases towards G_M, the separatrices and S approach each other as before. However, in this case the unstable equilibria, which lie on the separatrices, do not move.

The phase-space trajectory S can be characterized as the only periodic solution of the differential equation[33,34]

$$\frac{d\dot{\theta}}{d\theta} = -G + \frac{I - \sin\theta}{\dot{\theta}} , \tag{2.5}$$

which is equivalent to (2.1). Expanding that solution in powers of G/I we get the following expression, valid for $I > I_{\min}(G)$,

$$\dot{\theta}(\theta) = \frac{I}{G} + \frac{G}{I}\cos\left[\theta + \frac{G^2}{I}\right] - \frac{1}{4}\left[\frac{G}{I}\right]^3 \cos 2\theta + O\left[\frac{G^5}{I^5}\right] . \tag{2.6}$$

For $I = I_{\min}(G)$, a first approximation to the critical stable periodic trajectory S_c, i.e., the steady state that has just coalesced with the separatrix, is given by

$$\dot{\theta} = 2\left|\cos\frac{\theta - \Delta}{2}\right| ,$$

where $\Delta = -\sin^{-1} I_{\min}(G)$ [cf. Fig. 2(a)].

In this paper we consider a parameter regime for which a stable nonequilibrium steady state exists, and calculate thermal fluctuations due to noise about that state. If, in addition, stable equilibrium states coexist with the stable nonequilibrium state, we also calculate the lifetime of the nonequilibrium state until it undergoes a transition to one of the equilibrium states.

III. DISTRIBUTION OF FLUCTUATIONS ABOUT THE NONEQUILIBRIUM STEADY STATE

We now consider the effects of thermal noise on the dynamics of the pendulum described in the previous section. We calculate the probability density of noise-induced fluctuations about the nonequilibrium steady state S. In contrast to previous results, our results are not restricted to a description of small fluctuations about the nonequilibrium steady state.

We assume a Langevin white-noise model, i.e., Eq. (2.1) is replaced by

$$\ddot{\theta} + G\dot{\theta} + U'(\theta) = L(t) , \tag{3.1}$$

where the white noise $L(t)$ satisfies

$$\langle L(t+\tau) L(t) \rangle = 2GT\delta(\tau) \tag{3.2}$$

and T is the temperature in the dimensionless units of (2.1). Under these conditions, the probability density of fluctuations in phase space $\rho(\theta,\dot{\theta},t)$ satisfies the Fokker-Planck equation,[1-3]

$$\frac{\partial\rho}{\partial t} = \hat{L}\rho \equiv -\dot{\theta}\frac{\partial\rho}{\partial\theta} + U'(\theta)\frac{\partial\rho}{\partial\dot{\theta}} + G\frac{\partial}{\partial\dot{\theta}}\left[\rho\dot{\theta} + T\frac{\partial\rho}{\partial\dot{\theta}}\right] . \tag{3.3}$$

We seek a stationary distribution ρ_s, such that

$$\hat{L}\rho_s = 0 . \tag{3.4}$$

The Boltzmann distribution

$$\rho_B \sim e^{-E/T} , \tag{3.5}$$

where E denotes energy, satisfies (3.4) and in fact represents the distribution of fluctuations about an equilibrium state. It does not however, represent the distribution of fluctuations about nonequilibrium steady states, since the density (3.5) is unbounded on S, and is not periodic in θ. Finally, the probability current

$$J \equiv (\dot{\theta}\rho, -U'(\theta)\rho - G\dot{\theta}\rho - GT\,\partial\rho/\partial\dot{\theta})$$

in phase space corresponding to ρ_B vanishes, whereas in the nonequilibrium steady state we expect a nonzero probability current flowing in the direction of decreasing $U(\theta)$. Therefore, instead of (3.5), we seek a solution of (3.4) that is bounded, periodic in θ with the same period as $U(\theta)$, and produces a nonzero current in the appropriate direction.

To this end, we adopt the WKB approximation for small T, writing

$$\rho_s=\rho_0 e^{-W/T}\ , \tag{3.6}$$

where the functions $W(\theta,\dot\theta)$ and $\rho_0(\theta,\dot\theta,T)$ remain to be determined. Thus W plays a role similar to that of energy in (3.5). Both W and ρ_0 must be periodic on S, and ρ_0 is assumed to be a regular function of T at $T=0$. Substituting (3.6) into (3.4) and expanding in powers of T, we find that W satisfies the Hamilton-Jacobi- (eikonal-) type equation

$$G\left[\frac{\partial W}{\partial\dot\theta}\right]^2+\dot\theta\frac{\partial W}{\partial\theta}-[G\dot\theta+U'(\theta)]\frac{\partial W}{\partial\dot\theta}=0\ . \tag{3.7}$$

At the same time to leading order in T, ρ_0 is the 2π-periodic (in θ) solution of the transport equation[35]

$$\left[2G\frac{\partial W}{\partial\dot\theta}-G\dot\theta-U'(\theta)\right]\frac{\partial\rho_0}{\partial\dot\theta}+\dot\theta\frac{\partial\rho_0}{\partial\theta}+G\left[\frac{\partial^2 W}{\partial\dot\theta^2}-1\right]\rho_0=0\ . \tag{3.8}$$

This approximation is valid throughout D_s, as long as $T<<1$, i.e., as long as the usual Boltzmann thermal energy k_BT is much less than the energy scale of the problem.

We first show that the contours of constant W in phase space correspond to the deterministic nonequilibrium steady-state trajectories of Eq. (2.1) for $0\le G\le G_M(I)$. This property will enable us to evaluate the function $W(\theta,\dot\theta)$.

To do so we will need to consider the following equations in the phase space $(\theta,\dot\theta)$ which we now denote by (x,y).

(1) The equations of motion in the absence of noise, given by

$$\begin{aligned}\dot x&=y\ ,\\ \dot y&=-Gy-U'(x)\ .\end{aligned} \tag{3.9}$$

(2) The parametric equations for the constant-W contours

$$\begin{aligned}\dot x&=y\ ,\\ \dot y&=-Gy-U'(x)+GW_y\ .\end{aligned} \tag{3.10}$$

(3) The parametric equations for the characteristic curves of W (see, e.g., Ref. 36)

$$\begin{aligned}\dot x&=y\ ,\\ \dot y&=-Gy-U'(x)+2GW_y\ ,\\ \dot W_y&=-W_x+GW_y\ ,\\ \dot W_x&=U''(x)W_y\ ,\\ \dot W&=GW_y^2\ .\end{aligned} \tag{3.11}$$

First we observe that $W=$const and $\nabla W=0$ on S. Indeed from (3.9) and (3.7), it follows that on S, $\dot W$ is given by

$$\dot W=-GW_y^2\le 0\ . \tag{3.12}$$

Hence, in order for W to be periodic on S, the right-hand side of (3.12) must vanish identically, so that $W=$const on S. Moreover, from (3.7) it follows that $W_x=0$ on S, and hence that $\nabla W=0$ on S.

Next we find W_y on the W contours. To this end, we consider the function

$$H(x,y)\equiv\frac{y^2}{2}+U(x)+G\int_{x_0}^{x}(y-W_y)dx\ , \tag{3.13}$$

where the integral is a line integral along the W contour that passes through the point (x,y). From (3.10), it is easy to see that $H=$const on any W contour. Using (3.11) we can now calculate the rate of change of $H(W)$ on a characteristic curve, given by

$$\dot H=GyW_y+G^2\int_{x_0}^{x}W_y(1-W_{yy})dx\ . \tag{3.14}$$

Hence, we obtain

$$\begin{aligned}H'(W)&=\frac{dH}{dW}\\ &=\frac{\dot H}{\dot W}=\frac{y}{W_y}+\frac{G}{W_y^2}\int_{x_0}^{x}W_y(1-W_{yy})dx\ .\end{aligned}$$

As shown in (3.24), $W_{yy}=1+O(G)$ for small G, so that

$$H'(W)=\frac{y}{W_y}+O(G^2)\equiv\frac{1}{K(W)}+O(G^2)\ . \tag{3.15}$$

Using this result, we can rewrite (3.10) for small G as

$$\begin{aligned}\dot x&=y\ ,\\ \dot y&\cong-\Gamma(W)y-U'(x)\ ,\end{aligned} \tag{3.16}$$

where $\Gamma(W)\equiv G[1-K(W)]$. The system (3.16) has the same form as (3.9), except that G has been replaced by Γ. Thus the W contours are nothing but the family of steady-state trajectories of (3.16) for $0\le\Gamma(W)\le G_M(I)$, and they are given by the approximate expression (2.6) with G replaced by Γ. In

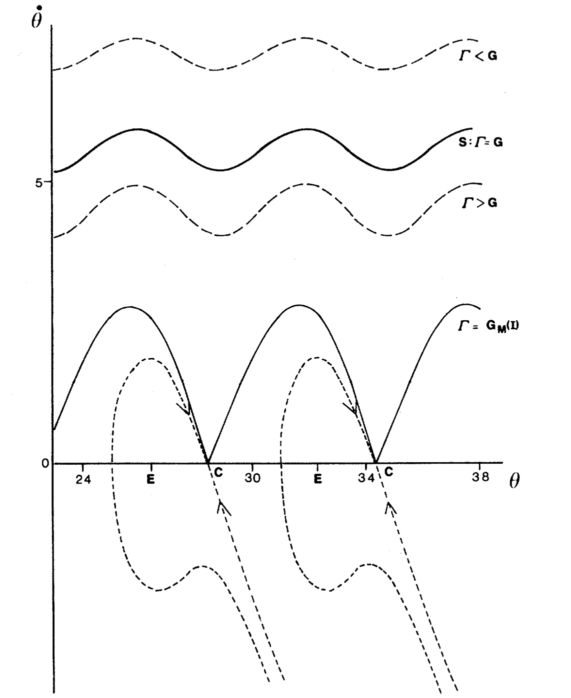

FIG. 3. Plot of W contours in phase space for $I=0.5$ and $G=0.1$.

Fig. 3, we show some of these W contours for various values of Γ, up to $\Gamma=G_M(I)$. We will show in Sec. IV, that $\Gamma=G_M(I)$ corresponds to the first W contour that touches the separatrix. The determination of this critical contour will enable us to determine the transition rate out of S. We observe that (3.15) is actually valid only for values of K which are $O(1)$. For large K, e.g., K corresponding to the critical contour, the $O(G^2)$ estimate of the integral in (3.14) which leads to (3.15) is no longer valid. However, the contribution of the integral to (3.16) is $O(1)$ only on short time intervals, since most of the time on the critical contour is spent near the unstable equilibrium point. The influence of the integral on the solution of (3.16) is therefore negligible.

In order to evaluate $W(x,y)$, we must first determine its relation to Γ. As coordinates in the x,y plane, we choose W and x. Employing the relation $W_y=Ky+O(G^2)$ and $\Gamma=G(1-K)$, we obtain to leading order in G,

$$W_\Gamma=\frac{(1-\Gamma/G)y}{\Gamma_y}\ . \tag{3.17}$$

Hence, to leading order in G,

$$W(\Gamma)=\tfrac{1}{2}\int_G^\Gamma\left[1-\frac{\Gamma}{G}\right](y^2)_\Gamma d\Gamma\ . \tag{3.18}$$

Integrating by parts we obtain

$$W(\Gamma)=\frac{1}{2}\left[\left[1-\frac{\Gamma}{G}\right]y^2(\Gamma,x)+\frac{1}{G}\int_G^\Gamma y^2(\Gamma,x)d\Gamma\right]\ . \tag{3.19}$$

We observe that although y depends on both Γ and x, the right-hand side of (3.19) is independent of x on a W contour. Finally, employing the asymptotic expression (2.6) for $y(G,x)$ in (3.19), we obtain

$$W(\Gamma)\approx\frac{1}{2}\left[\frac{I}{\Gamma}-\frac{I}{G}\right]^2,\quad G\le\Gamma\le G_M\ . \tag{3.20}$$

This equation can be understood in the following way. The generalized action $A(\Gamma)$ of each of the trajectories is given by

$$A(\Gamma)\equiv\frac{1}{2\pi}\int_0^{2\pi}y\,dx\cong\frac{I}{\Gamma}\ . \tag{3.21}$$

Thus Eq. (3.20) can be written in the form

$$W(\Gamma)\cong\tfrac{1}{2}[A(\Gamma)-A_0]^2\ , \tag{3.22}$$

where

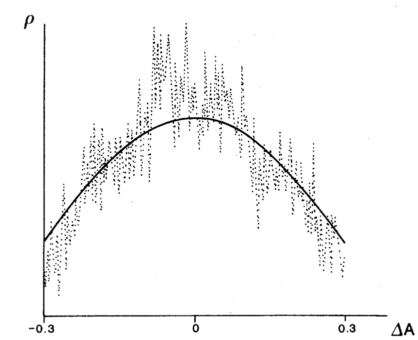

FIG. 4. Comparison of the stationary distribution according to Eq. (3.26) with that of numerical simulation of the Langevin equation (3.1) for $G=0.5$ and $T=0.1$, and various values of I between 0.6 and 0.9. A Runge-Kutta scheme was employed to solve Eq. (3.1), with the noise term $L(t_i)dt$ replaced by $n_i(dt/2GT)^{1/2}$, where $n_i=N(0,1)$ is a Gaussian random number. The phase space around the steady-state trajectory for each pair of values of (G,I) was divided into bins corresponding to equal intervals of the action A. (This was done with the help of an array of steady-state trajectories for different values of G.) The number of times $N(A_i)$ that the solution spent in each bin was then counted. What appears as a jagged dotted line in the graph is actually a discrete plot of these numbers versus the deviation of A from the steady-state value $\Delta A_i=A_i-A_s$. The smooth curve through the points is that predicted by Eq. (3.26).

$$A_0 = A(\Gamma = G) .$$

Finally, we show that ρ_0 is approximately constant for small G. We choose ρ_0 to have the average value 1 on S. Employing (3.11) in (3.8) it follows that along the characteristic curves

$$\dot{\rho}_0 = -G(W_{yy} - 1)\rho_0 . \tag{3.23}$$

By differentiating (3.20) and employing (2.6) differentiated with respect to $y = \dot{\theta}$, we show that

$$W_{yy} = 1 + O(G) . \tag{3.24}$$

Consequently, (3.23) implies that

$$\rho_0 = 1 + O(G^2) . \tag{3.25}$$

It follows that the stationary probability density of fluctuations about S is given by

$$\rho_s \sim \exp\left[-\frac{(\Delta A)^2}{2T}\right] , \tag{3.26}$$

where

$$\Delta A = A(\Gamma) - A(G) . \tag{3.27}$$

The quantity A is the difference between the action associated with S and the action associated with the steady-state trajectory through x,y (which corresponds to a different value of the friction constant, Γ instead of G). The approximate result (3.26) was obtained in our earlier paper.[26]

We have compared this distribution with the results of numerical solutions of (3.1) (see Fig. 4). The quality of the agreement evident in that figure indicates the power of our approach.

IV. MEAN LIFETIME OF THE NONEQUILIBRIUM STEADY STATE

In addition to fluctuations about the nonequilibrium steady state S, thermal noise also causes transitions from the basin of attraction D_s of S, into one of the basins of attraction D_E of a stable equilibrium state. We now calculate the mean lifetime $\bar{\tau}_s$ of S, i.e., the mean time spent inside D_s before undergoing such a transition.

Under the conditions necessary to justify our use of the WKB approximation, it is also true (as verified below) that $\bar{\tau}_S$ is sufficiently large so that the motion is a stationary diffusion process.[3] This means that in D_s, ρ is very nearly equal to the stationary distribution ρ_s of (3.6). As seen in Sec. III, ρ_s is essentially constant on the W contours. Thus the process can be considered as a one-dimensional stationary diffusion in the space of W contours. We recall that these contours are the nonequilibrium steady-state trajectories of the pendulum for values of Γ in the range $0 \le \Gamma \le G_M$. We shall derive a Smoluchowski-type equation for the probability distribution in this space. It is natural to introduce W and X as coordinates in the Fokker-Planck equation (3.3), and to average that equation over a W contour. The average of a function $F(x,y)$ over such a contour is defined as

$$\bar{F}(W) \equiv \frac{1}{T_p}\int_0^{T_p} F(x(t),y(t))dt . \tag{4.1}$$

Here t is the parameter of Eq. (3.16) which represents time along the steady-state trajectory corresponding to $\Gamma(W)$, while $T_p(W)$ is the period of that trajectory. Applying this averaging procedure to the Fokker-Planck equation leads to the one-dimensional Smoluchowski-type equation in W given by (cf. Appendix B)

$$\frac{\partial\rho}{\partial t} = 2G\left[TW\frac{\partial^2\rho}{\partial W^2} + (W + T/2)\frac{\partial\rho}{\partial W} + \frac{\rho}{2}\right] . \tag{4.2}$$

We recall that by employing the approximation (2.6) in (3.19), we showed that $W(\Gamma) \approx \frac{1}{2}[A(\Gamma) - A_0]^2 \equiv \frac{1}{2}(\Delta A)^2$, so that Eq. (4.2) can be approximated by (cf. Appendix B)

$$\frac{\partial\rho}{\partial t} = G\frac{\partial}{\partial A}\left[[A(\Gamma) - A_0]\rho + T\frac{\partial\rho}{\partial A}\right] . \tag{4.3}$$

Equation (4.3) was obtained in our previous work, where the action A and x were employed as coordinates. Here we see that it represents the leading-order term in an asymptotic expansion in powers of G. A more accurate expression than (4.3) can be obtained by employing more terms of the expansion (2.6) in (4.2). Equation (4.3) is similar in form to the Smoluchowski equation for motion in a potential well in the case of large dissipation, with an effective friction coefficient $1/G$. Kramers[3] also obtained a similar equation in action-angle variables to describe the transition from a stable equilibrium state in the case of small dissipation. We emphasize that although similar in form, our Eq. (4.3) describes a totally different process—the transition from a stable *nonequilibrium* steady state. Such a description has not been given before. Following the method employed by Kramers for the transition rate from an equilibrium state, we assume a quasistationary distribution for $A(\Gamma) < A(G_M)$ and a constant diffusion current J across the separatrix. That is,

$$[A(\Gamma)-A_0]\rho+T\frac{\partial\rho}{\partial A}=-\frac{J}{G} . \qquad (4.4)$$

Then, employing the stationary distribution $\rho_s=\exp\{-[A(\Gamma)-A_0]^2/2T\}$ in (4.4), we solve for J. Using the relation $\bar{\tau}_s^{-1}=J/N$, where N is the total number of particles, we obtain an expression for the lifetime $\bar{\tau}_s$ of the nonequilibrium steady state as

$$\bar{\tau}_s\cong\frac{\sqrt{\pi}}{G}\left[\frac{T}{\Delta W}\right]^{1/2}\exp\left[\frac{\Delta W}{T}\right] , \qquad (4.5)$$

where

$$\Delta W=\frac{1}{2}\left[\frac{I}{G}-\frac{I}{G_M(I)}\right]^2\cong\frac{1}{2}\left[\frac{I}{G}-\frac{4}{\pi}\right]^2$$

is the increment in W between S and the W contour that touches the separatrix. We observe that $\bar{\tau}_s$ is exponentially large, for $T/\Delta W \ll 1$, so that both the stationary diffusion assumption and the WKB approximation are justified. Finally, we remind the reader that these results only apply for $G<G_c$ and $I_{\min}(G)<I<1$, since only then do both types of steady states S and E coexist.

We show in Appendix B that the W contour that first touches the separatrix does so at the unstable equilibrium point C [see Fig. 2(a)], that it has a cusp at that point, and that $\nabla W=0$ there. Along the separatrix, W attains its minimum value at C, so that C is a saddle point of W. Furthermore, the characteristic curve through C can be shown to be the most probable path out of D_s.[37] Finally, we remark that the fact that the W contours reach all the way to the separatrix (in fact, they fill the entire basin D_s) closes a gap in the argument developed in Ref. 26. Specifically, the system of contours of constant I used there did not quite reach the separatrix.

V. APPLICATIONS

The pendulum model discussed here is directly applicable to a number of other physical systems. Here we describe two such systems.

A. The shunted Josephson junction

This system is often described, in dimensional units, by the phenomenological equation[29,30]

$$C\frac{dV}{dt}+\frac{V}{R}+I_J\sin\theta=I_{\rm dc} , \qquad (5.1)$$

where C, R, and I_J are the capacitance, resistance, and the critical current of the junction, respectively. $I_{\rm dc}$ is the external dc current source, while θ and V are the phase difference and voltage across the junction, respectively. Employing the Josephson relation[38]

$$\dot{\theta}=\frac{2e}{\hbar}V , \qquad (5.2)$$

Eq. (5.1) reduces to Eq. (2.1) with

$$I\equiv I_{\rm dc}/I_J , \quad G\equiv(\omega_J RC)^{-1} , \qquad (5.3)$$

where the Josephson plasma frequency ω_j is given by

$$\omega_J^2\equiv\frac{2eI_J}{\hbar c}$$

and the time is measured in units of ω_J^{-1}.

B. Classical transport of a charge-density-wave excitation

In a recent discussion[29,30] of electrical conduction by transport of charge-density-wave (CDW) excitations, the phenomenological equation

$$\ddot{x}+\frac{1}{\tau}\dot{x}+\frac{\omega_0^2}{Q}\sin Qx=\frac{eE}{m} , \qquad (5.4)$$

was introduced to describe the position x of a single excitation. Here $1/\tau=\gamma/m$, where γ is the damping constant, ω_0 is the natural frequency of oscillations of the excitation in its confining potential, Q is the wave number of the CDW, and E is the external electric field. Defining θ by

$$\theta=Qx , \qquad (5.5)$$

Eq. (5.4) reduces to Eq. (2.1) with

$$I\equiv E/E_T , \quad G\equiv(\omega_0\tau)^{-1} , \qquad (5.6)$$

where

$$E_T\equiv\frac{m\omega_0^2}{Qe} ,$$

and time is measured in units of ω_0^{-1}. In this system, the current density carried by CDW transport is given by

$$J=\frac{e\dot{x}\omega_0}{\sigma} , \qquad (5.7)$$

where σ is the cross section of the sample.

The results of Secs. III and IV can be used to predict various measurable quantities in these systems. Thus, using (3.26) we can calculate the mean-square fluctuation of $\dot{\theta}$. Consequently, using (5.2), one can show that the mean-square voltage fluctuations across a shunted Josephson junction are approximately equal to $k_B TR$.[39] Also, using (4.3) an expression for the lifetime of the nonequilibrium steady state of the Josephson junction, in dimensional units, is derived as

$$\bar{\tau}_s \cong \sqrt{\pi} RC \left[\frac{k_B T}{\Delta W} \right]^{1/2} \exp\left[\frac{\Delta W}{k_B T} \right] , \qquad (5.8)$$

where

$$\Delta W \cong (I_{\rm dc} = I_{\rm min})^2 R^2 C \ . \qquad (5.9)$$

This quantity can either be measured directly when it is sufficiently large, or observed as a peak in the power spectrum of the junction at frequency $1/\bar{\tau}_s$. Similarly we derive the expression for the mean-square CDW current fluctuations as $k_B T$, and the expression

$$\bar{\tau}_s \cong \sqrt{\pi} \tau \left[\frac{k_B T}{\Delta W} \right]^{1/2} \exp\left[\frac{\Delta W}{k_B T} \right] , \qquad (5.10)$$

where

$$\Delta W \cong \left[\frac{E\omega_0 \tau}{E_T} - \frac{4}{\pi} \right]^2 e E_T Q \qquad (5.11)$$

for the lifetime of a CDW excitation in dimensional units.

Finally, it is possible to use the values of $\bar{\tau}_s$ as a function of I in a calculation of the I-V characteristics of the Josephson junction. However, we remark that an expression for the lifetime of the equilibrium states must also be employed since transitions occur back and forth between them and the nonequilibrium steady state. This calculation was carried out in Ref. 26 and the results compared with the I-V characteristic as constructed from numerical simulations. The agreement was seen to be excellent.

VI. DISCUSSION AND SUMMARY

We begin by discussing the relationship of the present work to Refs. 25 and 26. In Ref. 25 an approximate formula for the distribution of fluctuations was presented. The approximation was based on the assumption that the dissipation and the constant driving torque balanced each other and consequently were ignored. Then the contours of constant probability were claimed to be the constant energy trajectories of the free pendulum, that is $E_p = \dot{x}^2/2 - \cos x$. In view of our analysis, this can be justified in the limit of small dissipation. The authors of Ref. 25 have also presented a numerical approach[40] to the calculation of ρ whose results are in qualitative agreement with ours. In our earlier work[26] approximate formulas for both the distribution of fluctuations and the transition rate were presented. In that approximation, the constant probability contours were approximated by steady-state trajectories corresponding to various values of I, or equivalently of action A. The present analysis justifies that the result in the limit of small dissipation.

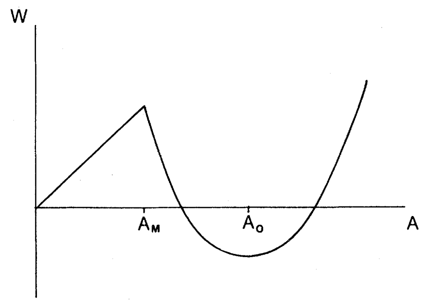

FIG. 5. Sketch of effective potential of forced pendulum in action space.

In summary, we have introduced a method to derive expressions for the distribution ρ of large fluctuations about a nonequilibrium steady state, and for the transition rate from that state at low temperatures. The expresssion for ρ has the same form as the Boltzmann distribution with the Boltzmann distribution with the energy replaced by the function W. The expression for the transition rate is of the form $\Omega e^{-\Delta W/T}$, where ΔW is the effective height of the barrier to be overcome in order to make a transition from the stable nonequilibrium steady state. For the case of small G/I, a simple analytical approximation to W, in terms of an action increment, i.e., $W \cong \frac{1}{2}(\Delta A)^2$ was derived. Thus we can represent the random motion of the pendulum as diffusion in action space, in a force field determined by a potential which is linear in D_E and harmonic in D_s, as shown in Fig. 5. In other cases, a straightforward numerical procedure[41] can be employed to solve the characteristic equations (3.11) for W. Our method is readily generalized to other types of noise, e.g., to shot noise, to nonlinear dissipation, and to higher-dimensional systems, e.g., to the dc SQUID.

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APPENDIX A; ASYMPTOTIC REPRESENTATION OF THE NONEQUILIBRIUM STEADY-STATE TRAJECTORY

The differential equation of the trajectories of (2.1) in phase space is

$$\frac{dy}{dx}=-G+\frac{I-\sin x}{y} \tag{A1}$$

and the trajectory S is the unique (cf. Ref. 34) 2π-periodic solution of (A1). For small G we construct this solution in the form

$$y\sim\frac{y_{-1}}{G}+y_0+Gy_1+\cdots \; . \tag{A2}$$

Inserting (A2) into (A1), comparing coefficients of like powers of G and using the periodicity condition, we find the coefficients $y_{-1},y_0,y_1\ldots$, thus obtaining (2.6).

To find $I_{\min}(G)$ and $G_M(I)$ we set $I=GI_0$ for G close to $G_M(I)$, write y in the form

$$y\sim y_0+Gy_1+\cdots \; , \tag{A3}$$

and insert (A3) into (A1). The first term y_0 satisfies the undamped and unforced pendulum equation, so that

$$y_0=[1(\cos x+1)]^{1/2}=2\left|\cos\frac{x}{2}\right| \; . \tag{A4}$$

In the next order we obtain the periodicity condition

$$2\pi I_0=\int_0^{2\pi} y_0 dx=8 \; .$$

Hence, on the critical trajectory $S=S_c$ that touches the x axis, we have

$$I_0=\frac{4}{\pi} \; ,$$

so that

$$I_{\min}(G)\approx\frac{4G}{\pi} \; , \qquad G_M(I)\approx\frac{\pi}{4}I \tag{A5}$$

for small G. The graph of $I_{\min}(G)$ for all values of G is given in Fig. 1(b).

Next we shall show that S_c touches the separatrix at the unstable equilibrium point and that it has a cusp there. To show this we note first that

$$\dot{W}=-GW_y^2 \tag{A6}$$

for motion on the separatrix. Therefore the minimum of W on the separatrix is achieved at the unstable equilibrium point, toward which the separatrix converges. It follow that the W contour that touches the separatrix does so at the unstable equilibrium point. At this point x_0 we have, by (A1) with $G=G_M$,

$$\begin{aligned} y'(x_0)&=-G_M+\lim_{x\to x_0}\frac{I-\sin x}{y}\\ &=-G_M-\frac{\cos x_0}{y'(x_0)}\\ &=-G_M+\frac{(1-I^2)^{1/2}}{y'(x_0)} \; . \end{aligned}$$

Hence

$$y'(x_0)=\frac{-G_M\pm[G_M^2+4(1-I^2)^{1/2}]^{1/2}}{2} \; , \tag{A7}$$

so the W contour that touches the separatrix has a cusp at x_0. We further note that the slope q of the separatrix at this point is different from either slope of the W contour, since q is given by (A7) with the negative square root, and with G_M replaced by G. The point $(x_0,0)$ is a saddle point for W since by (3.17) we have

$$W_y=0 \text{ at } (x_0,0)$$

so that

$$W_x\to 0$$

as $y \rightarrow 0$ and $x \rightarrow x_0$. Thus W achieves its maximal value at (x_0,y) along any curve that reaches that point from the side of S_c that includes the steady-state trajectory S, and W achieves its minimum value there along the separatrix. If one chooses a coordinate system at $(x_0,0)$, one of whose axes is the separatrix, while the other axis is a curve that satisfies the above-mentioned condition and then crosses smoothly into the domain D_E, then $(x_0,0)$ is clearly a saddle point of W.

APPENDIX B: DERIVATION OF THE SMOLUCHOWSKI-TYPE EQUATION IN W SPACE

It has been shown in Sec. III that ρ_s is essentially a function of W and therefore it is a constant on W contours. Writing the Fokker-Planck equation (3.3) in the form

$$\frac{\partial \rho}{\partial t} = \left\{ -y\frac{\partial \rho}{\partial x} - [-U'(x) - Gy + GW_y]\frac{\partial \rho}{\partial y} \right\} + GW_y\frac{\partial \rho}{\partial y} + GT\frac{\partial^2 \rho}{\partial y^2} + G\rho \,, \tag{B1}$$

we note that the expression in curly braces in (B1) is the time derivative of ρ along a W contour, and therefore it vanishes. Thus, choosing W and x as independent variables in D_s we can rewrite (B1) in the form

$$\frac{\partial \rho}{\partial t} = GTW_y^2\frac{\partial^2 \rho}{\partial W^2} + G(W_y^2 + TW_{yy})\frac{\partial \rho}{\partial W} + G\rho \,. \tag{B2}$$

Averaging (B2) on W contours as defined in (4.1) and noting that $\langle W_y^2 \rangle \cong 2W$ and $\langle w_{yy} \rangle \cong 1$, we obtain the Smoluchowski-type equation

$$\frac{\partial \rho}{\partial t} = 2G\left[TW\frac{\partial^2 \rho}{\partial W^2} + (W + \tfrac{1}{2}T)\frac{\partial \rho}{\partial W} + \frac{\rho}{2} \right] . \tag{B3}$$

A considerable simplification of (B3) is achieved if we choose the action $A(\Gamma)$ as the independent variable in (B3), rather than W. Using the relation $W \simeq \frac{1}{2}[A(\Gamma) - A_0]^2$ [see (3.22)] in (B3), we obtain

$$\frac{\partial \rho}{\partial t} = G\frac{\partial}{\partial A}\left[[A(\Gamma) - A_0]\rho + T\frac{\partial \rho}{\partial A} \right] . \tag{B4}$$

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