Theoretical hyperfine structure of the muonic ^{3}He and ^{4}He atoms

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The hyperfine structures of the muonic ^{3}He atom (^{3}He$\mu^- e^-$) and the muonic ^{4}He atom (^{4}He$\mu^- e^-$) are calculated with the use of variational wave functions. Relativistic and radiative corrections are included. The theoretical hyperfine structure interval is $\Delta\nu = 4166.8 \pm 0.3$ MHz for the muonic ^{3}He atom, and $\Delta\nu = 4465.0 \pm 0.3$ MHz for the muonic ^{4}He atom; these values are in good agreement with the experimental values.

I. INTRODUCTION

The hyperfine structure (hfs) interval $\Delta\nu$ for the muonic helium atom provides an interesting and unusual case of atomic hyperfine structure.[1–3] For the muonic ^{4}He atom (^{4}He$\mu^- e^-$) calculations at different levels of approximation have been carried out by many authors.[1–6] The theoretical value[3,5,6] of $\Delta\nu$, including relativistic and radiative contributions, is in good agreement with the experimental value[7,8] $\Delta\nu_{\text{expt}} = 4465.004(29)$ MHz (6.5 ppm) within the ~ 200 ppm accuracy of the theoretical evaluation of the leading Fermi term for $\Delta\nu$. In this paper, we report on the theoretical hyperfine structure of the muonic ^{3}He atom (^{3}He$\mu^- e^-$) and an improved value for that of the muonic ^{4}He atom.

In both the muonic ^{3}He and ^{4}He atoms, the negative muon is bound closely to the nucleus; in the ground state of the atom, the orbital radius of the muon is about 400 times smaller than that of the electron due to their mass ratio and different charge screenings. Therefore, in the simplest approximation, the muonic helium atom can be considered to be hydrogenlike with a pseudonucleus (^{3}Heμ)$^+$ or (^{4}Heμ)$^+$. The major difference in the hyperfine structure of the ground state of the muonic ^{3}He and ^{4}He atoms arises from the spin and associated magnetic moment of the ^{3}He nucleus. The ^{3}He nucleus (spin $I = \frac{1}{2}$) and μ^- ($I_\mu = \frac{1}{2}$) have a combined total spin $F_1 = 1$ or 0, which interacts magnetically with the electron spin ($J = \frac{1}{2}$) to yield hyperfine levels with different total angular momenta $\vec{F} = \vec{F}_1 + \vec{J}$. Figure 1 shows schematically the hyperfine splitting of the ground states of the ^{3}He$\mu^- e^-$ and ^{4}He$\mu^- e^-$ atoms. For ^{3}He$\mu^- e^-$ the hfs interval between states with $F_1 = 1$ and $F_1 = 0$ arises from the magnetic interaction of μ^- with the ^{3}He nucleus and is not discussed in this paper. For $F_1 = 1$, the hfs interval $\Delta\nu$ between states with $F = \frac{3}{2}$ and $F = \frac{1}{2}$, which arises from the magnetic interaction between the electron and the pseudonucleus (^{3}Heμ^-)$^+$, is the quantity calculated.

II. HYPERFINE SPLITTING OF THE MUONIC ^{3}He ATOM

The method of calculation in this paper follows closely that of our previous paper[3] on the hyperfine structure of muonic ^{4}He. In the Pauli approximation, the hyperfine interaction operator for the ground state of the ^{3}He$\mu^- e^-$ atoms is given in atomic units as

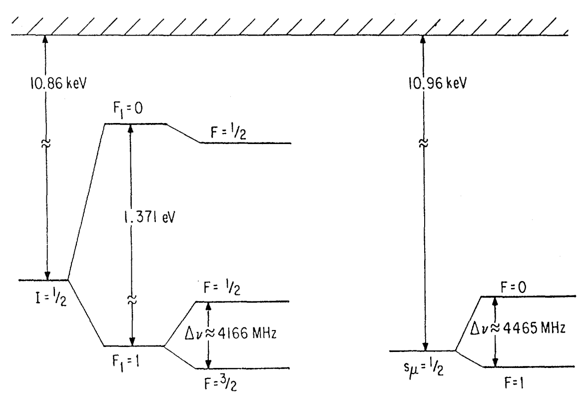

FIG. 1. Schematic level diagrams of the ground states of the ^{3}He$\mu^- e^-$ and ^{4}He$\mu^- e^-$ atoms.

$$H_{\rm hfs} = -\frac{8\pi\alpha^2}{3}\left[\frac{1}{m_\mu}(\vec{\rm J}\cdot\vec{\rm I}_\mu)\delta^3(\vec{\rm r}_{e\mu}) + \mu_N(\vec{\rm I}\cdot\vec{\rm J})\delta^3(\vec{\rm r}_e) + \frac{\mu_N}{m_\mu}(\vec{\rm I}\cdot\vec{\rm I}_\mu)\delta^3(\vec{\rm r}_\mu)\right] . \quad (1)$$

Here m_μ is the muon rest mass, and the magnetic moment μ_N of the ^{3}He nucleus[9] is given in the unit of Bohr magneton $\mu_B(=e\hbar/2m_ec)$. From the hyperfine interaction operator $H_{\rm hfs}$, we can easily deduce the lowest-order hyperfine splitting of the muonic ^{3}He atom as

$$\Delta\nu_F = \Delta\nu_F^{(e\mu)} + \Delta\nu_F^{(e)} , \quad (2)$$

where

$$\Delta\nu_F^{(e\mu)} = \frac{2\pi\alpha^2}{m_\mu}\langle\delta^3(\vec{\rm r}_{e\mu})\rangle , \quad (3)$$

$$\Delta\nu_F^{(e)} = 2\pi\alpha^2\mu_N\langle\delta^3(\vec{\rm r}_e)\rangle . \quad (4)$$

The term $\Delta\nu_F^{(e\mu)}$ accounts for the magnetic interaction between the electron and muon, and the term $\Delta\nu_F^{(e)}$ accounts for that between the electron and ^{3}He nucleus. The size of the first term is about four times that of the second term.

To evaluate the expectation values $\langle\delta^3(\vec{\rm r}_{e\mu})\rangle$ and $\langle\delta^3(\vec{\rm r}_e)\rangle$, we use variational functions which contain interparticle coordinates

$$\psi(\vec{\rm r}_e,\vec{\rm r}_\mu) = \sum^{l+m+n\le\omega} C_{lmn}U_{lmn} , \quad (5)$$

where ω is chosen to have certain selected values, and

$$U_{lmn} = \frac{1}{4\pi}e^{-ar_e/2-br_\mu/2}r_{e\mu}^l r_e^m r_\mu^n . \quad (6)$$

Here the parameters a and b and the coefficients C_{lmn} are determined variationally. As demonstrated in our previous paper,[3] by including more terms with high powers of $r_{e\mu}$ and dropping terms with high powers of r_e and r_μ, we achieved a better convergence of $\Delta\nu_F$ with a given number of terms in the variational wave function. In the present calculation we choose $m,n\le 2$. Allowing complete freedom in varying parameters a, b, and C_{lmn}, we obtain a sequence of variational wave functions with increasing ω. A sequence of $\Delta\nu_F^{(e\mu)}$ and $\Delta\nu_F^{(e)}$ are then evaluated, which are presented in Table I. The converged values are

$$\Delta\nu_F^{(e\mu)} = 3340.5\pm 0.3 \text{ MHz} , \quad (7)$$

$$\Delta\nu_F^{(e)} = 818.0\pm 0.3 \text{ MHz} , \quad (8)$$

where the uncertainties are estimated from the convergence of the values in Table I and from the reason given below. We have studied the convergence of the series with $m,n\le 1$ and $m,n=0$. When these series are used, the advantage in keeping terms $r_{e\mu}^l$ with high power l starts to deteriorate due to less terms with $r_e^m r_\mu^n$. Consequently, the convergence of the series with $m,n\le 1$ and $m,n=0$ becomes worse compared with the series with $m,n\le 2$. It is by studying the fluctuation of extrapolated values from series with $m,n\le 3$, $m,n\le 2$, $m,n\le 1$, and $m,n=0$ that we have assigned an error (0.3 MHz) which is much larger than the apparent degree of convergence of the chosen series ($m,n\le 2$). From the converged values (7) and (8), we obtain the lowest-order hyperfine structure as

TABLE I. Lowest-order hyperfine splittings of the ground states of the muonic ^{3}He atom and the muonic ^{4}He atom for variational wave functions with $l+m+n\le\omega$ and $m,n\le 2$.

ω with $m,n\le 2$	No. of terms	$\Delta\nu_F^{(e\mu)}$	^{3}Heμ^-e^- (MHz)	$\Delta\nu_F^{(e)}$	^{4}Heμ^-e^- $\Delta\nu_F$ (MHz)
25	216	3340.65		817.89	4455.31
27	234	3340.60		817.92	4455.23
29	252	3340.55		817.93	4455.17
31	270	3340.52		817.95	4455.13
33	288	3340.51		817.97	4455.11
35	306	3340.50		817.97	4455.10
37	324	3340.49		817.98	4455.10
39	342	3340.49		817.98	4455.09
41	360	3340.48		817.98	4455.09
Converged values		3340.5±0.3		818.0±0.3	4455.1±0.3

$$\Delta\nu_F = 4158.5 \pm 0.3 \text{ MHz} . \quad (9)$$

This value is consistent with our previous result[10,11] 4157.6±1.0 MHz, which was obtained with a 368-term wave function. Our value is to be compared with other calculations: 4156.2±3.0 MHz by Lakdawala and Mohr[12] and 4154 MHz by Drachman.[13] Including relativistic and radiative corrections, the theoretical hyperfine structure of the muonic ^{3}He atom may be written

$$\Delta\nu = \Delta\nu_F^{(e\mu)} \left[\frac{g_e}{2}\right]\left[\frac{g_\mu}{2}\right](1+\delta_{\rm rel}^{(e)}+\delta_{\rm rel}^{(\mu)}+\delta_{vp}^{(e)}+\delta_{vp}^{(\mu)}+\delta_{\rm binding}+\delta_{\rm recoil}) + \Delta\nu_F^{(e)}\left[\frac{g_e}{2}\right](1+\delta_{\rm rel}^{(e)}+\delta_{vp}^{(e)}+\delta_{\rm binding}+\delta_{\rm recoil}) , \quad (10)$$

where g_e and g_μ are the lepton g factors. We use the relativistic and radiative corrections given in Ref. 3 except for the recoil correction $\delta_{\rm recoil}$, which is obtained from Ref. 4. The theoretical hyperfine structure of the muonic ^{3}He atom is then calculated as

$$\Delta\nu = 4166.8 \pm 0.3 \text{ MHz} . \quad (11)$$

This value is in satisfactory agreement with the recent experimental result[14] 4166.3±0.2 MHz.

III. HYPERFINE SPLITTING OF THE MUONIC ^{4}He ATOM

The hyperfine splitting of the ground state of the muonic ^{4}He atom is[3]

$$\Delta\nu = \Delta\nu_F \left[\frac{g_e}{2}\right]\left[\frac{g_\mu}{2}\right](1+\delta_{\rm rel}^{(e)}+\delta_{\rm rel}^{(\mu)}+\delta_{vp}^{(e)}+\delta_{vp}^{(\mu)} +\delta_{\rm binding}+\delta_{\rm recoil}) , \quad (12)$$

where

$$\Delta\nu_F = \frac{8\pi\alpha^2}{3m_\mu}\langle\delta^3(\vec{r}_{e\mu})\rangle . \quad (13)$$

We note that the $\langle\delta^3(\vec{r}_{e\mu})\rangle$ for the muonic ^{4}He atom is slightly different from that for the muonic ^{3}He atom in (3) due to the reduced-mass effect. A sequence of $\Delta\nu_F$ is evaluated and presented in Table I. The converged value is given as

$$\Delta\nu_F = 4455.1 \pm 0.3 \text{ MHz} . \quad (14)$$

Our previously calculated value was $\Delta\nu_F = 4455.2 \pm 1.0$ MHz, based on the use of trial wave functions with $m,n \le 3$. The use here of wave functions with $m,n \le 2$ has significantly improved the convergence of the sequence. The theoretical hyperfine splitting of the ground state of the muonic ^{4}He atom including relativistic and radiative corrections is calculated as

$$\Delta\nu = 4465.0 \pm 0.3 \text{ MHz} , \quad (15)$$

which is in excellent agreement with the recent experimental value[8]

$$\Delta\nu = 4465.004(29) \text{ MHz } (6.5 \text{ ppm}) . \quad (16)$$

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[1]V. W. Hughes and S. Penman, Bull. Am. Phys. Soc. 4, 80 (1959).
[2]E. W. Otten, Z. Phys. 225, 393 (1969).
[3]K.-N. Huang and V. W. Hughes, Phys. Rev. A 20, 706 (1979); 21, 1071 (1980).
[4]E. Borie, Z. Phys. A 291, 107 (1979).
[5]S. D. Lakdawala and P. J. Mohr, Phys. Rev. A 22, 1572 (1980).
[6]R. J. Drachman, Phys. Rev. A 22, 1755 (1980).
[7]H. Orth, K. P. Arnold, P. O. Egan, M. Gladisch, W. Jacobs, J. Vetter, N. Wahl, M. Wigand, V. W. Hughes, and G. zu Putlitz, Phys. Rev. Lett. 45, 1483 (1980).
[8]C. J. Gardner, A. Badertscher, W. Beer, P. R. Bolton, P. O. Egan, M. Gladisch, M. Greene, V. W. Hughes, D. C. Lu, F. G. Mariam, P. A. Souder, H. Orth, J. Vetter, and G. zu Putlitz, Phys. Rev. Lett. 45, 1168 (1982).
[9]W. L. Williams and V. W. Hughes, Phys. Rev. 185, 1251 (1969).
[10]K.-N. Huang and V. W. Hughes, Bull. Am. Phys. Soc.

26, 547 (1981).
[11]K.-N. Huang and V. W. Hughes, The Ninth International Conference on High-Energy Physics and Nuclear Structure, Versailles, France, 1981 (unpublished).
[12]S. D. Lakdawala and P. J. Mohr, Phys. Rev. A 24, 2224 (1981).
[13]R. J. Drachman, J. Phys. B 14, 2733 (1981).
[14]H. Orth and G. zu Putlitz, private communication.