# Higher harmonic emission by a relativistic electron beam in a longitudinal magnetic wiggler

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(Received 5 February 1982)

The classical limit of the Einstein-coefficient method is used in the low-gain regime to calculate the stimulated emission from a tenuous relativistic electron beam propagating in the combined solenoidal and *longitudinal* wiggler fields $(B_0+\delta B\sin k_0 z)\hat{e}_z$ produced near the axis of a multiple-mirror (undulator) field configuration. Emission is found to occur at all harmonics of the wiggler wave number $k_0$ with Doppler upshifted output frequency given by $\omega=(lk_0V_b+\omega_{cb})(1+V_b/c)\gamma_b^2/(1+\gamma_b^2V_\perp^2/c^2)$, where $l\geq 1$. The emission is compared to the low-gain cyclotron maser with $\delta B=0$ and to the low-gain free-electron laser (operating at higher harmonics) utilizing a transverse linearly polarized wiggler field.

## I. INTRODUCTION

The Lowbitron (acronym for longitudinal wiggler beam interaction) is a novel source of coherent radiation in the centimeter-, millimeter-, and submillimeter-wavelength regions of the electromagnetic spectrum. The radiation is generated by a tenuous, thin, relativistic electron beam with average axial velocity $V_b$ and transverse velocity $V_\perp$ propagating along the axis of a multiple-mirror (undulator) magnetic field. It is assumed that the beam radius is sufficiently small that the electrons experience only the axial solenoidal and wiggler fields given by Eq. (2). The output frequency $\omega$ is upshifted in proportion to harmonics of $k_0V_b$, where $\lambda_0=2\pi/k_0$ is the wiggler wavelength. This offers the possibility of radiation generation at very short wavelengths.

Previously, we have considered this free-electron-laser (FEL) configuration in the high-gain regime using the Maxwell-Vlasov equations to study coherent emission at the fundamental harmonic,[1,2] and at higher harmonics.[3] In this paper, the classical limit of the Einstein-coefficient method is used in the low-gain regime to study stimulated emission at the fundamental and higher harmonics. In Sec. II, we determine the electron orbits in the magnetic field given by Eq. (2). These orbits are then used in Sec. III to determine the spontaneous energy radiated. In Sec. IV, the amplitude gain per unit length is calculated for a cold, tenuous, relativistic electron beam. For sufficiently large magnetic fields, we find that the emission is inherently broadband in the sense that many adjacent harmonics can exhibit substantial amplification. For a device operating as an oscillator, it would be possible to tune the output over a range of frequencies for fixed electron-beam and magnetic-field parameters by changing the optical-mirror separation to correspond to the different harmonics. The low-gain Lowbitron results are compared to the low-gain cyclotron maser and low-gain, higher harmonic FEL utilizing a transverse, linearly polarized wiggler field.

## II. CONSTANTS OF THE MOTION AND ELECTRON TRAJECTORIES

We consider a tenuous, relativistic electron beam propagating along the axis of a combined solenoidal magnetic field and multiple-mirror (undulator) magnetic field with axial periodicity length $\lambda_0=2\pi/k_0$. It is assumed that the beam radius $R_b$ is sufficiently small that $k_0^2R_b^2<1$ and that $k_0^2r^2<1$ is satisfied over the radial cross section of the electron beam. Here, cylindrical polar coordinates $(r,\theta,z)$ are introduced, where $r$ is the radial distance from the axis of symmetry and $z$ is the axial coordinate. For $k_0^2r^2<1$, the axial and radial magnetic field, $B_z^0(r,z)$ and $B_r^0(r,z)$, can be approximated near the axis by[1-3]

$$B_z^0=B_0\left[1+\frac{\delta B}{B_0}\sin k_0z\right]+\tfrac{1}{4}\delta Bk_0^2r^2\sin k_0z\ ,$$
$$B_r^0=-\tfrac{1}{2}\delta Bk_0r\cos k_0z\ , \tag{1}$$

 

where $B_0=\text{const}$ is the average solenoidal field, $\delta B=\text{const}$ is the oscillation amplitude of the multiple-mirror field, and $\delta B/B_0<1$ is related to the mirror ratio $R$ by $R=(1+\delta B/B_0)/(1-\delta B/B_0)$. For present purposes, it is assumed that $k_0R_b$ is sufficiently small that field contributions of the order $k_0r\delta B$ (and smaller) are negligibly small. Therefore, in the subsequent analysis, the axial and radial magnetic fields in Eq. (1) are approximated by

$$B_z^0=B_0\left[1+\frac{\delta B}{B_0}\sin k_0z\right],\qquad B_r^0=0\,. \tag{2}$$

That is, to lowest order, the electron experiences only the axial solenoidal and wiggler field components of the multiple-mirror field.

Assuming a sufficiently tenuous electron beam with negligibly small electron motion in the longitudinal wiggler field given by Eq. (2) is characterized by the four constants of the motion

$$\begin{aligned}&p_z\,,\\&p_\perp^2=(p_r^2+p_\theta^2)\,,\\&\gamma mc^2=(m^2c^4+c^2p_\perp^2+c^2p_z^2)^{1/2}\,,\\&P_\theta=r\left[p_\theta-\frac{e}{c}A_\theta^0(r,z)\right].\end{aligned} \tag{3}$$

Here, $p_z$ is the axial momentum, $p_\perp=(p_r^2+p_\theta^2)^{1/2}$ is the perpendicular momentum, $\gamma mc^2$ is the electron energy, $P_\theta$ is the canonical angular momentum, and $A_\theta^0=(rB_0/2)[1+(\delta B/B_0)\sin k_0z]$ is the vector potential for the axial field $B_z^0$ in Eq. (2). Also, $m$ is the electron rest mass, $-e$ is the electron charge, and $c$ is the speed of light *in vacuo*. Note that $\gamma mc^2=\text{const}$ can be constructed from the constants of the motion, $p_z$ and $p_\perp^2$, which are independently conserved.

For present purposes, it is assumed that the equilibrium electron distribution $f_b^0$ has no explicit dependence on $P_\theta$, and the class of beam equilibria

$$f_b^0=f_b^0(p_\perp^2,p_z) \tag{4}$$

is considered. In order to determine the detailed properties of the growth rate, we make the specific choice of beam equilibrium

$$f_b^0=\frac{n_b}{2\pi p_\perp}\delta(p_\perp-\gamma_b mV_\perp)\delta(p_z-\gamma_b mV_b)\,, \tag{5}$$

where $n_b=\int d^3p\, f_b^0=\text{const}$ is the beam density, the constants $V_b$ and $V_\perp$ are related to $\gamma_b$ by

$$\gamma_b=(1-V_b^2/c^2-V_\perp^2/c^2)^{-1/2}\,,$$

and

$$V_b=\left[\int d^3p\,(p_z/\gamma m)f_b^0\right]\Big/\left[\int d^3p\, f_b^0\right]$$

is the average axial velocity of the electron beam. For this choice of distribution function, the beam equilibrium is cold in the axial direction with effective axial temperature

$$T_\parallel=\frac{\left[\int d^3p\,(p_z-\langle p_z\rangle)(v_z-\langle v_z\rangle)\right]}{\left[\int d^3p\, f_b^0\right]}=0\,,$$

where

$$\langle\psi\rangle\equiv\left[\int d^3p\,\psi f_b^0\right]\Big/\left[\int d^3p\, f_b^0\right].$$

On the other hand, the effective transverse temperature is given by

$$\begin{aligned}T_\perp&=(\tfrac{1}{2})\left[\int d^3p\, p_\perp v_\perp f_b^0\right]\Big/\left[\int d^3p\, f_b^0\right]\\&=\gamma_b mV_\perp^2/2\,.\end{aligned}$$

This thermal anisotropy $T_\perp>T_\parallel$ provides the free-energy source to amplify the radiation.

In order to calculate the spontaneous energy radiated by an electron passing through the magnetic-field configuration given by Eq. (2), we first determine the electron orbits from

$$\frac{dp_x'}{dt'}=-\frac{e}{c}v_y'B_z^0(z')\,, \tag{6}$$

$$\frac{dp_y'}{dt'}=\frac{e}{c}v_x'B_z^0(z')\,, \tag{7}$$

$$\frac{dp_z'}{dt'}=0\,, \tag{8}$$

where $\vec{p}\,'(t')=\gamma m\vec{v}(t')$ and $\gamma=(1+\vec{p}\,'^2/m^2c^2)^{1/2}=\text{const}$. Here, the boundary conditions $\vec{x}\,'(t'=t)=\vec{x}$ and $\vec{p}\,'(t'=t)=\vec{p}$ are imposed, i.e., the particle trajectory passes through the phase-space point $(\vec{x},\vec{p})$ at time $t'=t$. From Eq. (8), the axial orbit is given by

$$p_z'=p_z,\quad z'=z+v_z\tau\,, \tag{9}$$

where $\tau=t'-t$ and $v_z=p_z/\gamma m$ is the constant axial velocity. In order to determine the transverse motion, Eqs. (6) and (7) are combined to give

$$\frac{d}{dt'}v_+'=i\omega_c\left[1+\frac{\delta B}{B_0}\sin(k_0z+k_0v_z\tau)\right]v_+'\,, \tag{10}$$

where $v'_+ = v'_x(t') + iv'_y(t')$, $\omega_c = eB_0/\gamma mc$ is the relativistic cyclotron frequency in the solenoidal field $B_0$, and use has been made of Eq. (9). Integrating Eq. (10) with respect to $t'$ and enforcing

$$v'_+(t'=t) = v_x + iv_y = v_\perp \exp(i\phi) ,$$

where

$$(v_x, v_y) = (v_\perp \cos\phi, v_\perp \sin\phi)$$

is the transverse velocity at $t'=t$, gives

$$v'_+(t') = v_\perp \exp\left[ i\phi + i\omega_c\tau + i\omega_c \frac{\delta B}{B_0} \frac{\cos k_0 z - \cos(k_0 z + k_0 v_z \tau)}{k_0 v_z} \right] . \tag{11}$$

From Eq. (11), it is evident that $p'_\perp(t') = \gamma m \,|\, v'_+(t') \,| = \gamma m v_\perp$ is independent of $t'$, although the individual transverse velocity components, $v'_x(t')$ and $v'_y(t')$, may be strongly modulated by the longitudinal wiggler field $\delta B \sin k_0 z$. Making use of

$$\exp(ib\cos\alpha) = \sum_{m=-\infty}^{\infty} J_m(b) \exp(-im\alpha + im\pi/2) ,$$

Eq. (11) becomes

$$v'_+(t') = v_\perp \exp(i\phi) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_m\left[ \frac{\omega_c}{k_0 v_z} \frac{\delta B}{B_0} \right] J_n\left[ \frac{\omega_c}{k_0 v_z} \frac{\delta B}{B_0} \right] (i)^{n-m} \exp[i(\omega_c\tau + mk_0 v_z \tau)] \exp[i(m-n)k_0 z] , \tag{12}$$

where $J_n(x)$ is the Bessel function of the first kind of order $n$. Integrating Eq. (12) with respect to $t'$ gives for the radius of the electron orbit

$$r'_+(t') - r_+ = v_\perp \exp(i\phi) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_m\left[ \frac{\omega_c}{k_0 v_z} \frac{\delta B}{B_0} \right] J_n\left[ \frac{\omega_c}{k_0 v_z} \frac{\delta B}{B_0} \right] (i)^{n-m} \times \exp[i(m-n)k_0 z] \left[ \frac{\exp[i(\omega_c\tau + mk_0 v_z\tau)] - 1}{i(\omega_c + mk_0 v_z)} \right] , \tag{13}$$

where $r'_+(t') \equiv x'(t') + iy'(t')$. In the absence of wiggler field ($\delta B = 0$), Eq. (13) gives the constant-radius orbit corresponding to simple helical motion in the solenoidal field $B_0$. In the absence of the solenoidal field ($B_0 = 0$), the $m=0$ term in Eq. (13) grows linearly with $\tau$, and the radius of the orbit increases without bound unless the argument of $J_0$ is near a zero of $J_0$, in which case the orbit remains bounded. Also, in the presence of both the solenoidal and wiggler fields, the radius of the orbit grows linearly in $\tau$ for $\omega_c = -mk_0 v_z$ exactly. In the following analysis, it is assumed that the value of $v_z \simeq V_b$ is such that $\omega_c + mk_0 V_b \neq 0$, and the radius of the electron orbit remains bounded.

We remind the reader that in the derivation the approximate orbits in Eqs. (9) and (12) have assumed that $k_0 r \ll 1$ and $\delta B/B_0 \ll 1$, and the (oscillatory) radial magnetic field $B_r^0 = -(\delta B/2) k_0 r \cos k_0 z$ has been approximated by $B_r^0 = 0$ [Eq. (2)]. To determine the range of validity of this approximation, we have also calculated (in an iterative sense) the leading-order corrections to the longitudinal and transverse orbits, treating the magnetic force $(-e/c)\vec{v}\,' \times B_r^0 \hat{e}_r$ as a small correction. It is found that Eqs. (9) and (12) constitute excellent approximations to the axial and transverse orbits provided the inequalities

$$\frac{1}{k_0^2 v_z^2}, \frac{1}{k_0^2 v_\perp^2} \gg \frac{1}{2} \left| \frac{\omega_c}{k_0 v_z} \frac{\delta B}{B_0} \right| \left| \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} J_m\left[ \frac{\omega_c}{k_0 v_z} \frac{\delta B}{B_0} \right] J_l\left[ \frac{\omega_c}{k_0 v_z} \frac{\delta B}{B_0} \right] \times (mk_0 v_z + \omega_c)^{-1} [k_0 v_z + (m+l)k_0 v_z + \omega_c]^{-1} \right|$$

are satisfied. These inequalities are readily satisfied in the regimes of practical interest, including $\delta B/B_0 \ll 1$ and $\omega_c^2/k_0^2 v_z^2 > 1$.

## III. SPONTANEOUS EMISSION COEFFICIENT

The spontaneous emission coefficient $\eta_\omega(\vec{x},\vec{p})$ is the energy radiated by an electron per unit frequency interval per unit solid angle divided by the time $T \simeq L/v_z$ that the electron is being accelerated. Here, $L$ is the axial distance over which the acceleration takes place. It is assumed that the radiation field is right-hand circularly polarized and propagating in the $z$ direction with frequency $\omega$ and wave number $k$ related by $\omega \simeq kc$ in the tenuous beam limit. For observation along the $z$ axis, the spontaneous emission coefficient in the classical limit is given by[4]

$$\eta_\omega = \frac{1}{T}\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c^3 T}\left| \int_0^T d\tau\, \hat{e}_z \times (\hat{e}_z \times \vec{v}\,') \exp i(kz' - \omega\tau) \right|^2 . \tag{14}$$

The orbits in Eqs. (9) and (12) are substituted into Eq. (14), and the integration over $\tau$ is carried out. This gives

$$\eta_\omega = \frac{e^2\omega^2 v_\perp^2}{8\pi^2 c^3 T} \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (i)^n(-i)^l \exp[i(l-n)k_0 z] J_l\left[\frac{\omega_c}{k_0 v_z}\frac{\delta B}{B_0}\right] J_n\left[\frac{\omega_c}{k_0 v_z}\frac{\delta B}{B_0}\right]$$
$$\times \left[\frac{\exp[i(kv_z + lk_0 v_z + \omega_c - \omega)T] - 1}{kv_z + lk_0 v_z + \omega_c - \omega}\right]$$
$$\times \left[\frac{\exp[-i(kv_z + nk_0 v_z + \omega_c - \omega)T] - 1}{kv_z + nk_0 v_z + \omega_c - \omega}\right] . \tag{15}$$

Equation (15) contains terms that (spatially) oscillate on the length scale of the wiggler wavelength $\lambda_0 = 2\pi/k_0$. Since our primary interest is in the *average* emission properties, we average Eq. (15) over a wiggler wavelength, which gives the average spontaneous emission coefficient $\bar{\eta}_\omega$:

$$\bar{\eta}_\omega = \frac{e^2\omega^2 v_\perp^2 T}{8\pi^2 c^3} \sum_{l=-\infty}^{\infty} J_l^2\left[\frac{\omega_c}{k_0 v_z}\frac{\delta B}{B_0}\right] (\sin^2\psi_l)/\psi_l^2 , \tag{16}$$

where

$$\psi_l = (kv_z + lk_0 v_z + \omega_c - \omega)T/2 .$$

In the absence of wiggler field ($\delta B = 0$), only the $l = 0$ term in Eq. (16) survives, and $\bar{\eta}_\omega$ is a maximum for $\psi_0 = 0$ corresponding to cyclotron resonance in the solenoidal field $B_0$. For $\delta B \neq 0$, spontaneous emission occurs at all harmonics of $k_0 v_z$. Maximum emission at each harmonic number $l$ occurs when $\psi_l = 0$ and the argument of $J_l$ is such that $J_l^2$ is a maximum. Even when the argument of the Bessel function gives a maximum value of $J_l^2$ for a particular choice of $l$, the emission in neighboring harmonics can be substantial. Also, for $\delta B \neq 0$, the $\psi_0 = 0$ contribution in Eq. (16) is reduced by the $J_0^2$ factor relative to the $\psi_0 = 0$ emission when $\delta B = 0$.

## IV. AMPLITUDE GAIN IN THE TENUOUS BEAM LIMIT

Making use of the expression for the spontaneous emission $\bar{\eta}_\omega$ in Eq. (16), the amplitude gain per unit length $\Gamma$ can be determined from the classical limit of the Einstein-coefficient method. The amplitude gain per unit length is given by[4] ($\Gamma > 0$ for amplification)

$$\Gamma = \frac{4\pi^3 cF}{\omega^2} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dp_z \int_0^{\infty} dp_\perp p_\perp \bar{\eta}_\omega \frac{\gamma m}{p_\perp} \left[ \left[\frac{\omega}{k} - v_z\right] \frac{\partial f_b^0}{\partial p_\perp} + v_\perp \frac{\partial f_b^0}{\partial p_z} \right] , \tag{17}$$

where $f_b^0(p_\perp^2, p_z)$ is the equilibrium distribution function, $\omega \simeq kc$ has been assumed, $v_z = p_z/\gamma m$ and $v_\perp = p_\perp/\gamma m$ are the axial and transverse velocities, and $\gamma mc^2 = (m^2c^4 + c^2p_z^2 + c^2p_\perp^2)^{1/2}$ is the electron energy. In Eq. (17), a phenomenological filling factor $F$ has been included which describes the coupling of the electron beam to the electromagnetic mode being amplified. The geometric factor $F$ is equal to unity for a uniform electromagnetic plane wave and electron beam with infinite radius. Moreover, for finite-beam cross section, $F$ is equal to unity when the electron beam and radial extent of the radiation field exactly overlap. On the other hand, $F<1$ when the beam radius is less than the radial extent of the radiation field.

Substituting Eqs. (5) and (16) into Eq. (17) and integrating by parts with respect to $p_z$ and $p_\perp$ gives the gain per unit length $\Gamma=\sum_{l=-\infty}^{\infty}\Gamma_l$, where

$$\Gamma_l=\frac{\omega_{pb}^2 LF}{8\gamma_b c^2}\left\{\frac{\sin^2\psi_l}{\psi_l^2}\left[b\left[\frac{V_\perp}{V_b}\right]^2 J_l(b)[J_{l-1}(b)-J_{l+1}(b)]+\left[\frac{V_\perp}{V_b}\right]^2 J_l^2(b)+2(1-c/V_b)J_l^2(b)\right]\right.$$
$$\left.+\frac{L}{2V_b}\left[\frac{V_\perp}{V_b}\right]^2 J_l^2(b)[\omega(-1+V_b/c)+\omega_{cb}]\frac{\partial}{\partial\psi_l}\left[\frac{\sin^2\psi_l}{\psi_l^2}\right]\right\}. \quad (18)$$

Here, $\omega_{pb}^2=4\pi n_b e^2/m$ is the nonrelativistic electron plasma frequency squared, $\omega_{cb}=eB_0/\gamma_b mc$,

$$b=(\omega_{cb}/k_0V_b)(\delta B/B_0),$$

and

$$\psi_l=(kV_b+lk_0V_b+\omega_{cb}-\omega)T/2.$$

Equation (18) is valid only for the case of low gain ($\Gamma L<1$) and $c/\omega L\ll 1$. In order for the line-shape factors proportional to $(\sin^2\psi_l)/\psi_l^2$ in Eq. (18) to be a valid representation of the emission for more general choice of $f_b^0$, it is necessary that any small axial spread in electron momentum ($\Delta p_z$) and small spread in transverse electron momentum ($\Delta p_\perp$) satisfy the inequalities

$$1/L\gg[\omega(1-V_b/c)/c+lk_0]\Delta p_z/\gamma_b mV_b$$

and

$$1/L\gg\omega V_\perp^2\Delta p_\perp/c^2\gamma_b mV_\perp V_b.$$

We first examine Eq. (18) in the absence of wiggler magnetic field, i.e., $\delta B=0$. In this limit, only the $l=0$ term survives, and Eq. (18) gives the gain per unit length for the cyclotron maser instability taking into account a finite interaction length $L$, i.e.,

$$\Gamma_{cm}=\frac{\omega_{pb}^2 LF}{8\gamma_b c^2}\left[[2(1-c/V_b)-V_\perp^2/V_b^2]\frac{\sin^2\psi_0}{\psi_0^2}+\left[\frac{V_\perp}{V_b}\right]^2\frac{\sin 2\psi_0}{\psi_0}\right]. \quad (19)$$

An expression similar to Eq. (19) has been derived previously using the single-particle equations of motion.[5] For exact resonance ($\psi_0=0$), Eq. (19) predicts only absorption of radiation. Also, for $V_\perp=0$ and arbitrary $\psi_0$, Eq. (19) predicts only absorption, as expected. The above expression for $\Gamma_{cm}$ has its maximum value[5] for $\psi_0\simeq\pm3.75$ with the final term in Eq. (19) giving the dominant contribution. Equation (19) is symmetric in $\psi_0$ and gives amplification on either side of $\psi_0=0$. Both transverse and axial electron bunching contribute to Eq. (19) with the axial bunching dominating for the maximum value of $\Gamma_{cm}$. The output frequency is approximately

$$\omega=\omega_{cb}(1+V_b/c)\gamma_b^2/(1+\gamma_b^2V_\perp^2/c^2),$$

which is limited to wavelengths in the centimeter and millimeter range for values of $B_0$ and $\gamma_b$ typically available. For moderately large values of $B_0$ and $\gamma_b$, it may be possible to reach submillimeter wavelengths.

We now examine Eq. (18) in the presence of the wiggler magnetic field, $\delta B\neq 0$. For finite values of $b, l\neq 0$, and assuming $(\partial/\partial\psi_l)(\sin^2\psi_l/\psi_l^2)$ is not negligibly small, the terms in Eq. (18) proportional to $L^2$ are dominant. This gives

$$\Gamma_l\simeq\frac{\omega_{pb}^2L^2F}{16\gamma_bV_bc^2}\left[\frac{V_\perp}{V_b}\right]^2 J_l^2(b)[\omega_{cb}-\omega(1-V_b/c)]\times\frac{\partial}{\partial\psi_l}\left[\frac{\sin^2\psi_l}{\psi_l^2}\right]. \quad (20)$$

Rewriting

$$[\omega_{cb}-\omega(1-V_b/c)]=(2\psi_l/L-lk_0)V_b$$

in Eq. (20) gives

$$\Gamma_l\simeq\frac{\omega_{pb}^2L^2F}{16\gamma_bc^2}\left[\frac{V_\perp}{V_b}\right]^2 J_l^2(b)(2\psi_l/L-lk_0)\times\frac{\partial}{\partial\psi_l}\left[\frac{\sin^2\psi_l}{\psi_l^2}\right]. \quad (21)$$

Typically, $|lk_0|\gg|2\psi_l/L|$. Moreover, since we are interested in output frequencies that are Doppler upshifted, we take $l>0$. As a function of $\psi_l$, the quantity $\Gamma_l$ in Eq. (21) then assumes its maximum value for $\psi_l\simeq1.3$, which gives

$$\Gamma_l^{\max} \simeq \frac{0.54}{16} \frac{\omega_{pb}^2 L^2 F}{\gamma_b c^2} \left[ \frac{V_\perp}{V_b} \right]^2 l k_0 J_l^2(b) , \quad (22)$$

with an output frequency of approximately

$$\omega = \frac{(l k_0 V_b + \omega_{cb})(1 + V_b/c)\gamma_b^2}{(1+\gamma_b^2 V_\perp^2/c^2)} .$$

In the presence of the wiggler magnetic field, it is evident from Eqs. (20) and (21) that the gain per unit length gives only amplification for $\psi_l > 0$. This is in contrast to the case $\delta B = 0$ where amplification occurs for both positive and negative $\psi_0$, symmetric about $\psi_0 = 0$.

Comparing the output frequency with and without the wiggler field, we find that the output frequency for $\delta B \neq 0$ is always greater than that for $\delta B = 0$ and can be substantially larger for $l k_0 V_b > \omega_{cb}$. Taking the ratio of Eq. (22) to the maximum value obtained from Eq. (19), and assuming that the final term in Eq. (19) is dominant, gives

$$\frac{\Gamma_l^{\max}}{\Gamma_{cm}} \simeq l k_0 L J_l^2(b) . \quad (23)$$

Depending on the size of $J_l^2(b)$ in Eq. (23), it is evident that for $k_0 L \gg 1$ and $\delta B \neq 0$, it is possible to obtain a larger or comparable gain to the cyclotron maser, but at a much higher output frequency.

From Eq. (22), depending on the size of $J_l^2$, it is clear that substantial amplification can occur simultaneously in several adjacent harmonics. If $b < 1$, then the small-argument expansion of the Bessel function appearing in Eq. (22) can be used, which shows that $l = 1$ gives the largest amplification. For sufficiently large magnetic field, $b$ can take on values greater than unity. In this case, for a specified value of $l$, several neighboring harmonics can give substantial amplification at different output frequencies. For operation as an oscillator, given values of $k_0$, $V_b$, $V_\perp$, and $\gamma_b$, it would be possible to tune the output over a narrow frequency range by adjusting the mirror locations to correspond to the frequency at a particular harmonic.

As a numerical example, for $b = 1.8$, $J_1^2$ is a maximum, and the first three harmonics can be excited simultaneously with $\Gamma_1/\Gamma_2 = 1.87$ and $\Gamma_1/\Gamma_3 = 11.68$. For $b = 4.2$, $J_3^2$ is a maximum, with $\Gamma_3/\Gamma_1 = 28.3$, $\Gamma_3/\Gamma_2 = 2.89$, $\Gamma_3/\Gamma_4 = 1.44$, and $\Gamma_3/\Gamma_5 = 4.33$. In this case, the first five harmonics can be excited to a significant level. The above values chosen for $b$ require substantial magnetic fields. For example, if $\gamma_b = 2$, $V_b/c = 0.71$, $V_\perp/c = 0.5$, $\delta B/B_0 = \frac{1}{3}$, then $b = 1.8$ requires $\omega_{cb}/ck_0 = 3.83$ or $B_0 = 12.8 k_0$ kg, where $k_0 = 2\pi/\lambda_0$ is expressed in cm$^{-1}$. For the above values of $\gamma_b$, $V_b$, $V_\perp$, and $\delta B/B_0$, the choice of $b = 4.2$ then requires $B_0 = 23 k_0$ kG.

We also note the condition on the electron-beam energy spread in order for Eq. (18) to be valid:

$$\frac{(V_b/c)^2}{k_0 L\, [\, l(1+V_b/c) + \omega_{cb}/ck_0 ]} > \frac{\Delta\gamma}{\gamma_b}$$

places a stringent limitation on the beam energy spread. This condition becomes increasingly difficult to satisfy as the harmonic number $l$ is increased. For the first numerical example given in the preceding paragraph, the beam energy spread must satisfy $0.3/k_0 L(l + 2.1) > \Delta\gamma/\gamma_b$, which for $k_0 L \sim 50$ and $l = 3$ requires $\Delta\gamma/\gamma_b \lesssim 10^{-3}$. For the second numerical example, $0.3/k_0 L(l + 5.2) > \Delta\gamma/\gamma_b$, which for $k_0 L \sim 50$ and $l = 5$, requires $\Delta\gamma/\gamma \lesssim 0.6 \times 10^{-3}$.

An FEL using a transverse, linearly polarized wiggler field with no solenoidal field has been shown theoretically to radiate at odd harmonics, $f = 1, 3, 5, \ldots$, of the wave number $k_0$. In the present notation, the corresponding gain per unit length and output frequency are given by[6]

$$\Gamma_f = \frac{0.54}{16} \frac{\omega_{pb}^2 L^2}{\gamma_b^3 c^2} f k_0 \kappa_f^2 ,$$

$$\omega = \frac{(1+V_b/c) f k_0 \gamma_b^2 V_b}{1 + b^2 \gamma_b^2 V_b^2/2c^2} , \quad (24)$$

where

$$\kappa_f = (-1)^{(f-1)/2} [J_{(f-1)/2}(f\zeta) - J_{(f+1)/2}(f\zeta)] V_b b \gamma_b / c ,$$

$$\zeta = V_b^2 b^2 \gamma_b^2 / 4c^2 (1 + V_b^2 b^2 \gamma_b^2 / 2c^2) .$$

Comparing the growth rate for the case of a longitudinal wiggler to Eq. (24) gives (assuming parameters otherwise the same)

$$\frac{\Gamma_l^{\max}}{\Gamma_f} = \left[ \frac{\gamma_b V_\perp}{V_b} \right]^2 \frac{l}{f} \frac{J_l^2(b)}{\kappa_f^2} , \quad (25)$$

where the longitudinal wiggler output frequency is given by

$$\omega = \frac{(l k_0 V_b + \omega_{cb})(1 + V_b/c)\gamma_b^2}{1 + \gamma_b^2 V_\perp^2/c^2} .$$

For $b < 1$, the $l = f = 1$ term is dominant with $\Gamma_1^{\max}/\Gamma_1 = (V_\perp c/2V_b^2)^2$. Therefore, the transverse wiggler gives a somewhat larger growth rate due to the fact that the longitudinal wiggler operates with an electron beam having larger initial transverse velocity $V_\perp$. Although the growth rate for the transverse wiggler is typically larger, for $\gamma_b^2 V_\perp^2/c^2 \leq 1$ the output frequency for the longitudinal wiggler can be substantially higher than the output frequency for the transverse wiggler FEL. Comparing the gain at higher harmonics, a similar conclusion holds when $\gamma_b^2 V_\perp^2/c^2 \leq 1$.

## V. CONCLUSION

In summary, we have used the classical limit of the Einstein-coefficient method to study in the low-gain regime stimulated emission from a cold, tenuous, thin, relativistic electron beam propagating in the combined solenoidal and longitudinal wiggler fields produced on the axis of a multiple-mirror (undulator) field [Eq. (2)]. The gain per unit length was calculated in Sec. IV and the maximum gain per unit length is given by Eq. (22). Emission was found to occur simultaneously in all harmonics of $k_0$ with the Doppler-upshifted output frequency given by

$$\omega = (lk_0 V_b + \omega_{cb})(1 + V_b/c)\gamma_b^2/(1+\gamma_b^2 V_\perp^2/c^2) \; .$$

For sufficiently large magnetic fields, the emission is inherently broadband in the sense that many adjacent harmonics can exhibit substantial amplification. For $\delta B \neq 0$, it is possible to obtain a larger or comparable growth rate to the low-gain cyclotron maser ($\delta B = 0$), at a much higher output frequency. For $\gamma_b^2 V_\perp^2 \leq c^2$, it was also found that the output frequency can be considerably higher than that of an FEL using a transverse wiggler, although the gain per unit length is typically somewhat smaller.

## ACKNOWLEDGMENTS

This research was supported in part by the Office of Naval Research, and in part by the Air Force Aeronautical Systems Division.

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