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Disappearance of stable convection between free-slip boundaries

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A new secondary instability has been found for nearly parallel rolls that eliminates all wavelengths previously thought to be stable for Prandtl numbers less than 0.301 or 0.782, depending on the circumstances. There is currently no known counterpart to this instability between rigid boundaries.

Rayleigh-Bénard convection is arguably the most thoroughly studied example of how turbulence (i.e., temporal aperiodicity), can arise through a sequence of continuous bifurcations.[1] A number of features of the convective transition are common to a wide class of statically stressed nonequilibrium systems. In particular, as the Rayleigh number R is increased through the first bifurcation point R_c, there is an exchange of stabilities leading to a time-independent state of lower spatial symmetry, i.e., convective rolls, that can assume any wave number $\tilde{q}$ within a band of order $\sqrt{\epsilon}$, $\epsilon=(R-R_c)/R_c$ about a critical value q_0.

The precise limits on the band of locally stable states are set by a variety of secondary instabilities which either add to the "richness" of the problem or constitute an annoying complication depending on one's perspective.[2-4] While careful consideration of such "imperfections" as roll curvature[5] or rigid lateral boundaries[6] can introduce a second rather long-time scale on which the $O(\epsilon^{1/2})$ band collapses, there has been no theoretical reason not to believe such a band exists in an infinite container for straight rolls. In this Brief Report we point out a new secondary instability that follows from the amplitude equations of Ref. 7 and eliminates all stable states between free-slip boundaries for sufficiently small Prandtl numbers P. A plausible but uncontrolled modification of our free-slip equations suggests that this instability might be of interest for rigid boundaries and $P \lesssim O(1)$ although in this case some stable states undoubtedly remain close to onset.

The correct amplitude equations[8] for free-slip boundaries are conveniently phrased in terms of a complex amplitude A, describing the modulation of the roll pattern, and the vertical vorticity Ω_z. In scaled units for rolls parallel to the y axis,

$$\partial_t A = A + (\partial_x - i\partial_y^2)^2 A - |A|^2 A - iB_x A \ , \tag{1a}$$

$$[\partial_y^2 + (\xi_y^2/\xi_x^2)\partial_x^2]\Omega_z = -g\,\partial_y[A^*(\partial_x - i\partial_y^2)A + \text{c.c.}] \ , \tag{1b}$$

where

$$g = 2(1+P)/P^2 \ , \tag{1c}$$

$$B_x = -\partial_y \Phi, \quad [\partial_y^2 + (\xi_y^2/\xi_x^2)\partial_x^2]\Phi = \Omega_z \tag{1d}$$

and we have introduced length scales $\xi_x^{-2} = 3\pi^2\epsilon/8$, $\xi_y^2 = \xi_x/(\sqrt{2}\pi)$. The scale factors applied to the time and field amplitudes are given in Ref. 7. To avoid an unphysical lack of uniformity in the limit of a y-independent perturbation, the full two-dimensional Laplacian appears in (1b) and (1d) even though the x variation is of higher order in ϵ (i.e., $\xi_y^2/\xi_x^2 \sim \epsilon^{1/2}$). There would be no error made by insisting on a strict ϵ expansion provided all variation in the y direction was on a scale set by ξ_y.

To study the linear stability of parallel rolls we set

$$A = (1-q^2)^{1/2}\exp[i(qx+\phi)](1+u) \ ,$$

and linearize in u, B, and the gradients of ϕ. Note that in our units $q=0$ corresponds to the first unstable mode and $q^2=1$ is the parabolic approximation to the curve of R_c versus wave number. The cross-roll and Eckhaus instabilities are unchanged by the inclusion of vertical vorticity and render unstable all $q^2 \gtrsim \frac{1}{3}$ with the precise value depending on the Prandtl number.[2] The zigzag instability eliminates a portion of the states with $q<0$ provided $P>10$ (Ref. 7).

The remaining instabilities have eigenmodes that involve both the x and y wave numbers k_x, k_y; the general condition being [with $L = k_y^2 + (\xi_y^2/\xi_x^2)k_x^2$],

$$[2(1-q^2)+k_x^2+2qk_y^2+k_y^4][k_x^2+2qk_y^2+k_y^4+2g(1-q^2)k_y^4/L^2] - 4k_x^2(q+k_y^2)[q+k_y^2+g(1-q^2)k_y^2/L^2] < 0 \ . \tag{2}$$

The skew-varicose instability found in Ref. 7 follows from the long-wavelength limit of (2), i.e., k_x, $k_y << \min(1,P^{-1/2})$. It destabilizes all $q>0$ but disappears in a finite box as $P\rightarrow\infty$. Both the skew-varicose and zigzag instabilities can be described in terms of an effective equation for the phase since u follows ϕ instantaneously.

One further instability that was overlooked in Ref. 7 occurs only for $k_x\sim O(1)$, $k_y\sim O(|q|)$, and P small. The time scales for u and ϕ are thus comparable. Considering only $q\leqslant 0$, one finds analytically that $q=0$ is unstable for $g\geqslant 3+2\sqrt{2}$ or $P\lesssim 0.782$. The corresponding eigenmode has $k_x^2=2+2\sqrt{2}$ and $k_y^2\sim O(\epsilon^{1/2})$ but otherwise resembles skew varicose since the induced velocity B_x tends to accentuate the bulges in the rolls. The other points in Fig. 1 have to be found numerically. The last stable state disappears at $P\simeq 0.301$. In the more conventional picture of ϵ versus the unscaled wave number, $\tilde{q}=q_0+q/\xi_x$, and for $0.782<P<10$, all $\tilde{q}\leqslant q_0$ within the parabola allowed by the cross-roll instability are stable. For $0.301<P<0.782$ a second parabolic boundary enters the $\tilde{q}<q_0$ region (points within it are unstable), and only values of q between the limits specified in Fig. 1 are stable. Note, too, that the wave vector of the most unstable mode lies well outside the band of stationary states suggesting a complete breakdown in the array of parallel rolls as is characteristic of convection in large containers.

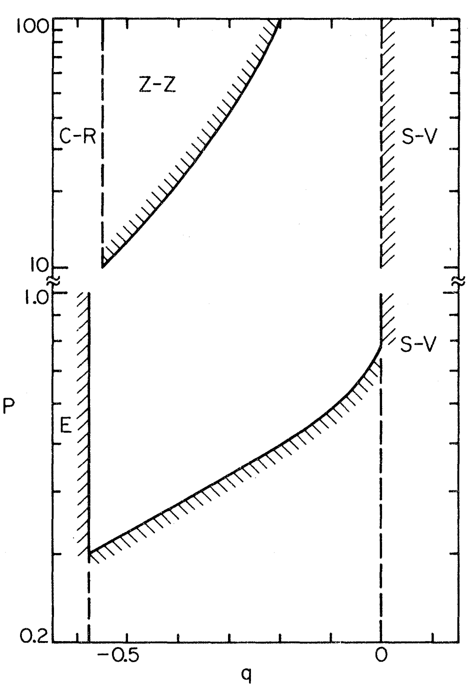

FIG. 1. Stability diagram for free-slip boundaries as a function of a Prandtl number P and a scaled wave number $q=(\tilde{q}-q_0)\xi_x$, where $\tilde{q}$ is in physical units, $q_0=\pi/\sqrt{2}$, and ξ_x is given below Eq. (1). The cross-hatched regions are eliminated by the indicated instabilities; ZZ is zigzag, SV is skew varicose, E is Eckhaus, and CR is cross roll. The skew-varicose line is dashed for large P since an infinitely large container is required as $P\rightarrow\infty$. The lower boundary is new to this paper.

It has by now been noted in a variety of ways that a roll pattern with an $O(1)$ variation in orientation is only stationary for a nearly unique wave number which tends to q_0 at onset.[5] Between free-slip boundaries such a pattern would be expected to become time dependent below $P=0.782$. The analogs of Eqs. (1a)–(1d) have been derived for an approximately circularly symmetric pattern in polar coordinates and yield the same critical Prandtl number at q_0 as in Fig. 1. This is reasonable and should generalize to arbitrary textures that vary on a scale greater than ξ_x since the instability in question occurs at a finite wave number.

To treat rigid boundaries we can modify the various scale factors to leave (1a) invariant[9] and then in a physically plausible but *ad hoc* fashion replace the left-hand side of (1b) by $-\Omega_z$ and replace g by

$$g'=c_1\epsilon^{1/2}\frac{(0.512+P)}{P^2}\times\left[0.699-\frac{0.005}{P}+\frac{0.0083}{P^2}\right]^{-1}, \qquad (3)$$

where c_1 is an unknown constant of order 1.

The strongest argument in favor of our ansatz is its ability to qualitatively reproduce the positions of the skew-varicose and zigzag instabilities computed by Busse and Clever for $P\lesssim O(1)$.[3,4] It also makes apparent why free-slip boundary conditions render nonideal roll patterns time dependent even for $P>O(1)$. If the orientation varies by an amount of order 1 on a scale L then (1b) suggests $\delta B_x\sim O(1)$ for free slip and $O(L^{-2})$ for rigid. A Reynolds number based on L and B_x can thus be large for free-slip but is unavoidably small for rigid boundaries.

When we repeat, for rigid boundaries, the calculation that leads to the lower boundary in Fig. 1, we find a new low-P instability for

$$(1-q^2)g'>-2q+2(1-q^2)^{1/2} \qquad (4)$$

that has no counterpart in the catalog of Clever and Busse.[3,4] At onset, this new instability occurs at $k_x^2=k_y^4=$ infinite so that in physical units $\tilde{k}_y^2/\tilde{k}_x=2q_0$. The most unstable wave number is finite (but large) just above onset so that the perturbation theory we are using is consistent.

According to Clever and Busse[3] for small P, $0.1\leqslant P\leqslant 1.0$, the stable states are confined to $q<0$ by skew-varicose and $\epsilon^{1/2}\lesssim O(P)$ by the oscillatory instability. Equation (4) suggests a further restriction to $\epsilon^{1/2} \lesssim O(P^2)$ with a constant that varies by no more than a factor of 2 for $q < 0$. It is therefore not out of the question that this new instability could be responsible for some of the time dependence seen in moderate-sized cells at small P (Ref. 10).

Lastly we recall that rotating convection between rigid boundaries[11] is analogous to nonrotating free-slip convection at low P in that there are no stable states above R_c. The turbulence that results in only weakly nonlinear and presumably spatially disorganized. Its dynamics may be amenable to theoretical study.

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[1]F. H. Busse, Rep. Prog. Phys. 41, 1929 (1978); and in *Hydrodynamic Instabilities and the Transition to Turbulence,* edited by J. Gollub and H. Swinney (Springer, New York, 1981).

[2]A. Schlüter, O. Lortz, and F. H. Busse, J. Fluid Mech. 38, 279 (1969).

[3]F. H. Busse and R. M. Clever, J. Fluid Mech. 91, 319 (1979); and R. M. Clever and F. H. Busse, Z. Angew. Math. Phys. 29, 711 (1978).

[4]R. M. Clever and F. H. Busse, J. Fluid Mech. 65, 625 (1974).

[5]This follows for the cylindrically symmetric case from the amplitude equations of S. N. Brown and K. Stewartson, Proc. R. Soc. London, Ser. A 360, 455 (1978); see also Y. Pomeau and P. Manneville, J. Phys. (Paris) 42, 1067 (1981); and M. C. Cross and A. Newell (private communication).

[6]M. C. Cross, P. G. Daniels, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. Lett. 45, 898 (1980); and Y. Pomeau and S. Zaleski, J. Phys. (Paris) 42, 515 (1981).

[7]E. D. Siggia and A. Zippelius, Phys. Rev. Lett. 47, 835 (1981).

[8]Our methodology follows A. C. Newell and J. A. Whitehead, J. Fluid Mech. 38, 279 (1969), but we reach different conclusions.

[9]M. C. Cross, Phys. Fluids 23, 1727 (1980).

[10]G. Ahlers and R. W. Walden, Phys. Rev. Lett. 44, 445 (1980); R. P. Behringer, C. Agoste, J. S. Jan, and J. N. Shaumeyer, Phys. Lett. 80A, 273 (1980); and A. Libchaber and J. Maurer, J. Phys. (Paris) Lett. 39, L369 (1978).

[11]K. E. Heikes and F. H. Busse, in *Nonlinear Dynamics,* edited by R. H. G. Helleman (New York Academy of Science, New York, 1980).