Diffusion in discrete nonlinear dynamical systems

S. Grossmann and H. Fujisaka
Fachbereich Physik, Philipps-Universität, D-3550 Marburg, Federal Republic of Germany
(Received 25 February 1982)

Diffusive processes in one-dimensional discrete chaotic systems are considered. Drift and diffusion coefficients are calculated, which show critical behavior including logarithmic corrections. A Kubo formula for the diffusion coefficient in terms of the time-correlation function is given. Nondiffusive states may exist in certain parameter windows showing drift with broken symmetry or strict localization. Period-doubling bifurcations and states of periodic chaos describing chaotic but nondiffusive drift occur.

One-dimensional discrete nonlinear mappings $X_{\tau+1}=F_a(X_\tau)$ of an interval I into itself may have solutions $\{X_\tau(a)\}_{\tau=0}^{\infty}$ which seem to model the dynamics of certain physical systems (for instance, Rayleigh-Bénard cells with small aspect ratio[1] or anharmonic *LRC* circuits[2]). As a function of the parameter a there are stationary states, period-doubling cascades of periodic states,[3,4] as well as chaotic states in which time correlations decay and spectra show broad-band noise.[4,5] Varying a one finds an infinity of nested hierarchies, each starting at some $a_0^{(m)}$ via tangent-type bifurcation with an m-periodic stable state in a certain a interval, followed by the period-doubling cascade $m2^n$, $n=1,2,\ldots$ in adjacent a intervals, converging geometrically[3,4] to an $a_\infty^{(m)}$. For $a>a_\infty^{(m)}$ there is a corresponding sequence of *discrete* parameters $\tilde{a}_n^{(m)}$ at which the trajectories are period-$m2^n$ cycles with superimposed pure chaos.[4] The $\tilde{a}_n^{(m)}$ are also the band-merging points of the corresponding hierarchy. $\tilde{a}_0^{(m)}$ is the end of that hierarchy. In the open intervals between these $\tilde{a}_n^{(m)}$ other such hierarchies appear, etc.

The "width" $\tilde{a}_0^{(m)}-a_0^{(m)}$ of a hierarchy decreases rapidly with increasing order m. For the parabolic map $F_a(X)=4aX(1-X)$ the basic hierarchy (see Fig. 10 of Ref. 4) extends from $a_0^{(1)}=0.25$ to $\tilde{a}_0^{(1)}=1$, the 3 hierarchy from $\hat{a}_0^{(3)}=0.958$ (superstable) to $\tilde{a}_0^{(3)}=0.964$, etc. There is a certain ordering between the different m hierarchies,[6] while each hierarchy itself is governed by scaling parameters δ and α, universally[3] connected with the order of the maximum of $F_a(X)$.

What happens if a is increased further, so that some X are mapped outside the basic interval I? That is the concern of this Brief Report for dynamical laws $F_a(X)$, which are periodic repetitions along the diagonal of the map in [0,1] extended to all real X (see Fig. 1).

If the maximum $a(\mu)$ of F is >1 the phase point can hop to adjacent boxes, $N=\pm1,\pm2,\ldots$; $|\Delta N|\leq[a]$.[7]

Considering the example b the trajectories are irregular not only within one box but also irregularly hopping forwards and backwards between the boxes for almost all initial points X_0. Due to the symmetry $F(1-X)=1-F(X)$ there is no net drift; hopping in either direction occurs equally often. In contrast, the map c gives a mean drift with velocity $0\leq v\leq[a]$. The ergodic and even mixing character of the map (since $|F'|>1$ a.e.) yields a diffusive broadening of the trajectories. The variance increases linearly with t. This will be traced back to the correlation decay by a quantitative treatment of this qualitative description.

But even qualitatively this is not the whole story of deterministic diffusion. Let us consider the map a. Its diffusive behavior was described by Geisel and Nierwetberg[8] at the threshold $a_c(\mu)=1$ (correspond-

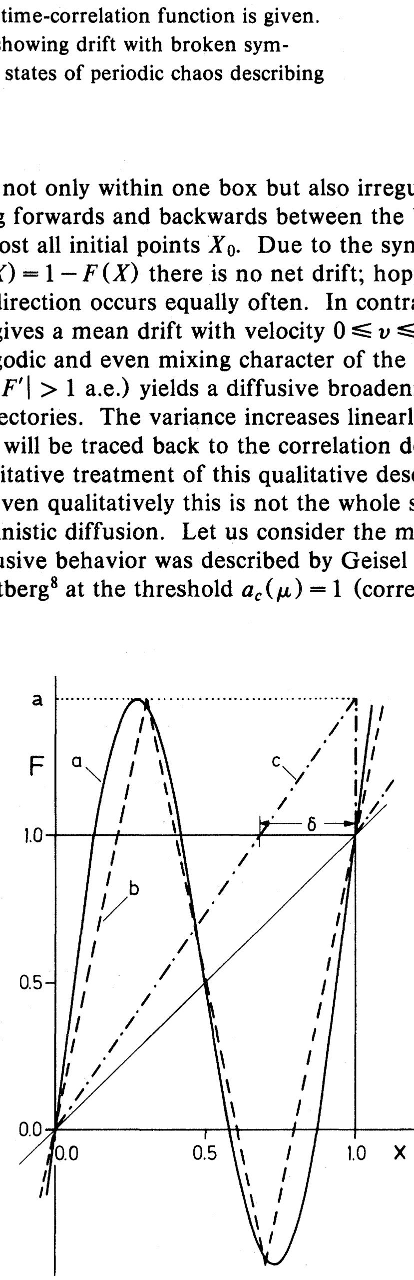

FIG. 1. Unit box of some periodic maps. (a) $F(X)=X+\mu\sin2\pi X$. (b) Sawtooth map, slope $\pm\mu$. (c) Nonsymmetric sawtooth, slope a.

ing to $\mu_c = 0.732\,644\ldots$). They calculated the diffusion coefficient by adding external noise. This hides the vivid structure of the trajectories if $a(\mu) > 1$, in particular the singular nature of a_c near which very regular motions can be stable.

We describe the trajectories by $X_\tau = N_\tau + x_\tau$. N_τ is the box number at time τ, $x_\tau \in [0,1)$ is the position within the box. Each map $F_a(X)$ is equivalent to the coupled dynamical laws

$$N_{\tau+1} = N_\tau + \Delta_a(x_\tau)\,, \quad x_{\tau+1} = g_a(x_\tau) \ . \qquad (1)$$

$\Delta_a(x_\tau)$ is an integer number, describing the magnitude of the jump. $g_a(x_\tau)$ is the fractional part of the phase at $\tau+1$; $0 \leq g_a(x) < 1$. Δ and g are independent of the box number by translational invariance. Note Fig. 2.

The reduced map $g(x)$ determines the character of a trajectory starting from x_0. ($N_0 = 0$ by definition.) If $g(x)$ has stable fixed points, say x^*, there is regular drift $v = \Delta(x^*)$. If it has stable periodic orbits, say period 2, there is regular drift $v = [\Delta(x^*) + \Delta(y^*)]/2$, etc. If $g(x)$ is chaotic it depends on the attractor whether there is regular drift again or $\Delta(x_\tau)$ is an integer stochastic variable, leading to irregular hopping with diffusive broadening of the trajectory. We will give examples for both situations.

In the sinusoidal mapping there are precisely five bands b_i, $i = 1, \ldots, 5$ containing stable period-2 cycles as indicated in Fig. 2. One finds them by looking carefully at $g(x)$ or by considering the second iterate $g^{(2)}(x)$.

b_1 starts with period 2 via tangent-type bifurcation. $x_+^* \in I_1$, $y_+^* \in I_2$ so $v = \frac{1}{2}$ or $x_-^* \in I_5$, $y_-^* \in I_4$ so $v = -\frac{1}{2}$. There is regular drift, a hopping each second step, but no diffusive broadening. There are two attractors each displaying broken drift symmetry; stochastically both together give zero drift, expected from the symmetry of the dynamical law $F_\mu(X)$. Increasing μ leads to a slope-type bifurcation cascade and the corresponding noisy-periodic states. Evidently $g^{(2)}(x)$ near these attractors is conjugate[4] to the parabola map, so the whole family of nested hierarchies governed by δ, α appears and constitutes the band b_1. *All* states are nondiffusive and show broken symmetry $v = 0.5$ or -0.5. The width of the band is $\Delta\mu \approx 0.004$.

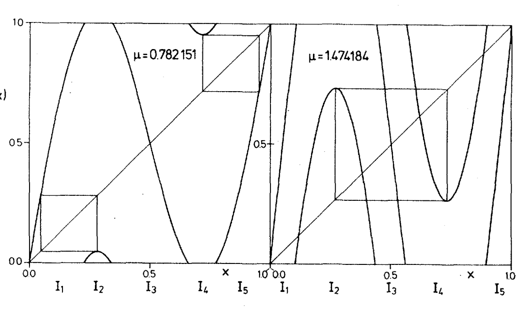

FIG. 2. $g(x)$ for two $a(\mu)$. Stable cycles are indicated. For $1 < a < 2$ the unit interval is decomposed into five I_j.

b_2 starts with an interval $1 \leq \mu < (1+\pi^{-2})^{1/2}$ displaying a fixed point in I_2 and another one in I_4 as separate attractors. We have symmetry breaking, $v = +1$ or -1, respectively. For example, the solution hops forwards from x^* in box N_τ to x^* in box $N_\tau + 1$, etc. x^* bifurcates to a period-2 cycle, etc. The hierarchy band b_2 ends with a purely noisy period-1 state with attractor $(\underline{x} = 0.179\,076$, $\bar{x} = 0.369\,678) \subset I_2$ or $(1-\bar{x}, 1-\underline{x}) \subset I_4$. There is deterministic, regular hopping from $N_\tau \rightarrow N_\tau + 1 \rightarrow N_\tau + 2 \ldots$ but superimposed small scale noise with amplitude $\bar{x} - \underline{x} = 0.190\,602$. The width of the whole hierarchy band b_2 is $\Delta\mu = 0.108\,230$ or $\Delta a = 0.106\,986$.

The structure of the hierarchy bands b_3, b_5 is qualitatively similar to that of b_1. All three are very narrow. The position of all b_i is summarized in Fig. 3.

The hierarchy band b_4 shows an interesting new feature. It starts (tangent-type bifurcation) with a *symmetric* period-2 cycle. Since $\Delta(x^*) = 1$, $\Delta(y^*) = -1$, the phase X_τ is hopping forwards and backwards between two neighboring boxes. The trajectory is strictly *localized*. There is no diffusion. (This μ regime is the analog of the first stable b_2 interval with two separate period-1 fixed points.) At $\mu = \frac{3}{2}$ by slope-type bifurcation two period-2 cycles emerge, being asymmetric after the bifurcation. These cycles correspond to the two period-2 cycles which in b_2 bifurcated off the two fixed points. The trajectory remains localized. The total width of b_4 is $\Delta\mu = 0.076\,604$ or $\Delta a = 0.076\,173$.

It is clear that from considering $g_\mu^{(m)}(x)$ one finds hierarchy bands based on cycles of higher periods m. They are expected to be increasingly small with increasing order m. For instance, there is a small period-4 band near $\mu = 0.86$, lying between b_1 and b_2. It is not yet clear how large the measures of the a sets are that belong either to hierarchy bands (localized or with $v \neq 0$, broken symmetry) or to diffusive trajectories [$v = 0$ due to the symmetry of $F_\mu(X)$, but nonzero diffusion coefficient]. Adding noise[8] and averaging over initial values will hide the bands, as it does for maps *within* an interval.[9]

If a is increased beyond $a = 2$, similar hierarchy bands show up again and again, being increasingly smaller (roughly $\propto \mu^{-1}$). The drift velocity may get larger, $|v| \leq [a(\mu)]$, localized states may be more extended.

In contrast to the sinusoidal map the sawtooth maps do not have stable periods if $|F'(X)| > 1$ a.e. To study deterministic diffusion they are particularly suited. The $g(x)$ statistic is ergodic and mixing, certainly at integer a and particular other choices of a. We find the following formulas for the drift and dif-

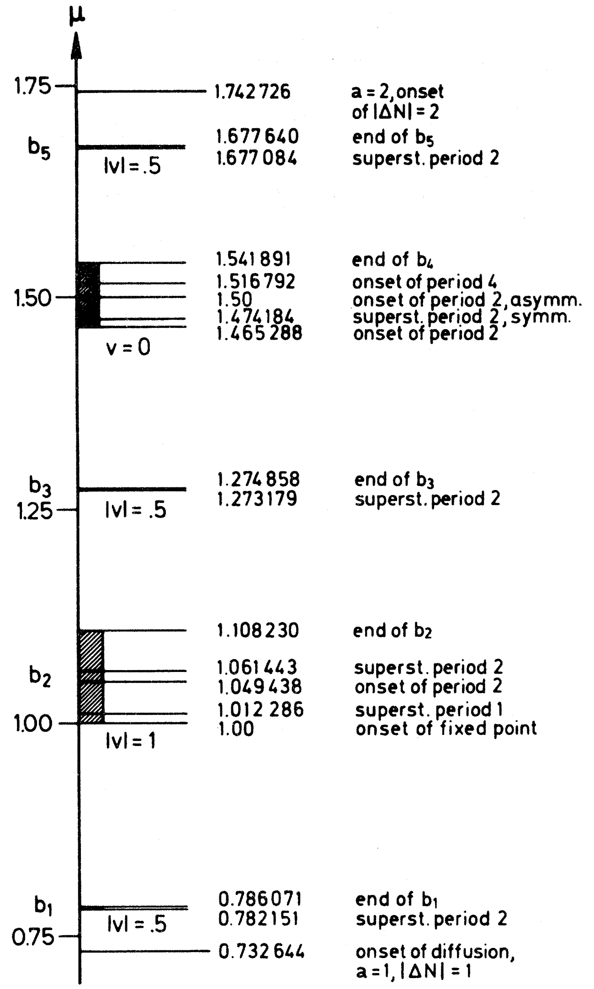

FIG. 3. The nondiffusive hierarchy bands b_i for $1 < a < 2$ containing stable period-2 cycles.

fusion:

$$v = \lim (1/t) \sum_{\tau=0}^{t-1} (X_{\tau+1} - X_\tau) = \langle \Delta(x) \rangle \quad . \tag{2}$$

The average $\langle \cdots \rangle$ has to be taken over the stationary distribution $\rho^*(x)$ of the g map, defined by

$$\rho^*(x) = H\rho^* \equiv \int \delta(g(y) - x)\rho^*(y)dy \quad .$$

H is the Frobenius-Perron operator.

The variance is $\langle (X_t - \langle X_t \rangle)^2 \rangle \cong 2Dt$ for not too small t. The diffusion coefficient can be calculated by the formula ($\delta\Delta \equiv \Delta - \langle \Delta \rangle$)

$$2D = \lim (1/t) \sum_{\tau=0}^{t-1} \sum_{\lambda=0}^{t-1} \langle \delta\Delta(x_\tau)\delta\Delta(x_\lambda) \rangle \quad . \tag{3}$$

In particular, if the jump number $\Delta(x)$ is random with zero memory we find

$$2D = \langle [\delta\Delta(x)]^2 \rangle \quad . \tag{4}$$

A Kubo relation relates the transport coefficient D with the correlation function of $\Delta(x)$. Introducing the normalized correlation function and its Laplace transform for discrete processes[10]

$$C_\tau = \frac{\langle \delta\Delta(x_\tau)\delta\Delta(x_0) \rangle}{\langle [\delta\Delta(x_0)]^2 \rangle} \, , \quad C_z = \sum_{\tau=0}^{\infty} e^{-iz\tau} C_\tau \ ,$$

one obtains

$$2D = \langle [\delta\Delta(x)]^2 \rangle \left[2C_{z=0} + i \int_{-\pi-i\eta}^{+\pi-i\eta} C_z \cot(z/2) dz/2\pi \right] \ .$$

For the map b, due to its symmetry, one has $v = 0$. If a is an integer, $\rho^*(x) = 1$, we find zero memory, and the calculation of the diffusion coefficient by means of (4) is simple. $D = (a-1)a(2a-1)/[3(4a-1)] \approx a^2/6$.

The asymmetric map c shows many interesting properties of deterministic diffusion very clearly. For integer a again $\rho^*(x) = 1$; we find from (2) and (4) that $v = (a-1)/2$ and $D = (a-1)(a+1)/24$. For large a D is $\frac{1}{4}$ of the diffusion coefficient for the symmetric map b. This reflects the reduction of the $\delta\Delta$ scale by 2 in (4).

The onset of diffusion happens at $a_c = 1$. It is $\langle (\delta\Delta)^2 \rangle = \langle \Delta^2 \rangle - \langle \Delta \rangle^2 = \langle \Delta \rangle - \langle \Delta \rangle^2$ since $\Delta = 0$ or 1 if $1 \le a < 2$. Therefore $2D \cong v = \langle \Delta \rangle$.

The diffusion coefficient is given by the length $\delta = (a-1)/a$ of the x interval leading to a jump and the stationary probability to hit it.

$$2D = \delta \times \rho^*(x \in I_\delta) \quad . \tag{5}$$

This general formula covers many special cases. For a parabolic map (as, e.g., a) $\delta \propto (a-1)^{1/2}$. If the maximum is of order $|x - x_m|^z$ we get $\delta \propto (a-1)^{1/z}$. For the sawtooth map $z = 1$, $\delta = (a-1)/a$. The probability density $\rho^*(x \in I_\delta)$ may tend to a finite constant with $\delta \to 0$, i.e., $a \to 1$, as happens for the maps, a,b. We then recover the result[8] $D = \text{const} \times \rho^*(x_m)\,(a - a_c)^{1/z}$.

Corrections show up if ρ^* gets singular for $a \to 1$. This happens for the sawtooth map c.

$$2D = v = (a-1)/\{a[1 - \ln(a-1)]\} \quad .$$

This is correct to lowest order in $a-1$. Corrections due to memory of $\Delta(x)$ are still negligable. [The expression is based on an exact calculation of ρ^* (see Refs. 4 and 10) if $-\ln(a-1)/\ln a$ is integer.]

Note added in proof. While this paper was in press we received a manuscript by M. Schell, S. Fraser, and R. Kapral [Phys. Rev. A 26, 504 (1982)] in which similar conclusions for the sinusoidal map are reported.

One of us (H.F.) thanks the Alexander von Humboldt Stiftung for a grant to stay and study in Germany.

[1]J. Maurer and A. Libchaber, J. Phys. 41, C3-51 (1980).
[2]P. S. Linsay, Phys. Rev. Lett. 47, 1349 (1981); J. Testa, J. Pérez, and C. Jeffries, *ibid.* 48, 714 (1982).
[3]M. J. Feigenbaum, J. Stat. Phys. 19, 25 (1978).
[4]S. Grossmann and S. Thomae, Z. Naturforsch. 32a, 1353 (1977).
[5]S. Thomae and S. Grossmann, J. Stat. Phys. 26, 485 (1981).
[6]N. Metropolis, M. L. Stein, and P. R. Stein, J. Comb. Theor. A 15, 25 (1973).
[7]$[a]$ denotes the largest integer $\leq a$.
[8]T. Geisel and J. Nierwetberg, Phys. Rev. Lett. 48, 7 (1982).
[9]G. Mayer-Kress and H. Haken, J. Stat. Phys. 26, 149 (1981); J. P. Crutchfield and B. A. Huberman, Phys. Lett. 77A, 407 (1980).
[10]H. Fujisaka and T. Yamada, Z. Naturforsch. 33a, 1455 (1978).