Exact dynamical polarizability for one-component classical plasmas

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We calculate the exact lowest-order collisional contribution to the polarizability of a one-component classical plasma. Both approximate analytic and numerical solutions are obtained over a wide range of frequencies and at long wavelengths, with a full taking into account of previously ignored dynamical effects in the screening. We have established a reliable standard against which plasma approximation schemes can be compared in order to assess their accuracy in the weak-coupling limit. We compare the exact solutions with those calculated in our earlier velocity-average-approximation scheme updated to take account of dynamical effects in the screening.

I. INTRODUCTION

A number of years ago a considerable effort was expended in calculating the first-order (in the plasma parameter γ) collisional correction α_1 to the frequency- and wave-number-dependent polarizability $\alpha(\vec{k}\omega)=\epsilon(\vec{k}\omega)-1$ [$\epsilon(\vec{k}\omega)$ is the dielectric response function].[1-6] This effort was motivated by the discovery that, rather than Landau damping, it is the higher-order collisional process which is the dominant mechanism for plasmon decay at long wavelengths.

Oberman, Ron, and Dawson[5] were the first to calculate the conductivity $\sigma=-i\omega\alpha/4\pi$ based on a solution of the first two equations of the Born-Bogolubov-Green-Kirkwood-Yvon (BBGKY) hierarchy; their result, which is restricted to high frequencies, involves only electron-ion correlations in the $k\rightarrow 0$ limit. It was Coste[6] who most accurately resolved the problem over the entire range of ω,k values; he formulated an exact (through order γ) expression for the electron-electron correlational correction to the one-component plasma (OCP) polarizability based on a systematic perturbation expansion of the BBGKY hierarchy.[7]

The renewed interest in Coste's work arises from recent advances in the dynamical theory of strongly coupled plasmas: there is now a need to assess the accuracy of strong-coupling approximation schemes through comparison with known exact results in the weak-coupling limit. For example, the OCP dynamical polarizability formulated in the velocity-average-approximation (VAA)[8] and then evaluated in this limit has been compared with the polarizability calculated from Coste's formulation.[8] In particular, the question as to whether the slope of the plasmon dispersion curve increases or decreases with γ near $\gamma=0$ (Refs. 9–11) emerges as an important new aspect of the exact theory. In the light of this, accurate small γ data are needed, not only for the imaginary part of $\alpha(\vec{k}\omega)$, but for its real part as well. The main objective of the present paper then is to calculate exact values for the OCP polarizability over a wide range of frequencies and at long wavelengths. This work completely generalizes our earlier (exact and VAA) polarizability calculations where only the static effects of screening were considered; here, we shall fully retain the dynamical effects in the screening. The calculation is based on the long-wavelength $k/\omega\rightarrow 0$ expansion. Even though the results are formally given down to $\omega=0$, the approximation is certainly invalid in the domain $\omega<k$. This explains the apparent violation of the

$$\int_0^\infty d\omega\,\omega\,\mathrm{Im}\alpha_1(\omega)=0$$

sum rule by the exhibited positive definite character of $\omega\,\mathrm{Im}\alpha_1(\omega)$. As discussed elsewhere,[11] in this domain the calculated $\mathrm{Im}\alpha_1(\omega)$ values have to be supplemented by a singular contribution that ensures the correct overall behavior.

The outline of the paper is as follows: In Sec. II, the exact OCP polarizability is exhibited in a form more illuminating than Coste's original expression. We briefly review in Sec. III the procedure followed in Ref. 8 where dynamical-screening effects are approximated by static screening to obtain analytical solutions for the first-order polarizability correction in both the exact and VAA cases. In Sec. IV, we numerically evaluate the exact and VAA polarizabilities now, however, with all of the dynamical-screening effects fully retained. The resulting long-wavelength solutions, displayed over a wide range of frequencies, are compared with the analytical static screening results of Sec. III. We shall also see that the VAA reproduces reasonably well almost all the qualitative features of the exact theory.

II. POLARIZABILITY FORMULATION

The Coste expression for the first-order (in γ) correction $\alpha_1(\vec{k}\omega)$ to the OCP random-phase approximation (RPA) polarizability $\alpha_0(\vec{k}\omega)$ can be derived from a perturbation expansion of the first two equations of the BBGKY hierarchy.[7] From Eqs. (3)–(9) of Coste's paper II,[6] one obtains the following convenient formulation, expanded for $k\to 0$ and valid through $O(k^2)$:

$$\alpha_1(\vec{k}\omega)=\gamma\frac{k^2}{\kappa^2}\frac{\omega_0^4}{\omega^4}(V_{\text{stat}}+V_{\text{dyn}})\ , \tag{1}$$

$$V_{\text{stat}}=\tfrac{2}{15}\ , \tag{2}$$

$$V_{\text{dyn}}=V_d+V_e\ , \tag{3}$$

$$V_d(\omega)=\frac{46}{15\pi}\int_0^\infty dx\,x^2\int_{-\infty}^\infty d\mu\,\delta_-(\mu)\hat{\alpha}_0(\vec{p}\mu)\hat{\alpha}_0(\vec{p}\omega-\mu)\ , \tag{4}$$

$$V_e(\omega)=-\frac{16}{15\pi}\frac{\omega_0^2}{\omega^2}\int_0^\infty dx\left[\frac{x^2}{1+x^2}-x^4\int_{-\infty}^\infty d\mu\,\delta_-(\mu)[\hat{\alpha}_0(\vec{p}\mu)\alpha_0(\vec{p}\omega-\mu)+\alpha_0(\vec{p}\mu)\hat{\alpha}_0(\vec{p}\omega-\mu)]\right]\ , \tag{5}$$

where $x=p/\kappa$, κ^{-1} is the Debye length, $\omega_0=(4\pi ne^2/m)^{1/2}$ is the plasma frequency and $\hat{\alpha}_0(\vec{p}\mu)=\alpha_0(\vec{p}\mu)/\epsilon_0(\vec{p}\mu)$ is the external RPA polarizability.

The exact expressions (2), (4), and (5) can be compared with their VAA counterparts

$$V_{\text{stat}}(\text{VAA})=\tfrac{2}{15}\ ,$$

$$V_d(\text{VAA})=\frac{6}{5\pi}\int_0^\infty dx\,x^2\int_{-\infty}^\infty d\mu\,\delta_-(\mu)\hat{\alpha}_0(\vec{p}\mu)\hat{\alpha}_0(\vec{p}\omega-\mu)=\tfrac{9}{23}V_d\ , \tag{6}$$

$$V_e(\text{VAA})=0\ , \tag{7}$$

quoted from Ref. 8 in the weak-coupling limit. Hence, on a structural level, while both the VAA and exact formulas for V_{dyn} share the V_d integral (with different coefficients), the major difference comes from the absence of the V_e integral in the VAA formulation.

III. STATIC SCREENING APPROXIMATION

Analytical, explicitly ω-dependent expressions for V_d and V_e can be derived by invoking the static screening approximation. This approximation, which amounts to the replacement of $\hat{\alpha}_0(\vec{p}\mu)$ and $\hat{\alpha}_0(\vec{p}\omega-\mu)$ in (4) and (5) by

$$\alpha_0(\vec{p}\mu)/\epsilon_0(\vec{p}0)\ \text{and}\ \alpha_0(\vec{p}\omega-\mu)/\epsilon_0(\vec{p}0) \tag{8}$$

has been successfully used by the authors[8] for the calculation of V_d and V_e and had been previously used by Baus[9] in his calculation of correlational effects on OCP plasmon dispersion. It is instructive to elaborate here on the work of Ref. 8, especially for the purpose of gaining insight into what will be the actual behavior of V_d and V_e when dynamical-screening effects are properly accounted for.

Introducing (8) into (4) and (5) gives

$$V_d(\omega)=\frac{46}{15\pi}\int_0^\infty dx\frac{x^6}{(1+x^2)^2}H(\vec{p}\omega)\ , \tag{9}$$

$$V_e(\omega)=-\frac{16}{15\pi}\frac{\omega_0^2}{\omega^2}\int_0^\infty dx\frac{x^2}{1+x^2}[1-2x^4H(\vec{p}\omega)]\ , \tag{10}$$

where the expression for

$$H(\vec{p}\omega)=\int_{-\infty}^{\infty} d\mu\, \delta_{-}(\mu)\alpha_0(\vec{p}\mu)\alpha_0(\vec{p}\omega-\mu)$$

$$=\frac{\kappa^4}{2p^4}\left[1+\frac{1}{\sqrt{2}}\frac{\omega}{\omega_0}\frac{\kappa}{p}Z\left[\frac{\omega/\omega_0}{\sqrt{2}p/\kappa}\right]\right] \qquad (11)$$

in terms of the plasma dispersion function

$$Z(\mu)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz\frac{\exp(-z^2/2)}{z-u-io}$$

results when the μ-integration is carried out according to a procedure originally suggested by Coste[6] (see Appendix A for details). Substituting (11) into (9) and (10) readily gives

$$V_d(\omega)=\frac{23}{15\pi}\left[\frac{\pi}{4}+\frac{1}{2\sqrt{2}}\left[\frac{\omega}{\omega_0}\right]^3 \times\int_0^{\infty}\frac{du\, uZ(u)}{[u^2+\frac{1}{2}(\omega/\omega_0)^2]^2}\right], \qquad (12)$$

$$V_e(\omega)=\frac{16}{15\pi}\frac{1}{2\sqrt{2}}\frac{\omega}{\omega_0}\int_0^{\infty}\frac{du}{u}\frac{Z(u)}{[u^2+\frac{1}{2}(\omega/\omega_0)^2]}, \qquad (13)$$

with

$$u=\frac{1}{\sqrt{2}}\frac{\omega}{\omega_0}\frac{\kappa}{p}.$$

The integrals in (12) and (13) have been further evaluated in Appendix B. We obtain

$$V_d(\omega)=\tfrac{23}{60}\{1-2\eta^2[1-\sqrt{\pi}\eta\,(\exp\eta^2)(1-\mathrm{erf}\eta)]\}-\frac{i}{\sqrt{\pi}}\frac{23}{30}\eta[\eta^2(\exp\eta^2)E_1(\eta^2)-1], \qquad (14)$$

$$V_e(\omega)=-\tfrac{2}{15}\frac{\sqrt{\pi}}{\eta}[1-(\exp\eta^2)(1-\mathrm{erf}\eta)]+\frac{i}{\sqrt{\pi}}\frac{2}{15}\frac{1}{\eta}[E_1(\gamma^2\eta^2)-e^{\eta^2}E_1(\eta^2)], \qquad (15)$$

where $\eta=\dfrac{\omega}{2\omega_0}$,

$$E_1(\eta^2)=\int_{\eta^2}^{\infty}(dt/t)e^{-t}$$

is the modified exponential function, and

$$\mathrm{erf}\eta=(2/\sqrt{\pi})\int_0^{\eta}\exp(-x^2)dx$$

is the error function. The appearance of the γ dependence in the imaginary part of V_e is the result of imposing the cutoff κ/γ on a logarithmically divergent integral on physical grounds. Thus, $\mathrm{Im}V_e$ contains a term proportional to $(\omega_0/\omega)\ln\gamma^{-1}$, which is dominant as $\gamma\rightarrow 0$. This important term, which is present for the same reason when dynamical-screening effects are fully accounted for (Sec. IV) is, however, absent in the VAA.

Figures 1 and 2 show how the real and imaginary parts of the exact and VAA V_{dyn}'s behave as functions of ω in the static screening approximation.

(i) For small values of ω up to about $0.5\omega_0$, the values of $\mathrm{Im}V_{\mathrm{dyn}}$(VAA) are very nearly identical to

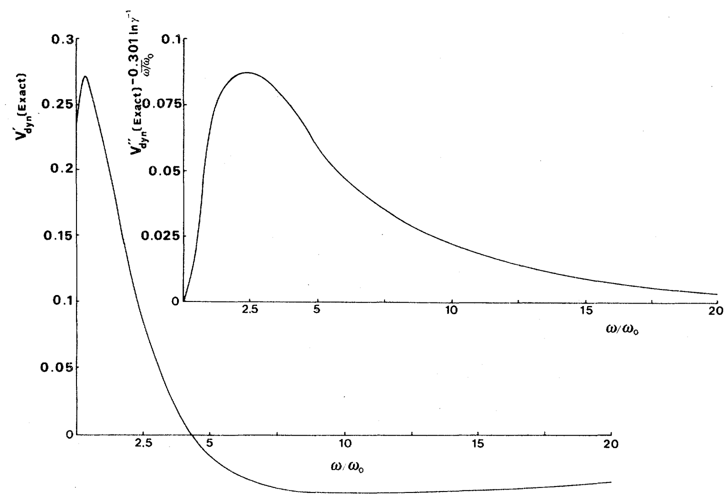

FIG. 1. Plot of the real and imaginary parts of V_{dyn}(exact) as functions of ω/ω_0 in the static screening approximation.

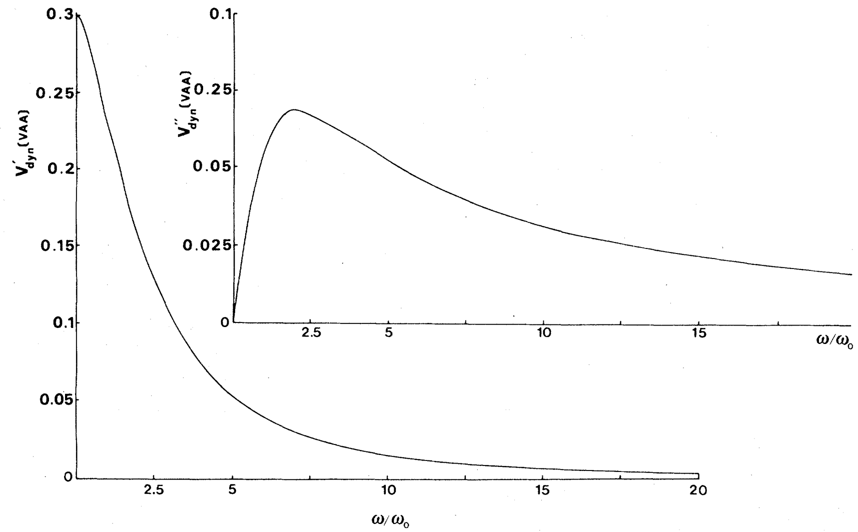

FIG. 2. Plot of the real and imaginary parts of $V_{\rm dyn}$(VAA) as functions of ω/ω_0 in the static screening approximation.

those of

$$\mathrm{Im}\,V_{\rm dyn}(\text{exact})-0.301(\omega_0/\omega)\ln\gamma^{-1}\ .$$

Table I facilitates the comparison between the VAA and exact values of $V_{\rm dyn}$ over the entire frequency range. The absence of the $0.301(\omega_0/\omega)\ln\gamma^{-1}$ term in the imaginary part of $V_{\rm dyn}$(VAA) and, therefore, in the subsequent formula for the damping of the collective modes near $\omega=\omega_0$ is the principal defect of the VAA.

(ii) Near $\omega=\omega_0$, however, the $\mathrm{Re}V_{\rm dyn}$ (exact) and $\mathrm{Re}V_{\rm dyn}$(VAA) are nearly identical. Thus, the change in the dispersion of the plasma oscillations due to finite γ effects is virtually identical in the VAA to what is predicted by the exact theory; both indicate that

$$\Delta=\left.\frac{\partial\omega}{\partial k}\right|_{\gamma\ll 1}-\left.\frac{\partial\omega}{\partial k}\right|_{\gamma=0}<0\ .$$

The negativeness of Δ has been observed as well in recent OCP molecular-dynamics experiments[12] where, however, data are still lacking below $\gamma=4.9$—far above the weak-coupling limit. On the other hand, Baus,[9] in a calculation based on the memory function formalism, found that $\Delta>0$ as a result of his approximate treatment of the BBGKY hierarchy. The positiveness of Δ seemed to be further corroborated through a work by Ichimaru, Totsuji, Tange, and Pines (ITTP)[10] who argued that model-independent considerations require $\Delta>0$ on quite general grounds. We have, nevertheless, demonstrated elsewhere[8,11] that the proofs of ITTP are incorrect and that there is indeed no *a priori* reason to accept $\Delta>0$.

(iii) Going beyond the plasma frequency, it appears that

$$\mathrm{Im}\,V_{\rm dyn}(\text{exact})-0.301(\omega_0/\omega)\ln\gamma^{-1}$$

and $\mathrm{Im}V_{\rm dyn}$(VAA) exhibit peaks near $\omega=2\omega_0$ as expected. Around $\omega=4\omega_0$, the contribution from the real parts of V_d (exact) and V_e (exact) are equal and opposite and $V_{\rm dyn}=0$. For larger values of ω, $\mathrm{Re}V_e$ (exact) dominates since it drops off as $1/\omega$—slower than the $1/\omega^2$ dropoff exhibited by $\mathrm{Re}V_d$ (exact) [and by $V_{\rm dyn}$(VAA) as well]. Near $\omega=10\omega_0$, the real part of V_e (exact) reaches its minimum and then slowly approaches zero. It should be noted that the asymptotic $1/\omega$ dependence of $\mathrm{Re}V(\omega)$

TABLE I. Static screening approximation: Exact and VAA values of $V_{\rm dyn}$ at $\omega=0$, ω_0, $2\omega_0$, and $\omega\gg\omega_0$.

ω	Exact	VAA
$\to 0$	$0.117+i(\omega_0/\omega)0.301\ln\gamma^{-1}$	$0.150+i0.085(\omega/\omega_0)$
ω_0	$0.115+i(0.064+0.301\ln\gamma^{-1})$	$0.116+i0.056$
$2\omega_0$	$0.060+i(0.087+0.150\ln\gamma^{-1})$	$0.076+i0.068$
$\to\infty$	$-0.472(\omega_0/\omega)+i(\omega_0/\omega)(0.778+0.301\ln 2\omega_0\gamma^{-1}/\omega)$	$0.900(\omega_0/\omega)^2+i(\omega_0/\omega)0.338$

which results in the appearance of a $1/|\omega|^5$ term in the asymptotic behavior of $\mathrm{Re}\alpha(\omega)$ violates the expected high-frequency structure which requires that the expansion contain even powers of ω only. This pathological feature is probably a consequence of the inapplicability of the perturbation expansion for $\omega > \omega_0/\gamma$; since it comes from the V_e term, it is absent in $V_{\rm dyn}$(VAA).

IV. DYNAMICAL-SCREENING EFFECTS INCLUDED

We now come to the central task of this paper: the calculation of V_d and V_e when all of the dynamical-screening effects are taken into account. The integrals in (4) and (5) have been evaluated numerically with an accuracy better than one part in one thousand over most of the range in ω. The accuracy of the numerical integration routines was also checked by applying them to (4) and (5) in the static screening approximation and noting excellent agreement between the resulting numerical solutions and the analytical solutions of Sec. III.

Figures 3 and 4 show how $\mathrm{Re}V_{\rm dyn}$(exact) and

$$\mathrm{Im}V_{\rm dyn}(\mathrm{exact}) - 0.301(\omega_0/\omega)\ln\gamma^{-1}$$

behave as functions of ω when dynamical-screening effects are included. The new features are the following.

(i) The real part of $V_{\rm dyn}$(exact) has a local maximum at $\omega = 2.325\omega_0$ and a local minimum at $\omega = 2.425\omega_0$. The imaginary function peaks very sharply at $\omega = 2.375\omega_0$ and has another very shallow peak around $\omega = 3\omega_0$. The sharp peaks in both the real and imaginary parts of $V_{\rm dyn}$(exact) evidently are due to the inclusion of dynamical effects in the screening. More precisely, these structures arise because of the convolution nature of the $\epsilon(\vec{p}\mu)\epsilon(\vec{p}\omega - \mu)$ denominator product in V_d with the p integration averaging the peaks of the integral.

(ii) For small values of ω, the contribution from the V_d term dominates $\mathrm{Re}V_{\rm dyn}$(exact). It consistently exceeds the static screening values by about a factor of 2.

(iii) As $\omega \to 0$,

$$\mathrm{Im}V_{\rm dyn}(\mathrm{exact}) - 0.301(\omega_0/\omega)\ln\gamma^{-1}$$

exhibits a $1/\omega$ dependence originating from the V_e term. This is not especially significant, since the $\omega \to 0$ behavior of $\mathrm{Im}\alpha_1(\omega)$ is already dominated by the similar $1/\omega$ dependence of the dominant $\ln\gamma^{-1}$ term.

The unphysical behavior of the dominant part and of $V_{\rm dyn}$(exact) as $\omega \to 0$ can be explained in terms of the well-known nonuniformity of the γ expansion which is expected to fail both for $\omega < O(\gamma)\omega_0$ and $\omega > O(1/\gamma)\omega_0$.[11]

Figures 5 and 6 illustrate the behavior of the real and imaginary parts of $V_{\rm dyn}$(VAA). Again, the real

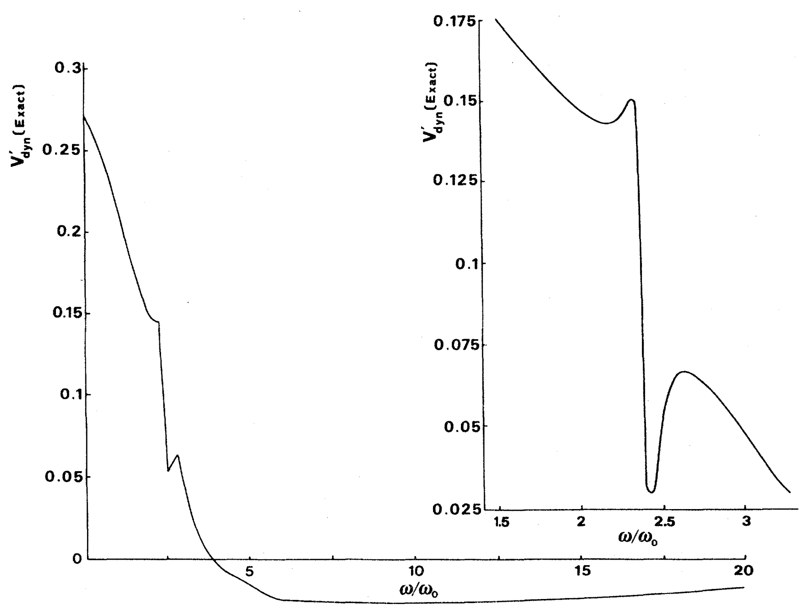

FIG. 3. Plot of the real part of $V_{\rm dyn}$(exact) as a function of ω/ω_0 when dynamical effects are fully included in the screening. Insert shows the more detailed structure around $\omega/\omega_0 = 2$.

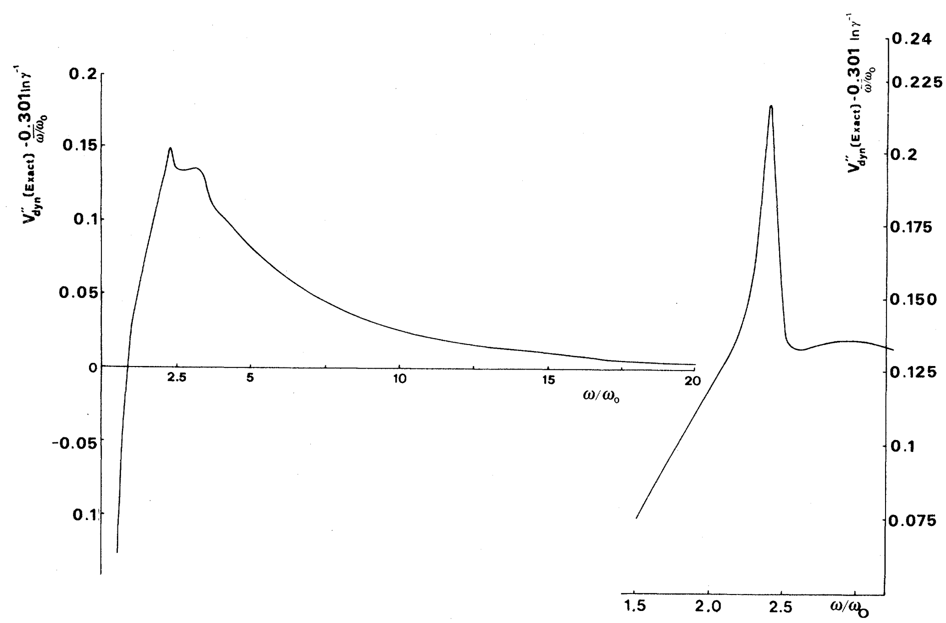

FIG. 4. Plot of the imaginary part of $V_{dyn}(\text{exact})-0.301(\omega_0/\omega)\ln\gamma^{-1}$ as a function of ω/ω_0 when dynamical effects are fully included in the screening. Insert shows the more detailed structure around $\omega/\omega_0=2$.

part of V_{dyn}(VAA) has a local maximum and minimum at $\omega=2.325\omega_0$ and $2.425\omega_0$, respectively, and the imaginary part peaks at $\omega=2.375\omega_0$ in perfect agreement with V_{dyn}(exact). The shapes of both VAA curves around the peaks are qualitatively the same as in the exact case and for the same reason as stated above; but there is a marked difference in the amplitudes.

(iv) For small values of ω, including $\omega=\omega_0$ the contribution from the V_d(VAA) term dominates and there is a difference between the ReV_{dyn}(VAA) and ReV_{dyn}(exact) values which is essentially due to

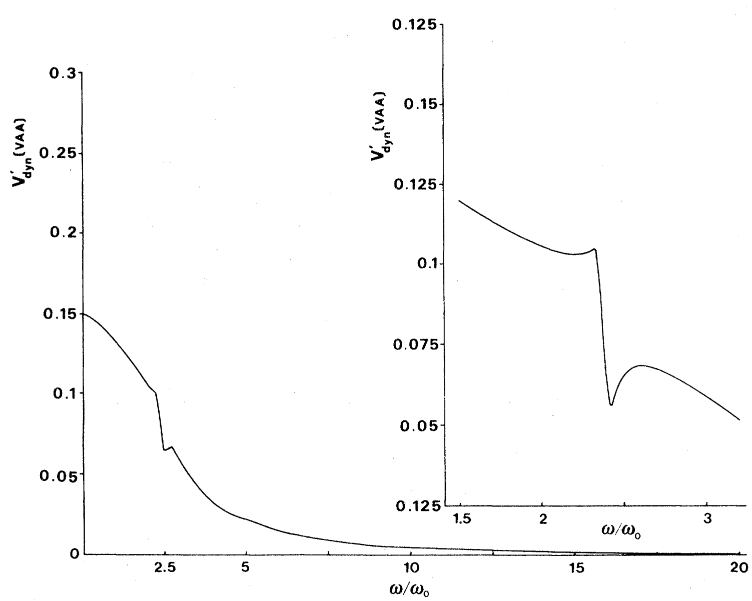

FIG. 5. Plot of the real part of V_{dyn}(VAA) as a function of ω/ω_0 when dynamical effects are fully included in the screening. Insert shows the more detailed structure around $\omega/\omega_0=2$.

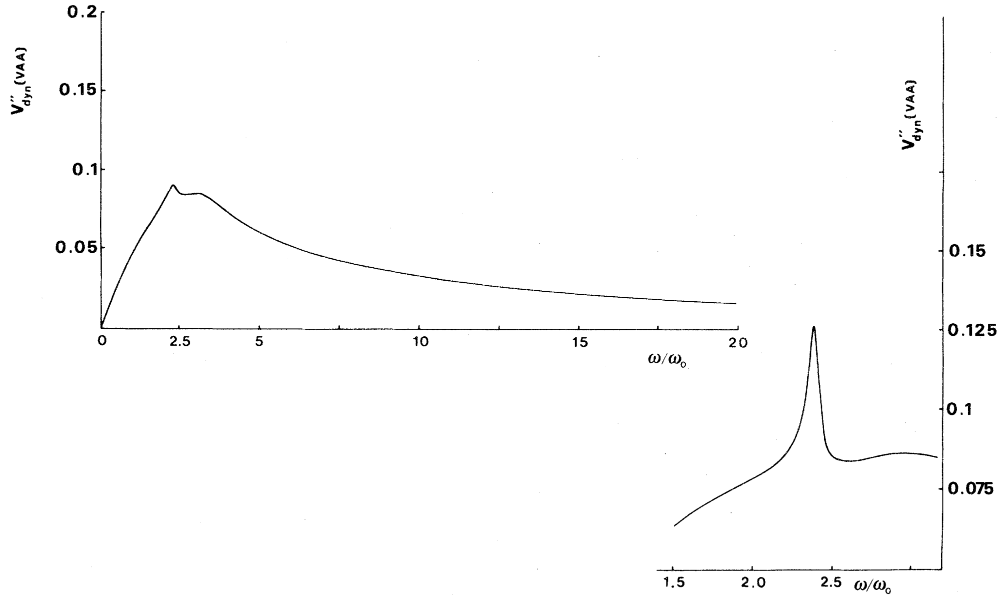

FIG. 6. Plot of the imaginary part of $V_{\rm dyn}$(VAA) as a function of ω/ω_0 when dynamical effects are fully included in the screening. Insert shows the more detailed structure around $\omega/\omega_0=2$.

the difference in the numerical coefficients of the V_d terms.

(v) $\mathrm{Im}V_{\rm dyn}(\mathrm{VAA})\rightarrow 0$ as $\omega\rightarrow 0$ as a consequence of the absence of the V_e term.

Table II facilitates the comparison between the VAA and exact values of $V_{\rm dyn}$ when all of the dynamical-screening effects are retained.

The absence of the 0.301 $(\omega_0/\omega)\ln\gamma^{-1}$ term in the imaginary part of $V_{\rm dyn}$(VAA), and therefore in the VAA description of the damping of collective modes near $\omega=\omega_0$, still persists. When dynamical effects are included in the screening, the change in the dispersion of the plasma oscillations due to finite γ effects is no longer identical in the VAA to what is predicted by the exact theory. However, inclusion of the dynamical effects does not alter the important fact that $\Delta<0$ in both theories.

Finally, we might also compare our results for $\mathrm{Im}V_{\rm dyn}$ with the work of DuBois and Gilinsky.[3] The equivalence of their notation with ours is established by

$$\mathrm{Im}V_{\rm dyn}(\omega)=\frac{8}{15\pi}\frac{\omega_0}{\omega}[I(\omega)+\tfrac{23}{16}J(\omega)]\ ,$$

where $I(\omega)$ and $J(\omega)$ are the integrals calculated by DuBois and Gilinsky; the "dominant" term resides in $I(\omega)$. For $\omega\rightarrow 0$ and $\omega\rightarrow\infty$ because of the different handlings of the k cutoff (DuBois and Gilinsky apply an $\hbar$ dependent cutoff), care has to be exercised in identifying the dominant terms. At $\omega=\omega_0$, DuBois and Gilinsky give the calculated values for $\ln\gamma^{-1}=\ln 10$. The comparison is now effected by Table III.

TABLE II. Dynamical-screening effects fully included: Exact and VAA values of $V_{\rm dyn}$ at $\omega=0$, ω_0, $2\omega_0$, and $\omega\gg\omega_0$.

ω	Exact	VAA
$\rightarrow 0$	$0.271+i(\omega_0/\omega)(-0.087+0.301\ln\gamma^{-1})$	0.149
ω_0	$0.214+i(0.030+0.301\ln\gamma^{-1})$	$0.133+i0.046$
$2\omega_0$	$0.148+i(0.119+0.150\ln\gamma^{-1})$	$0.106+i0.078$
$\rightarrow\infty$	$-0.457(\omega_0/\omega)+i(\omega_0/\omega)[0.78+0.301\ln(2\gamma^{-1}\omega_0/\omega)]$	$0.41(\omega_0/\omega)^2+i(\omega_0/\omega)0.338$

TABLE III. Dynamical-screening effects fully included: Exact and DuBois-Gilinsky values of $\mathrm{Im}V_{\mathrm{dyn}}$ at $\omega=0$, ω_0, and $\omega \gg \omega_0$.

ω	Exact	DuBois-Gilinsky
$\rightarrow 0$	$(-0.087+0.301\ln\gamma^{-1})\frac{\omega_0}{\omega_0}$	$(-0.030+0.301\ln\gamma^{-1})\frac{\omega_0}{\omega}$
ω_0	0.723	0.888
$\rightarrow\infty$	$\frac{\omega_0}{\omega}[+0.780+0.301\ln(2\gamma^{-1}\omega_0/\omega_0)]$	$[0.898+0.301\ln(2\gamma^{-1}\omega_0/\omega)]\frac{\omega_0}{\omega_0}$

V. CONCLUSIONS

We have evaluated an exact first-order (in γ) expression for the correlational correction to the OCP RPA polarizability. Our numerical solutions, obtained over a wide range of frequencies and valid through $O(k^2)$ at long wavelengths, fully take into account the previously ignored dynamical effects in the screening. In addition to the intrinsic interest of the results, we have established a reliable standard against which plasma approximation schemes can be compared in order to assess their accuracy in the weak-coupling limit. Comparison of the exact solutions with updated (to account for dynamical effects in the screening) VAA solutions reveals that, apart from the absence of the $\gamma\ln\gamma^{-1}$ term in the damping, almost all the other important correlational and long-time effects are reasonably well reproduced by the VAA. While the change in the dispersion of the plasma oscillations due to finite γ effects is no longer identical in the updated VAA to what is predicted by the similarly updated exact theory, inclusion of the dynamical effects, nevertheless does not alter the important fact that

$$\Delta=\left.\frac{\partial\omega}{\partial k}\right|_{\gamma\ll 1}-\left.\frac{\partial\omega}{\partial k}\right|_{\gamma=0}<0$$

in both theories.

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APPENDIX A: EVALUATION OF $H(\vec{p}\omega)$

Exploiting the fact that

$$\alpha(\vec{p}\mu)=\alpha'(\vec{p}\mu)+i\alpha''(\vec{p}\mu)$$
$$=\alpha^*(\vec{p}\mu)+2i\alpha''(\vec{p}\mu)$$

is a plus function and therefore $\alpha^*(\vec{p}\mu)$ as well as $(\vec{p}\omega-\mu)$ are minus functions of μ, and that $\alpha''(\vec{p}0)=0$, we have from Eq. (11) that

$$H(\vec{p}\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\mu}{\mu}\alpha_0''(\vec{p}\mu)\alpha_0(\vec{p}\omega-\mu)\ . \tag{A1}$$

Introducing the convenient dimensionless variable $w=(\kappa/p)(\mu/\omega_0)$ and noting the RPA polarizability formula

$$\alpha_0(\vec{p}\mu)=(\kappa^2/p^2)[1+wZ(w)] \tag{A2}$$

in terms of the plasma dispersion function $Z(w)=Z'+iZ''$ [see below (11)], Eq. (A1) can be written as

$$H(\vec{p}\omega)=\frac{\kappa^4}{\pi p^4}\left[I_1+\frac{\omega\kappa}{\omega_0 p}I_2\left[\frac{\omega\kappa}{\omega_0 p}\right]+I_3\left[\frac{\omega\kappa}{\omega_0 p}\right]\right]\ , \tag{A3}$$

where

$$I_1=\int_{-\infty}^{\infty}dw\,Z''(w)=\pi\ , \tag{A4}$$

$$I_2(y)=\int_{-\infty}^{\infty}dw\,Z''(w)Z(w+y)\ , \tag{A5}$$

$$I_3(y)=\int_{-\infty}^{\infty}dw\,wZ''(w)Z(w+y)\ . \tag{A6}$$

Following Coste,[6] I_2 and I_3 can be evaluated through the use of the integral representation of Z and interchanging the order of integration

$$I_2(y)=\tfrac{1}{2}\int_{-\infty}^{\infty}dw\,e^{-w^2/2}\int_{-\infty}^{\infty}dt\frac{e^{-t^2/2}}{t-w-y-io}\ , \tag{A7}$$

where o is a positive infinitesimal quantity. With

$$w=\tfrac{1}{2}(z-s)\,,\quad t=\tfrac{1}{2}(z+s)\,,$$

this becomes

$$I_2(y)=\int_{-\infty}^{\infty}dz\,e^{-z^2/4}\int_{-\infty}^{\infty}ds\frac{e^{-s^2/4}}{s-y-io}=\frac{\pi}{\sqrt{2}}Z\left[\frac{y}{\sqrt{2}}\right]. \tag{A8}$$

Similarly,

$$I_3(y)=\tfrac{1}{2}\int_{-\infty}^{\infty}dw\,we^{-w^2/2}\int_{-\infty}^{\infty}dt\frac{e^{-t^2/2}}{t-w-y-io}$$
$$=\tfrac{1}{8}\int_{-\infty}^{\infty}dz\,ze^{-z^2/4}\int_{-\infty}^{\infty}ds\frac{e^{-s^2/4}}{s-y-io}-\tfrac{1}{8}\int_{-\infty}^{\infty}dz\,e^{-z^2/4}-\tfrac{1}{8}y\int_{-\infty}^{\infty}dz\,e^{-z^2/4}\int_{-\infty}^{\infty}ds\frac{e^{-s^2/4}}{s-y-io}$$
$$=-\frac{\pi}{2}\left[1+\frac{y}{\sqrt{2}}Z\left[\frac{y}{\sqrt{2}}\right]\right]. \tag{A9}$$

Eqs. (A4), (A8), and (A9), when substituted into (A3), give Eq. (11).

APPENDIX B: p INTEGRATION OF EQ. (12)

The p integration of Eq. (12) involves the integral

$$J_1(a)=\int_0^{\infty}du\frac{u}{(u^2+a^2)^2}Z(u)=\tfrac{1}{2}\int_0^{\infty}du\frac{1}{u^2+a^2}\frac{d}{du}Z(u)\,, \tag{B1}$$

where $a=\omega/(\sqrt{2}\omega_0)$. By referring to the differential equation

$$\frac{d}{du}Z(u)+uZ(u)+1=0 \tag{B2}$$

and the alternative integral representation

$$Z'(u)=-e^{-u^2/2}\int_0^u dt\,e^{t^2/2} \tag{B3}$$

that the real part of $Z(u)$ satisfies, we transform (B1) into

$$J_1'(a)=\frac{\pi^{3/2}}{2^{5/2}}-\frac{1}{2a}\left[\frac{\pi}{2}+\frac{a^3}{\sqrt{2}}\int_0^{\infty}\frac{du}{u}\frac{e^{-u^2}}{u^2+(a^2/2)}\int_0^u dt\,e^{t^2}\right]. \tag{B4}$$

Baus[9] quotes the integration formula given by Turner

$$\int_0^{\infty}\frac{du}{u}\frac{e^{-u^2}}{u^2+\eta^2}\int_0^u dt\,e^{t^2}=\frac{\pi^{3/2}}{4\eta^2}[1-e^{\eta^2}(1-\mathrm{erf}\eta)]\,. \tag{B5}$$

Substitution of (B5) into (B4) gives the desired result for the real part of the integral

$$J_1'(a)=-\frac{\pi}{4a}+\frac{\pi^{3/2}}{2^{5/2}}e^{a^2/2}\left[1-\mathrm{erf}\left[\frac{a}{\sqrt{2}}\right]\right]. \tag{B6}$$

The imaginary part of $J_1(a)$

$$J_1''(a)=\int_0^{\infty}du\frac{u}{(u^2+a^2)^2}Z''(u)=\left[\frac{\pi}{2}\right]^{1/2}\int_0^{\infty}du\frac{u}{(u^2+a^2)^2}e^{-u^2/2} \tag{B7}$$

is integrable in terms of the modified exponential integral $E_1(\eta)$

$$E_1(\eta)=\int_{\eta}^{\infty}\frac{dt}{t}e^{-t} \tag{B8}$$

with the result

$$J_1''(a)=-\frac{\pi^{1/2}}{2^{3/2}}\frac{1}{a^2}\left[\frac{a^2}{2}e^{a^2/2}E_1\left[\frac{a^2}{2}\right]-1\right]. \tag{B9}$$

Utilizing (B1) and substituting (B6) and (B9) into

(12) yields

$$V_d = \tfrac{23}{60}\{1-2\eta^2[1-\sqrt{\pi}\eta e^{\eta^2}(1-\text{erf}\eta)]\} - \frac{i}{\sqrt{\pi}}\frac{23}{30}\eta[\eta^2 e^{\eta^2}E_1(\eta^2)-1] , \qquad \text{(B10)}$$

where $\eta=(\omega/2\omega_0)=a/\sqrt{2}$. Equation (B10) is the result quoted in Eq. (14). Since Eq. (6) differs from (4) in coefficient only, it follows that

$$V_{\text{dyn}}(\text{VAA}) = \tfrac{3}{20}\{1-2\eta^2[1-\sqrt{\pi}\eta e^{\eta^2}(1-\text{erf}\eta)]\} - \frac{i}{\sqrt{\pi}}\frac{3}{10}\eta[\eta^2 e^{\eta^2}E_1(\eta^2)-1] . \qquad \text{(B11)}$$

Next, the p integration of (13) for V_e involves the integral

$$J_2(a)=\int_0^\infty \frac{du}{u}\frac{1}{u^2+a^2}Z(u) . \qquad \text{(B12)}$$

Through the use of the alternative integral representation for Z', the real part of (B12) becomes

$$J_2'(a)=-\frac{1}{\sqrt{2}}\int_0^\infty \frac{du}{u(u^2+a^2/2)} \times e^{-u^2}\int_0^u dt\, e^{t^2} , \qquad \text{(B13)}$$

which may be integrated through the use of Turner's formula. The result is

$$J_2'(a)=-\frac{\pi^{3/2}}{2\sqrt{2}a^2}\left[1-e^{a^2/2}\left[1-\text{erf}\frac{a}{\sqrt{2}}\right]\right] . \qquad \text{(B14)}$$

The imaginary part of $J_2(a)$

$$J_2''(a)=\int_0^\infty \frac{du}{u(u^2+a^2)}Z''(u) = \left[\frac{\pi}{2}\right]^{1/2}\int_0^\infty \frac{du}{u(u^2+a^2)}e^{-u^2/2} , \qquad \text{(B15)}$$

can be written in the form

$$J_2''(a)=\left[\frac{\pi}{2}\right]^{1/2}\frac{1}{a^2}\left[\int_0^\infty \frac{du}{u}e^{-u^2/2} - \int_0^\infty du\frac{u}{u^2+a^2}e^{-u^2/2}\right] . \qquad \text{(B16)}$$

The first integral in (B16) is divergent and must therefore be cut off at $u_{\min}=\gamma\omega/(\sqrt{2}\omega_0)$ (i.e., $p_{\max}=\kappa/\gamma$) on physical grounds. This is the origin of the familiar $\ln\gamma^{-1}$ term. With the cutoff imposed and with a little algebra, $J_2''(a)$ becomes

$$J_2''(a)=\left[\frac{\pi}{2}\right]^{1/2}\frac{1}{2a^2}\left[\int_{\gamma^2a^2/2}^\infty \frac{dz}{z}e^{-z} - e^{a^2/2}\int_{a^2/2}^\infty \frac{dx}{x}e^{-x}\right] \qquad \text{(B17)}$$

which may then be integrated in terms of the modified exponential integral and yields

$$J_2''(a)=\left[\frac{\pi}{2}\right]^{1/2}\frac{1}{2a^2} \times[E_1(\gamma^2a^2/2)-e^{a^2/2}E_1(a^2/2)] . \qquad \text{(B18)}$$

Utilizing the definition of $J_2(a)$ [Eq. (B12)] and the results (B14) and (B18) in the Eq. (13) expression for V_e gives

$$V_e=-\frac{2}{15}\frac{\sqrt{\pi}}{\eta}[1-e^{\eta^2}(1-\text{erf}\eta)] + \frac{i}{\sqrt{\pi}}\frac{2}{15}\frac{1}{\eta}[E_1(\gamma^2\eta^2)-e^{\eta^2}E_1(\eta^2)] , \qquad \text{(B19)}$$

where $\eta=\omega/2\omega_0$. Equation (B19) is the result quoted in Eq. (15).

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