Stabilization by multiplicative noise

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(Received 25 February 1982)

We consider a dynamical model containing a short-time scale τ_s and a long-time scale τ_l and exhibiting a continuous instability depending on a control parameter. We study how the threshold is shifted if the control parameter is noisy with a correlation time $\tau_c \ll \tau_l$ with the use of both qualitative and systematic methods of adiabatic elimination. We find a noise-induced increase of the threshold of instability depending on the ratio $\lambda = \tau_s/\tau_c$. If the fluctuations of the control parameter are Gaussian, and if $\tau_c/\tau_l \to 0$, $\tau_s/\tau_l \to 0$, on the scale τ_l, the fluctuations act as a Stratonovich noise source for $\tau_c \gg \tau_s$ and as an Itô noise source for $\tau_c \ll \tau_s$. The intermediate regime $\tau_s = \lambda\tau_c$ with arbitrary λ is analyzed and found to be observable by the noise-induced shift of the threshold associated with it.

I. INTRODUCTION

Dynamical systems exhibiting a continuous instability as a function of a certain control or bifurcation parameter have found widespread interest, both theoretically and experimentally, in the current effort to understand the behavior of systems in nonequilibrium steady states. Examples are the careful studies of the photon statistics of lasers near threshold or the investigations of hydrodynamic instabilities near onset, like the Bénard instability or the Taylor instability.

It is well known that close to the threshold of instability such systems are extremely sensitive to even small perturbations. One source of perturbations is frequently of particular importance, since it is associated with the mechanism of instability itself— the perturbations of the control parameter. In many cases, one has only an indirect handle on this parameter, and it is then subject to inevitable random fluctuations, with experimental control only over its mean value and its variance. The question arises how such random fluctuations of the control parameter influence the behavior of the system near instability.

Experimentally, this question has been studied in various systems[1–4] by applying artificial and experimentally controlled broad-banded noise to the control parameter. Even though very different systems were investigated in this way, the results obtained established the following experimental facts.

(i) The transition, or instability, remained sharp after noise was superimposed on the control parameter, the new bifurcation parameter now being the average of the fluctuating control parameter.

(ii) The threshold of instability was shifted to larger values of the bifurcation parameter by the application of noise, with a rough proportionality between shift of threshold and noise. In other words, the noise acted to stabilize the system for a certain regime of the bifurcation parameter. Keeping the bifurcation parameter fixed in this regime, a lowering of the noise intensity therefore drives the system through the threshold of instability.

(iii) The transition point is also distinguished as the point where relaxation and fluctuations of a macroscopic variable exhibit slowing down, i.e., the associated rates vanish at this point.

The theoretical work on this problem can be divided in two classes. The first class consists of numerous papers[5–10] in which the sharp transition in the presence of noise is interpreted as the bifurcation of a "most probable value" of a macroscopic variable described by the maximum of an otherwise broad distribution. It remained unexplained, however, why only the maximum of the distribution should be observed in experiments. Disregarding this serious difficutly, an explanation of the shift of the threshold has been attempted on this basis. In the same line of reasoning, one was then forced to attribute the observed slowing-down effects to the dynamics of the maxima of probability densities.

In the second class of papers[11–15] the transition point was defined as the bifurcation point of the mean value of a macroscopic variable. In these papers simple models with noisy control parameter were studied. These models can be derived from realistic equations by the methods of bifurcation theory, and they are known to describe systems near instabilities, like lasers, Bénard convection, and Couette flow, in a satisfactory way. The noise was simply superimposed on the control parameter and, in view of its broad-banded nature, it was described by a Stratonovich-Gaussian white-noise source.[16] It has been established for these models that the transition remained sharp, and was associated with slowing down under the usual assumption of statistical mechanics that observed values of macroscopic variables correspond to ensemble averages, in agreement with the first and third experimental facts mentioned above. However, it was not yet possible to explain in this way the shift of the threshold. In the models considered, the threshold remained unaffected by noise.

The present paper now offers an explanation of the mechanism by which the threshold is shifted, but remains sharp. It should not be difficult to test this explanation experimentally. The explanation proposed here turns out to be intimately related to the general question of how a physically given broad-banded multiplicative noise source in a system with widely separated times scales should be represented in stochastic calculus. Close to the threshold, only the long-time scales associated with the instability are of interest. It may then be safely assumed that a broad-banded noise source acts like white noise on these long-time scales. Usually, the actual finite band width of the noise source is then invoked in order to justify its *a priori* interpretation as a Stratonovich source.[16] In the present paper we show that this procedure is not always correct. We find that the correct *a priori* interpretation of a physical broad-band noise source on the long-time scales also depends on the comparison of its band width with the short-time scales of the system, even if one is only interested in a reduced description on long-time scales.

In order to obtain these results, we study a simple model with a long-time scale τ_l, associated with a continuous instability, and a short-time scale $\tau_s \ll \tau_l$. The correlation time τ_c of the fluctuations of the control parameter is given by its inverse bandwidth $(\Delta\omega)^{-1}$ and satisfies $\tau_c \ll \tau_l$. For our model we establish that the Gaussian broad-banded noise of the control parameter acts like a Stratonovich noise source on the time scale τ_l, if $\tau_s/\tau_c \rightarrow 0$ for fixed spectral noise intensity Q. In this case no shift of the threhold of instability due to the noise is obtained, in agreement with the earlier theoretical work mentioned above. In the opposite case $\tau_c \ll \tau_s$, for fixed spectral noise intensity Q, the fluctuations of the control parameter act like an Itô noise source on the time scale τ_l, and the threshold of instability is shifted to larger values. All intermediate cases are also found to occur if τ_c and τ_s are of comparable size. The experimental investigation of these different cases seems therefore feasible and very interesting.

In the remainder of this paper, we proceed as follows. In Sec. II we motivate and introduce our model and its stochastic equations of motion. In Sec. III we present a simple qualitative argument which explains, for this model, why the superposition of noise leads to an increase of the threshold of instability proportional to the noise intensity, and why the shift of the threshold depends on the product of the noise bandwidth and the short time τ_s of the sytem. In Sec. IV we present a systematic analysis of the model in the lowest order of an expansion in τ_s/τ_l with τ_s/τ_c fixed and arbitrary. Two methods of adiabatic expansions are used, which agree with each other but differ in complexity and the scope of their results. The results of the second method are presented in the Appendix. Section V contains our conclusions.

II. THE MODEL

We are interested in systems undergoing a continuous instability as a control parameter d_0 is increased. The simplest dynamical model of such a situation is described by an equation of the form

$$\dot{x} = d_0 x - b x^3 , \tag{2.1}$$

where x is a macroscopic variable, $d_0 \lesseqgtr 0$ is the control parameter, and b is a positive constant providing nonlinear stabilization. The threshold of instability of the state $x = 0$ is at $d_0 = 0$. The instability is associated with symmetry breaking, since the states $x = \pm(d_0/b)^{1/2}$ for $d_0 > 0$ are not invariant under the $x \rightarrow -x$ symmetry of (2.1).

Equation (2.1) may be derived for many systems from more basic equations of motion by methods of bifurcation theory (more specifically by the method of adiabatic approximation) which make use of the fact that the time scale $\tau_l = 1/d_0$ close to instability is much larger than all other time scales in the sytem. In the following, it will be necessary to consider explicitly also the effects of the fast relaxation mechanisms on the behavior of the system close to threshold. We therefore extend (2.1) by allowing for inertia of the macroscopic variable and consider the dynamical model

$$\frac{1}{\gamma}\ddot{x} + \dot{x} = dx - bx^3 . \tag{2.2}$$

The parameter d will itself become a dynamical quantity below. For the time being we replace it by the constant bifurcation parameter d_0. Equation

(2.2) then contains the two scales $\tau_s = 1/\gamma$ and $\tau_l = |1/d_0|$. If d_0 is sufficiently close to its threshold value $d_0 = 0$, the relation $\tau_l \gg \tau_s$ is satisfied. Equation (2.1) clearly is an approximation to Eq. (2.2) and is obtained from Eq. (2.2) by adiabatic approximation, if only the dynamics on the long-time scale τ_l are considered.

We are now interested in the case where the parameter d in Eq. (2.2) is subject to fluctuations. We assume that these fluctuations are Gaussian and have a bandwidth $\Delta\omega$ around zero frequency. The bandwidth $\Delta\omega$ is assumed to be large compared to $\tau_l^{-1} = d_0$, i.e., $\Delta\omega \gg d_0$. A convenient model for these fluctuations is a stationary Ornstein-Uhlenbeck process. Thus we have, in addition to Eq. (2.2), the stochastic equations

$$d(t) = d_0 + y(t) , \qquad (2.3)$$
$$\dot{y}(t) = -\Delta\omega y + \sqrt{2\Delta\omega q}\, \xi(t) ,$$

with the new bifurcation parameter $d_0 = \langle d(t) \rangle$, the Gaussian white-noise source $\xi(t)$, satisfying

$$\langle \xi(t) \rangle = 0 , \qquad (2.4)$$
$$\langle \xi(t)\xi(0) \rangle = \delta(t) ,$$

and the correlation function

$$\langle y(t)y(0) \rangle = q e^{-\Delta\omega |t|} . \qquad (2.5)$$

We remark that the Gaussian process $y(t)$ with bandwidth $\Delta\omega$ becomes itself a Gaussian white-noise process at $\Delta\omega \to \infty$ with $q/\Delta\omega$ fixed and has the properties

$$\langle y(t) \rangle = 0 , \qquad (2.6)$$
$$\langle y(t)y(0) \rangle = Q\delta(t)$$

with

$$Q = \frac{2q}{\Delta\omega} . \qquad (2.7)$$

We note that in experiments q and $\Delta\omega$ are the independently controlled parameters. Equations (2.2)–(2.5) completely specify our model. In this paper we are interested only in the long-time behavior of the model on time scales of the order of $|d_0|^{-1}$. In Sec. III we first give a simple qualitative discussion, which contains the essential physics. In Sec. IV a systematic formal analysis is presented.

III. QUALITATIVE ANALYSIS OF THE MODEL

We now give a simple qualitative discussion of the long-time behavior of the model (2.2)–(2.4). The main goal is to show that the threshold of instability remains sharp, but is shifted in the presence of noise. We start by making the hypothesis $x(t) \neq 0$, divide Eq. (2.2) by $x(t)$ and take the average in the time independent ensemble of the statistical steady state. Using the fact that $\langle (d^2/dt^2)\ln x \rangle = 0$, $\langle (d/dt)\ln x \rangle = 0$, in the steady state, we obtain for arbitrary $y(t)$ with $\langle y(t) \rangle = 0$

$$b\langle x^2 \rangle + \frac{1}{\gamma}\left\langle \left[\frac{d\ln x}{dt}\right]^2 \right\rangle = d_0 . \qquad (3.1)$$

Equation (3.1) clearly has no solution for $d_0 < 0$, hence $x(t) \neq 0$ is contradictory in this case, and it follows that $\langle x^2 \rangle = 0$ and

$$x(t) \equiv 0 \quad \text{for } d_0 < 0 \qquad (3.2)$$

is the only solution in the steady state. The same conclusion still holds in the range

$$0 \le d_0 \le \frac{1}{\gamma}\left\langle \left[\frac{d\ln x}{dt}\right]^2 \right\rangle . \qquad (3.3)$$

For

$$d_0 > \frac{1}{\gamma}\left\langle \left[\frac{d\ln x}{dt}\right]^2 \right\rangle \qquad (3.4)$$

Eq. (3.1) is solvable with $\langle x^2 \rangle > 0$. Thus, the threshold remains sharp, but is shifted to values $d_0 \ge 0$ in the presence of noise. We now give an estimate of this shift in two limiting cases.

Case (i): $\gamma \gg \Delta\omega$. In this case the short-time scale $\tau_s = \gamma^{-1}$ of the dynamics governed by (2.2) is much shorter than the correlation time $(\Delta\omega)^{-1}$ of the noise. Therefore, even if the fluctuations of $x(t)$ induced by its coupling to $y(t)$ are allowed for, there exists a long-time regime with $t \gg \tau_s$ where $x(t)$ is independent of transient effects and $|(1/\gamma)\ddot{x}| \ll |\dot{x}|$. In this regime, Eq. (2.2) may therefore be replaced by

$$\dot{x} = d_0 x - b x^3 + x y(t) + O\left[\frac{\Delta\omega}{\gamma}\right] , \qquad (3.5)$$

where terms of order $\Delta\omega/\gamma$ are neglected. Equation (3.5) can now be used to evaluate the right-hand side of (3.4). We obtain

$$\frac{1}{\gamma}\left\langle \left[\frac{d\ln x}{dt}\right]^2 \right\rangle = \frac{1}{\gamma}\langle [d_0 - b x^2 + y(t)]^2 \rangle . \qquad (3.6)$$

Neglecting terms of order d_0/γ on the right-hand side we obtain

$$\frac{1}{\gamma}\left\langle \left[\frac{d\ln x}{dt}\right]^2 \right\rangle = \frac{1}{\gamma}\langle y(t)^2 \rangle = q/\gamma , \qquad (3.7)$$

where use has been made of Eq. (2.5). Thus, the threshold condition (3.4) now reads

$$d_0 \geq \frac{1}{\gamma}\langle y(t)^2\rangle = q/\gamma = \tfrac{1}{2}Q\frac{\Delta\omega}{\gamma} . \quad (3.8)$$

In the limit $d_0/\Delta\omega \to 0$, $y(t)$ in (3.5) becomes the white-noise process (2.6) and (2.7), which must be interpreted in the sense of Stratonovich, because of its finite bandwidth. For this case, the model (3.5) has been solved exactly in earlier work.[11-13] In the limit $1/\gamma \to 0$ for fixed $\Delta\omega$ the shift of the threshold (3.8) vanishes, in agreement with these earlier results. The opposite limit $\Delta\omega \to \infty$ for fixed γ, Q is not yet contained in this analysis and requires separate discussion.

Case (ii): $\Delta\omega \gg \gamma$. Our goal is again to reduce the second-order Eq. (2.2) to a first-order equation of the type (3.5) by making use of $\gamma \gg d_0$. However, there is an obstacle for this procedure: The noise $y(t)$ contains fluctuations of all frequencies up to $\Delta\omega$. In the second-order Eq. (2.2) the fluctuations of $x(t)$ with frequencies larger than γ are damped out on scales $\gamma^{-1} \ll t \ll d_0^{-1}$, i.e., only fluctuations $y(t)$ on time scales longer than $1/\gamma$ are then effective. This important effect would be lost completely if the second-order term in (2.2) is simply dropped. However, we may take this effect into account by cutting off the frequency spectrum of $y(t)$ at γ. That is, we make the replacement

$$y(t) \to y_\gamma(t) \quad (3.9)$$

with

$$\langle y_\gamma(t)\rangle = 0, \quad \langle y_\gamma^2(t)\rangle = \frac{\gamma}{\Delta\omega}\langle y^2(t)\rangle . \quad (3.10)$$

If a first-order equation of the type (3.5) is now anticipated to hold on the basis of $d_0 \ll \gamma$, we obtain the estimate

$$\frac{1}{\gamma}\left\langle \left[\frac{d\ln x}{dt}\right]^2\right\rangle = \frac{1}{\gamma}\langle y_\gamma^2(t)\rangle = \frac{1}{\Delta\omega}\langle y^2(t)\rangle . \quad (3.11)$$

Thus, the threshold condition now reads

$$d_0 \geq \frac{1}{\Delta\omega}\langle y^2(t)\rangle = \frac{q}{\Delta\omega} = \tfrac{1}{2}Q . \quad (3.12)$$

We note the different dependence on the bandwidth compared with (3.8), which should be easy to detect experimentally. The first-order equation incorporating the shift (3.12) and the noise $y_\gamma(t)$ reads

$$\dot{x} = \left[d_0 - \frac{Q}{2}\right]x - bx^3 + xy_\gamma(t) , \quad (3.13)$$

where $y_\gamma(t)$ on the scale d_0 may again be approximated by white noise in the sense of Stratonovich, with

$$\langle y_\gamma(t)\rangle = 0 ,$$
$$\langle y_\gamma(t)y_\gamma(0)\rangle = Q_\gamma\delta(t) . \quad (3.14)$$

Actually $Q_\gamma = Q$. In order to see this we denote

$$q_\gamma = \langle y_\gamma^2(t)\rangle = \frac{\gamma}{\Delta\omega}q ,$$

where $y_\gamma(t)$ is cut off at frequency γ. We then have in analogy to (2.7)

$$Q_\gamma = \frac{2q_\gamma}{\gamma} = \frac{2}{\gamma}\frac{\gamma}{\Delta\omega}q = Q . \quad (3.15)$$

In other words, Q is the noise intensity per frequency interval, which is not affected by the cutoff at γ.

We note, finally, that the long-time behavior in cases (i) and (ii) may be represented by the same formal equation,

$$\dot{x} = d_0 x - bx^3 + xy(t) , \quad (3.16)$$

if Eq. (3.16) is interpreted as a stochastic differential equation in the sense of Stratonovich in case (i), and in the sense of Itô in case (ii). A systematic derivation of this result and the intermediate cases is given in Sec. IV and the Appendix.

IV. SYSTEMATIC ADIABATIC EXPANSION

We now proceed to a systematic analysis of the long-time behavior of the model (2.2)–(2.4). It is useful to introduce the scaled variables

$$\bar{x} = \left[\frac{b}{Q}\right]^{1/2} x, \quad \bar{y} = \left[\frac{1}{\gamma Q}\right]^{1/2} y, \quad \bar{t} = Qt \quad (4.1)$$

and the new parameters

$$a = \frac{d_0}{Q}, \quad \lambda = \frac{\Delta\omega}{\gamma}, \quad \epsilon = \left[\frac{Q}{\gamma}\right]^{1/2} . \quad (4.2)$$

The model is then given by the scaled equations

$$\frac{d\bar{x}}{d\bar{t}} = \frac{1}{\epsilon}p ,$$
$$\frac{dp}{d\bar{t}} = \frac{1}{\epsilon}k(\bar{x}) - \frac{1}{\epsilon^2}[p - g(\bar{x})\bar{y}] , \quad (4.3)$$
$$\frac{d\bar{y}}{d\bar{t}} = -\frac{1}{\epsilon^2}\lambda\bar{y} + \frac{1}{\epsilon}\lambda\xi(\bar{t}) ,$$

with

$$k(\bar{x})=a\bar{x}-\bar{x}^3,\quad g(\bar{x})=\bar{x}\,, \tag{4.4}$$

$$\langle \xi(\bar{t})\rangle=0,\quad \langle \xi(\bar{t})\xi(0)\rangle=\delta(\bar{t})\,.$$

Henceforth, the scaled equations will be used, but all bars are omitted from the notation for simplicity. We are interested in the behavior of the system for large γ and arbitrary λ, d_0, i.e., we consider

$$a=O(1),\quad \lambda=O(1),\quad \epsilon \ll 1\,. \tag{4.5}$$

The Fokker-Planck equation corresponding to Eqs. (4.3) and (4.4) may be written in the form

$$\dot{W}=\frac{1}{\epsilon^2}L(\epsilon)W\,, \tag{4.6}$$

where $W=W(x,y,p,t)$ is the probability density of the process at time t, and $(1/\epsilon^2)L(\epsilon)$ is the Fokker-Planck operator, given by

$$\begin{aligned} L(\epsilon)&=L_0+\epsilon L_1\,,\\ L_0&=\frac{\partial}{\partial p}[p-g(x)y]+\lambda\frac{\partial}{\partial y}y+\tfrac{1}{2}\lambda^2\frac{\partial^2}{\partial y^2}\,,\\ L_1&=-p\frac{\partial}{\partial x}-k(x)\frac{\partial}{\partial p}\,. \end{aligned} \tag{4.7}$$

In the following we will primarily be interested in the properties of the stochastic process $x(t)$ alone, which can be described by the reduced probability distribution $P(x,t)$ defined as

$$P(x,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(x,y,p,t)dp\,dy\,. \tag{4.8}$$

Integrating over the Fokker-Planck equation [Eqs. (4.6) and (4.7)] for the joint probability density W, one immediately obtains the following equation of motion for $P(x,t)$:

$$\frac{\partial}{\partial t}P(x,t)=-\frac{1}{\epsilon}\frac{\partial}{\partial x}j_{1,0}(x,t)\,, \tag{4.9}$$

where the moments $j_{n,m}(x,t)$ are defined as

$$j_{n,m}(x,t)=\int p^n y^m W(x,y,p,t)dp\,dy\,. \tag{4.10}$$

Starting from Eq. (4.9) one can derive a closed equation for $P(x,t)$ in the form of a perturbation expansion in the parameter ϵ. This can be achieved, e.g., by generalizing the method developed by Wilemski[17] for a systematic derivation of the Smoluchowsky equation from the Fokker-Planck equation of Brownian motion. An alternative method is given in the Appendix. It leads to the same conclusions, but also allows for the perturbative construction of the joint probability distribution.

Using Eq. (4.6) and (4.7) we can write down equations of motion for the moments $j_{i,k}$ which assume the following form:

$$\begin{aligned}\frac{\partial}{\partial t}j_{n,m}(x,t)=&-\frac{1}{\epsilon^2}(n+\lambda m)j_{n,m}+\frac{1}{\epsilon}\left[nk(x)j_{n-1,m}-\frac{\partial}{\partial x}j_{n+1,m}\right]\\ &+\frac{1}{\epsilon^2}\left[ng(x)j_{n-1,m+1}+\frac{\lambda^2}{2}m(m-1)j_{n,m-2}\right]\,.\end{aligned} \tag{4.11}$$

Using the notation of Eq. (4.10) we identify

$$j_{0,0}(x,t)=P(x,t) \tag{4.12}$$

and Eq. (4.9) is contained in Eq. (4.11).

In the limit $\epsilon\to 0$ Eq. (4.11) describes a rapidly damped time evolution of the moments $j_{n,m}$ for $n,m\neq 0$ on a time scale ϵ^2 as indicated by the diagonal term in Eq. (4.11). This observation allows us to integrate Eq. (4.11) in the asymptotic time regime, where the arbitrary initial conditions $j_{n,m}(t=0)$ have already disappeared. This can be done conveniently by Laplace transformation or repeated partial integration and one finds

$$j_{n,m}(t)=\sum_{l=0}^{\infty}\left[\frac{-\epsilon^2}{n+\lambda m}\frac{\partial}{\partial t}\right]^l\frac{1}{n+\lambda m}\left[\epsilon\left[nk(x)j_{n-1,m}-\frac{\partial}{\partial x}j_{n+1,m}\right]+\left[ng(x)j_{n-1,m+1}+\frac{\lambda^2}{2}m(m-1)j_{n,m-2}\right]\right]. \tag{4.13}$$

This equation allows us to evaluate the desired moment $j_{1,0}$ up to a given order in ϵ by expressing it in terms of $j_{0,0}$ and its derivatives. The time derivatives which appear in Eq. (4.13) in approximation higher than first order can formally be eliminated by the use of Eq. (4.9). This approach finally produces a series for the moment $j_{1,0}$ of the following formal structure:

$$j_{1,0}(x,t)=\sum_{l=1}^{\infty}\epsilon^l \mathscr{L}^{(l)}\left[x,\frac{\partial}{\partial x}\right]P(x,t) . \tag{4.14}$$

The calculation of the operators $\mathscr{L}^{(l)}(x,\partial/\partial x)$ is straightforward, but can become very tedious for higher orders in ϵ. When we restrict ourselves, however, to the first order in ϵ the calculation becomes extremely simple.

In this order Eq. (4.13) simplifies to

$$j_{1,0}(x,t)=g(x)j_{0,1}(x,t)+\epsilon\left[k(x)j_{0,0}(x,t)-\frac{\partial}{\partial x}j_{2,0}(x,t)\right] . \tag{4.15}$$

In order to evaluate the right-hand side of (4.15), one needs to evaluate $j_{2,0}$ in zeroth order of ϵ and $j_{0,1}$ in first order of ϵ again using Eq. (4.13). The evaluation of $j_{0,1}$ in first order then requires the evaluation of $j_{1,1}$ and $j_{0,2}$ in zeroth order. We find

$$j_{1,0}(x,t)=\epsilon k(x)P(x,t)-\frac{1}{2}\frac{\epsilon}{1+\lambda}\left[g(x)\frac{\partial}{\partial x}+\frac{\partial}{\partial x}g(x)\right]g(x)P(x,t)+O(\epsilon^3) . \tag{4.16}$$

Inserting Eq. (4.16) into Eq. (4.9) generates the desired equation of motion for the reduced probability density $P(x,t)$:

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}\left[k(x)+\frac{1}{2}\frac{1}{1+\lambda}g(x)\frac{\partial g(x)}{\partial x}\right]P(x,t)+\frac{1}{2}\frac{\partial^2}{\partial x^2}g^2(x)P(x,t) . \tag{4.17}$$

Eq. (4.17) has the structure of a Fokker-Planck equation with a somewhat unusual spurious drift. As is clear from the derivation, Eq. (4.17) holds for general $g(x)$ and $k(x)$. It is the basic observation of this paper that the form of the Fokker-Planck equation (4.16), describing the statistical properties of the process $x(t)$ in lowest order in ϵ, is not consistent with either the Itô or the Stratonovich interpretation of the process of Eqs. (2.2) and (2.3) for $\gamma=\infty$, $g(x)=x$, $k(x)=ax-x^3$. These two standard interpretations, however, are contained in Eq. (4.17) as special cases and correspond to the limits $\lambda\to\infty$ and $\lambda\to 0$, respectively.

For the special choice (4.4)

$$k(x)=x(a-x^2) \text{ and } g(x)=x \tag{4.18}$$

the Fokker-Planck equation assumes the form

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}\left[\left[a+\frac{1}{2}\frac{1}{1+\lambda}\right]x-x^3 -\frac{1}{2}\frac{\partial}{\partial x}x^2\right]P(x,t) . \tag{4.19}$$

We note that after returning to unscaled variables via Eq. (4.1), Eq. (4.19) in the limits $\lambda\to 0,\lambda\to\infty$ is stochastically equivalent to Eq. (3.16), interpreted in the sense of Stratonovich and Itô, respectively. A Fokker-Planck equation of the form of Eq. (4.19) has already been solved exactly by analytic methods.[11,12]

We summarize the basic results.

(i) The stationary moments after returning to the "old" unbarred variables (cf Eq. 4.1) are given by

$$\lim_{t\to\infty}\langle x^n(t)\rangle=\left[\frac{Q}{b}\right]^{n/2}\begin{cases}\dfrac{\Gamma\left[\dfrac{d_0}{Q}-\dfrac{1}{2}\dfrac{\lambda}{1+\lambda}+\dfrac{n}{2}\right]}{\Gamma\left[\dfrac{d_0}{Q}-\dfrac{1}{2}\dfrac{\lambda}{1+\lambda}\right]}, & d_0>\dfrac{1}{2}\dfrac{\lambda Q}{1+\lambda}\\[2ex] 0, & d_0\le\dfrac{1}{2}\dfrac{\lambda Q}{1+\lambda}\end{cases} \tag{4.20}$$

and especially

$$\lim_{t\to\infty}\langle x^2(t)\rangle=\frac{1}{b}\begin{cases} d_0-\frac{1}{2}\frac{\lambda Q}{1+\lambda}, & d_0>\frac{1}{2}\frac{\lambda Q}{1+\lambda} \\ 0, & d_0\le\frac{1}{2}\frac{\lambda Q}{1+\lambda}\,. \end{cases} \tag{4.21}$$

When d_0 is lowered from large positive values all moments decrease and finally vanish at the transition point characterized by

$$d_0=d_0^t=\frac{1}{2}\frac{\lambda Q}{1+\lambda} \tag{4.22}$$

which depends linearly on the intensity of the noise.

(ii) From the dynamical properties of this model we want to mention here only that the eigenvalue spectrum of the process (4.18) consists of a discrete as well as a continuous branch with slowing down at the bifurcation point

$$d_0=d_0^t=\frac{1}{2}\frac{\lambda Q}{1+\lambda}\,. \tag{4.23}$$

For details of the dynamical behavior we refer to.[11,12]

V. CONCLUSIONS

We have presented here a qualitative and a systematic adiabatic analysis of a dynamical model with two widely separated internal time scales, which is subject to multiplicative Gaussian noise. The model exhibits a sharp symmetry breaking transition, accompained by critical slowing down at $d_0=d_0^t$, Eq. (4.23), even in the presence of noise. The threshold is shifted proportional to the noise intensity in the direction corresponding to a stabilization of the symmetrical state $x=0$. The shift of the threshold also depends in an interesting way on the ratio $\lambda=\Delta\omega/\gamma$ of the noise bandwidth and the fast relaxation rate γ of the model. If the bandwidth of the noise is large compared to the slow relaxation rate of the system the noise is white on the long-time scale.

One might have thought that Eqs. (2.2) and (2.3) for $\gamma\to\infty$ then reduce to the stochastic differential equation

$$\dot{x}=d_0x-bx^3+xy(t) \tag{5.1}$$

with $y(t)$ given by the white noise (2.6). However, it turned out in Sec. IV that Eq. (5.1) interpreted either in the sense of Itô or in the sense of Stratonovich is not obtained in the limit $\gamma\to\infty$, if $\Delta\omega/\gamma=\lambda$ is kept fixed. Rather, these two interpretations of Eq. (5.1) are only obtained in two limiting cases, more specifically, the Itô interpretation of (5.1) is obtained for $\lambda\to\infty$, the Stratonovich interpretation of (5.1) is obtained for $\lambda\to 0$. For finite λ, an intermediate interpretation must be given to Eq. (5.1), which was systematically derived in Sec. IV and the Appendix, by deriving the stochastically equivalent Fokker-Planck equation. Another convenient way to express the stochastic interpretation of Eq. (5.1) is to write down the equaivalent stochastic differential equation in the sense of Stratonovich for fixed λ

$$\dot{x}=\left[d_0-\frac{1}{2}\frac{\lambda Q}{1+\lambda}\right]x-bx^3+xy(t)\,. \tag{5.2}$$

In a sense the smooth change of the stochastic interpretation which must be given to Eq. (5.1) with increasing λ may be regarded as the origin of the shift of the threshold d_0^t, which is observable. In terms of the measurable quantities γ, $\Delta\omega$, and

$$\langle y(t)^2\rangle=q \tag{5.3}$$

the threshold condition (4.23) reads

$$d_0=d_0^t=\frac{q}{\gamma+\Delta\omega}\,. \tag{5.4}$$

For $\Delta\omega\ll\gamma$ and fixed q, the transition point becomes insensitive to $\Delta\omega$.

Various experiments have been performed on systems with continuous instabilities, in which an increase of the threshold proportional to the applied noise intensity of the control parameter was reported. These experiments were performed for fixed bandwidth $\Delta\omega\gg d_0$ and varying values of q. However, a comparison of the chosen $\Delta\omega$ with the fast time scale of the system was not made in the experimental work we know of. In as much as the shift of threshold was found to be insensitive to $\Delta\omega$, our result (5.4) suggests that $\Delta\omega\ll\gamma$ was satisfied in these experiments. Obviously, experiments of the kind reported in Refs. 1–4, but performed with widely different values of $\Delta\omega$, could check relation (5.4) and the mechanism for the stabilization by multiplicative noise proposed in this paper.

ACKNOWLEDGMENTS

One of us (R. G.) wishes to acknowledge stimulating discussions with Manfred Lücke and Crispin Gardiner at the Institute for Theoretical Physics in Santa Barbara.

APPENDIX

In this appendix we present an alternative adiabatic analysis of Eqs. (4.6) and (4.7) which is algebraically more involved than the method presented in Sec. IV, but leads to additional results. In particular, we obtain approximate expressions for the probability density $W(x,p,y,t)$ of the three-dimensional process $x(t)$, $p(t)$, $y(t)$. The perturbative method used here closely follows Papanicolaou.[18] Recent use of this method has also been made in Ref. 19.

Inserting the expansion

$$W=W_0+\epsilon W_1+\epsilon^2 W_2+\cdots \tag{A1}$$

in Eqs. (4.6) and (4.7) we obtain the system

$$L_0 W_0=0 \tag{A2}$$

in order ϵ^{-2},

$$L_0 W_1=-L_1 W_0 \tag{A3}$$

in order ϵ^{-1}, and

$$L_0 W_2=-L_1 W_1+\frac{\partial W_0}{\partial t} \tag{A4}$$

in order ϵ^0. Higher orders in ϵ will not be considered here.

Equation (A2) is easily solved for natural boundary conditions at infinity. We obtain

$$W_0(x,p,y,t)=P(x,t)G(x,p,y)\ , \tag{A5}$$

where we introduced the Gaussian

$$G(x,p,y)=\frac{1+\lambda}{\pi\lambda^{3/2}|x|}\times\exp\left[-\frac{1+\lambda}{\lambda^2}\left[y^2-\frac{2pu}{x}+\frac{1+\lambda}{x^2}p^2\right]\right] \tag{A6}$$

and where $P(x,t)$ is an arbitrary function of x,t, related to $W_0(x,p,y,t)$ by

$$P(x,t)=\int dp\, dy\, W_0(x,p,y,t) \tag{A7}$$

and is assumed to be normalized to 1. In addition to (A2) we also need the solution of the adjoint equation

$$L_0^+ W_0^+=0\ . \tag{A8}$$

With natural boundary conditions we find

$$W_0^+(x,p,y,t)=F(x,t)\ , \tag{A9}$$

where $F(x,t)$ is an arbitrary function of its arguments.

We now turn to the solution of Eq. (A3). Its multiplication with $W_0^+=F$ and integration over x,p,y yields the solvability condition

$$\int dy\, dp\, L_1 W_0=0\ . \tag{A10}$$

Integration over x has disappeared from (A10) because of the arbitrariness of $F(x,t)$ in (A9). Inserting the explicit form of L_1 in (A10), the solvability condition is seen to be satisfied for arbitrary $P(x,t)$. The solution of (A3) is then well defined and may be written in the form

$$W_1(x,p,y,t)=P(x,t)[\alpha+P^{(1)}(x,p,y,t)]\,G(x,p,y)\ . \tag{A11}$$

The constant α in the square brackets on the right-hand side of (A11) is arbitrary and represents the homogeneous solution of (A3). It is fixed by normalization. The remaining term in (A11) represents a particular integral of (A3) and must be calculated. In view of the form of the right-hand side of (A3) the particular integral is a polynomial of third order in p/x and y, multiplied by the Gaussian G. Moreover, the polynomial must be odd under $p,y\to -p,-y$. Therefore

$$P^{(1)}(x,p,y,t)=A_{1,0}\frac{p}{x}+A_{3,0}\frac{p^3}{x^3}+A_{0,1}y+A_{0,3}y^3+A_{1,2}\frac{py^2}{x}+A_{2,1}\frac{p^2}{x^2}y\ . \tag{A12}$$

It is straightforward, but requires some labor, to determine the coefficients. We only list the results

$$\begin{aligned}
A_{10}&=-2(1+\lambda)A_{01}+4\frac{1+\lambda}{1+2\lambda}-\frac{2(1+\lambda)^2}{\lambda^2 x}k(x)-1+x\frac{\partial}{\partial x}\ln P(x,t)\ ,\\
A_{30}&=-\tfrac{2}{3}(1+\lambda)^2(14+13\lambda+2\lambda^2)[\lambda^2(1+2\lambda)(2+\lambda)]^{-1}\ ,\\
A_{01}&=\frac{4(1+\lambda)^2}{\lambda(1+2\lambda)(2+\lambda)}-\frac{1}{\lambda}-\frac{2(1+\lambda)}{\lambda^2}\frac{k(x)}{x}+\frac{1}{\lambda}x\frac{\partial}{\partial x}\ln P(x,t)\ ,\\
A_{03}&=\frac{4(1+\lambda)}{3\lambda^2(1+2\lambda)(2+\lambda)},\quad A_{12}=-\frac{4(1+\lambda)}{\lambda^2(1+2\lambda)},\quad A_{21}=\frac{2(1+\lambda)}{\lambda(2+\lambda)}\left[1+\frac{8(1+\lambda)}{\lambda(1+2\lambda)}\right]\ .
\end{aligned} \tag{A13}$$

Finally, we proceed to Eq. (A4). The solvability condition is obtained as before and reads

$$\frac{\partial P(x,t)}{\partial t}=\int dp\, dy\, L_1 W_1 \;. \tag{A14}$$

The right-hand side of (A14) may be simplified by inserting (A11) and observing that under $p,y \to -p,-y$ the operator L_1 and $P^{(1)}$ are odd and the Gaussian G is even. In addition, the $\partial/\partial p$ term in L_1 may be eliminated by partial integration. We obtain, therefore,

$$\int dp\, dy\, L_1 W_1 = -\frac{\partial}{\partial x}\left[\left[\int dp\, dy\, p P^{(1)}(x,p,y,t) G(x,p,y,) \right] P(x,t) \right] . \tag{A15}$$

Using Eqs. (A12) and (A13) the large parentheses in (A15) may be evaluated. The integrals over p,y involved are given by the moments up to fourth order of the normalized Gaussian (A6). After algebraic rearrangements of the terms and combining (A14) and (A15) we obtain

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[k(x)+\frac{x}{2(1+\lambda)} \right] P(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} x^2 P(x,t) \;. \tag{A16}$$

Equation (A16) coincides with Eq. (4.18) obtained by a more straightforward and economical procedure in Sec. IV. However, the present calculation has, in addition, yielded an approximation for the joint probability density $W_0(x,p,y,t)$, given by Eq. (A5). Corrections to this expression have also been obtained in Eqs. (A11)–(A13).

We close this appendix by displaying the two asymptotic forms of the probability density (A5) in the limits $\Delta\omega \ll \gamma$, $\lambda \to 0$ and $\Delta\omega \gg \gamma$, $\lambda \to \infty$. For $\lambda \to 0$,

$$W_0(x,p,y,t) = P(x,t)\frac{1}{(\pi^2\lambda^3 x^2)^{1/2}} \exp\left[-\frac{y^2}{\lambda} \right] \times \exp\left[-\frac{(\frac{p}{x}-y)^2}{\lambda^2} \right] . \tag{A17}$$

For $\lambda \to \infty$,

$$W_0(x,p,y,t) = P(x,t)\frac{1}{(\pi^2\lambda x^2)^{1/2}} \exp\left[-\frac{y^2}{\lambda} \right] \times \exp\left[-\frac{p^2}{x^2} \right] . \tag{A18}$$

Thus, for small λ, p/x is distributed very tightly around y, and y is distributed somewhat less tightly around 0. For large λ, the distribution of y around 0 becomes very broad, and p/x is distributed around 0 with width $1/\sqrt{2}$.

These conclusions are independent of the form of $P(x,t)$, and are not affected by the bifurcation. In particular, it is interesting to remark that according to (A5) and (A6)

$$\left\langle \left[\frac{\dot{x}}{x} \right]^2 \right\rangle \sim \left\langle \frac{p^2}{x^2} \right\rangle = \frac{\lambda}{2(1+\lambda)} \tag{A19}$$

is different from zero to lowest order in ϵ, even in the steady state with

$$a < \frac{1}{2}\frac{\lambda}{1+\lambda} \;,$$

where $P(x,t)=\delta(x)$. As is shown by Eq. (3.7), the nonzero value of $\langle (\dot{x}/x)^2 \rangle$ provides for additional stabilization of the state $x=0$ and is responsible for the increase of the threshold of instability.

[1] S. Kabashima, S. Kogure, T. Kawakubo, and T. Okada, Appl. Phys. 50, 6296 (1979).
[2] S. Kai, T. Kai, M. Takata, and K. Hirakawa, J. Phys. Soc. Jpn. 47, 1379 (1979).
[3] P. De Kepper and W. Horsthemke, C. R. Acad. Sci., Ser. C 287, 251 (1978).
[4] T. Kawakubo, A. Yanagita, and S. Kabashima, J. Phys. Soc. Jpn. 50, 1451 (1981).
[5] W. Horsthemke and M. Malek Mansour, Z. Phys. B 24, 307 (1976).
[6] W. Horsthemke and R. Lefever, Phys. Lett. 64A, 19 (1977).
[7] L. Arnold, W. Horsthemke, and R. Lefever, Z. Phys. B 29, 367 (1978).
[8] H. Mori, T. Morita, and K. T. Mashiyama, Prog. Theor. Phys. 63, 1865 (1980).
[9] T. Morita, H. Mori, and K. T. Mashiyama, Prog. Theor. Phys. 64, 500 (1980).

[10]H. Brand and A. Schenzle, J. Phys. Soc. Jpn. 48, 1382 (1980).
[11]A. Schenzle and H. Brand, Phys. Rev. A 20, 1628 (1979).
[12]R. Graham and A. Schenzle, Phys. Rev. A 25, 1731 (1982).
[13]R. Graham, Phys. Rev. A 25, 3234 (1982).
[14]H. Fujisaka and S. Grossmann, Z. Phys. 43, 69 (1981).
[15]H. Brand, A. Schenzle, and G. Schröder, Phys. Rev. A 25, 2324 (1982).
[16]R. E. Mortensen, J. Stat. Phys. 1, 271 (1969).
[17]G. Wilemski, J. Stat. Phys. 14, 153 (1976).
[18]G. Blankenship and G. C. Papanicolaou, SIAM J. Appl. Math. 34, 437 (1978).
[19]W. Horsthemke and R. Lefever, Z. Phys. B 40, 241 (1980).