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Derivation of the static correlation function of the classical electron plasma by Mori's scaling method

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By the application of Mori's scaling method to the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy equation of the classical electron plasma in thermal equilibrium, the static correlation functions are derived. These correlation functions are used to calculate the internal energy, which is found to be different from that of R. Abe [Prog. Theor. Phys. 22, 213 (1958)]. This difference comes from the short-range correlation function and its implications are discussed.

I. INTRODUCTION

Mayer's cluster-expansion method[1] has been very powerful in calculating thermodynamical properties of classical electron plasma and the internal energy up to the other ϵ^2 was calculated[2,3] by numerous investigators, where

$$\epsilon=[4\pi\rho_0(e^2/k_BT)^3]^{1/2}$$

is the plasma parameter. ρ_0 is the mean electron density, T the equilibrium temperature, e the electronic charge, and k_B the Boltzmann constant. The higher-order contribution of the internal energy[4,5] up to the order ϵ^4 was calculated by extending the Mayer cluster expansion[6] combined with Meeron resummation technique.[7]

Another approach to calculate thermodynamic properties of the classical electron plasma is through the correlation function. The correlation function at the order ϵ was calculated by Debye and Hückel.[8] The correlation functions of the higher order in ϵ were calculated either by the Mayer cluster expansion method or the plasma parameter expansion of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy.[9] In the latter method, O'Neil and Rostoker[10] introduced the cut in the length and defined the two different correlation functions, i.e., the short- and long-range correlation functions. These two correlation functions are matched approximately at the cut. One of the interesting features of this work is that the effect associated with the three-particle correlation function on the (two-particle) correlation function is taken into account. This feature is also considered in the recent work by Shima *et al.*[11] They reproduce the O'Neil-Rostoker structure to order ϵ^2 and further extended the analysis to obtain the ϵ^3 contribution of three- and four-particle correlation function to the (two-particle) correlation function. Furthermore, they proposed a new method of calculating the correlation function in which the generalized nonlinear fluctuation-dissipation theorems are used.

Recently, Mori has formulated the systematic method of the coarse graining in space and time with the aid of the projection-operator method[12] and the asymptotic evaluation for large systems.[13,14] This asymptotic evaluation for large systems, which we call Mori's scaling method, is a generalization of the well-known asymptotic evaluation of the thermodynamic functions and their fluctuations of equilibrium systems, inspired by the hydrodynamic similarity laws.[15] Mori's scaling method can also be regarded as the generalized small parameter expansions[16] which systematically combine the expansions in the spatial gradient $\partial/\partial\vec{r}$ and the slowness parameter $\partial/\partial t$ with the conventional small parameter expansion such as the density expansion. Making use of this method, the divergence-free kinetic equation is derived from the BBGKY hierarchy equation of the classical electron plasma and the properties of fluctuations in μ space are clarified.[17] Furthermore, the kinetic equation of the dilute nonuniform electron plasma[18] is also derived which cannot be obtained from the conventional density expansion since the correction term associated with the nonuniformity is found to be proportional to $\rho_0^{3/2}$. Thus, the usefulness of the Mori's scaling method on the theory of plasma physics has been demonstrated.

We consider a classical electron plasma in thermal equilibrium with a small mean particle density ρ_0 in a neutralizing smeared-out background of positive charge with density $\rho_0 e$. In this plasma the coherent region is defined by the space cutoff b

which satisfies

$$\lambda_D \gg b \gg r_0 , \quad (1)$$

where λ_D and r_0 are Debye and Landau lengths, respectively, which are defined as

$$\lambda_D = (k_B T / 4\pi\rho_0 e^2)^{1/2}, \quad r_0 = \epsilon\lambda_D .$$

Mori's scaling method leads to the following scaling of the characteristic quantities for the coherent region[16,17]:

$$\lambda_D \to \epsilon^{-1}\lambda_D, \quad r_0 \to r_0, \quad \rho_0 \to \epsilon^2\rho_0 . \quad (2)$$

In thermal equilibrium the correlation function $G(1,2)$ can be written as

$$G(1,2) = G(r_{21}) , \quad (3)$$

where $r_{21} = |\vec{r}_2 - \vec{r}_1|$ is the distance between the electrons one and two. In the coherent region, the correlation function $G(r_{21})$ consists of two different correlation functions and is distinguished by the characteristic length λ_D and r_0, i.e.,

$$G(r_{21}) = G_f(r_{21}) + G_0(r_{21}) . \quad (4)$$

The correlation function $G_f(r_{21})$ is called the long-range correlation function, and $G_0(r_{21})$, the short-range correlation function. They have different scaling properties. Applying Mori's scaling method to the equilibrium BBGKY hierarchy equation, these correlation functions are expanded as follows:

$$G_f(r_{21}) = \epsilon G_f^{(1)}(r_{21}) + \epsilon^2 G_f^{(2)}(r_{21}) + \cdots , \quad (5)$$

$$G_0(r_{21}) = G_0^{(0)}(r_{21}) + \epsilon^2 G_0^{(2)}(r_{21}) + \cdots , \quad (6)$$

where $G_f^{(1)}$, $G_f^{(2)}$, $G_0^{(0)}$, and $G_0^{(2)}$ are scale invariant under the coherent scaling shown in Eq. (2). $G_f^{(1)}(r_{21})$ is the well-known Debye-Hückel correlation function and $G_f^{(2)}(r_{21})$ is found to be identical to that derived by O'Neil and Rostoker.[10] These authors calculated the internal energy from the correlation functions and found it to be identical to that given by Abe.[2] We have also calculated the internal energy from the correlation functions and obtained a different result from that of Abe. This difference comes from the short-range correlation function $G_0^{(0)}(r_{21})$. O'Neil and Rostoker have shown through the plausible argument that the short-range correlation function $\phi_{\text{II}}(r_{21})$ (in their notation) is of the following nonlinear Debye-Hückel form:

$$\phi_{\text{II}}(r_{21}) = -1 + \exp\left[-\frac{r_0}{r_{21}} e^{-r_{21}/\lambda_D} \right] . \quad (7)$$

Making use of Mori's scaling method, the corresponding short-range correlation function $G_0^{(0)}(r_{21})$ to Eq. (7) is found as

$$G_0^{(0)}(r_{21}) = -1 + e^{-r_0/r_{21}} . \quad (8)$$

The details of the above results and their implications are discussed in the subsequent sections.

II. DERIVATION OF THE CORRELATION FUNCTIONS

BBGKY hierarchy equation[19] for classical electron plasma can be written as

$$\vec{\nabla}_1 \rho^{(n)}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n) = -\sum_{j=2}^{n} [\vec{\nabla}_1 u(r_{1j})]\rho^{(n)} - \int d\vec{r}_{n+1} \rho^{(n+1)}(\vec{r}_1, \ldots, \vec{r}_{n+1}) \vec{\nabla}_1 u(\vec{r}_{1,n+1}) , \quad (9)$$

where $u(r_{ij}) = r_0/r_{ij}$ is the Coulomb potential (normalized by temperature) between the electron i and j. $\rho^{(n)}(\vec{r}_1, .., \vec{r}_n)$ is the n-particle distribution function. Ursell-Mayer expansion of the n-particle distribution function with Eq. (9) yields

$$\rho^{(1)}(\vec{r}_1) = \rho_0 , \quad (10a)$$

$$\rho^{(2)}(\vec{r}_1, \vec{r}_2) = \rho_0^2 [1 + G(r_{21})] , \quad (10b)$$

$$\rho^{(3)}(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \rho_0^3 [1 + (1 + P_{12} + P_{13}) G(r_{32}) + H(\vec{r}_{21}, \vec{r}_{31})] , \quad (10c)$$

$$\begin{aligned} \rho^{(4)}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) = \rho_0^4 [& 1 + (1 + P_{13} + P_{14} + P_{23} + P_{24} + P_{13}P_{24}) G(r_{43}) \\ & + (1 + P_{13} + P_{23}) G(r_{21}) G(r_{43}) + (1 + P_{12} + P_{13} + P_{14}) H(\vec{r}_{32}, \vec{r}_{42}) + I(\vec{r}_{21}, \vec{r}_{31}, \vec{r}_{41})] , \end{aligned} \quad (10d)$$

where P_{ij} is the exchange operator between i and j. Substituting Eqs. (10) into Eq. (9), we find the following equations:

$$\vec{\nabla}_1 G(r_{21})+[1+G(r_{21})]\vec{\nabla}_1 u(r_{21})=-\rho_0\int d\vec{r}_3[G(r_{32})+H(\vec{r}_{21},\vec{r}_{31})]\vec{\nabla}_1 u(r_{31}) , \tag{11}$$

$$\begin{aligned}\vec{\nabla}_1 H(\vec{r}_{21},\vec{r}_{31})&+H(\vec{r}_{21},\vec{r}_{31})\vec{\nabla}_1[u(r_{21})+u(r_{31})]\\ &=-[G(r_{31})+G(r_{32})]\vec{\nabla}_1 u(r_{21})-[G(r_{21})+G(r_{32})]\vec{\nabla}_1 u(r_{31})-\rho_0\int d\vec{r}_4\\ &\quad\times[G(r_{21})G(r_{43})+G(r_{31})G(r_{42})+H(\vec{r}_{32},\vec{r}_{42})+I(\vec{r}_{21},\vec{r}_{31},\vec{r}_{41})]\vec{\nabla}_1 u(r_{41}) .\end{aligned} \tag{12}$$

A. Short-range correlation function

We define $r_{21}=r$, $r_{31}=R$, $\mu=(\vec{r}_{21}\cdot\vec{r}_{31}/r_{21}r_{31})$, and normalize lengths by the Landau length r_0. After some algebra, Eq. (11) can be written as

$$\frac{dG}{dr}-\frac{1}{r^2}[1+G(r)]=\frac{\epsilon^2}{2}\int_0^{a/r_0}dR\int_{-1}^{1}d\mu\,\mu\{G[(R^2+r^2-2Rr\mu)^{1/2}]+H(r,R,\mu)\} , \tag{13}$$

where we introduced the cutoff length a satisfying $r_0<a\ll\lambda_D$. The scaling properties of cutoff length a is $a\rightarrow a$ (Ref. 17) and $a/r_0\rightarrow 1$ in the scaling limit.

It is clear from Eq. (13) that the short-range correlation function $G_0(r)$ can be expanded as

$$G_0(r)=G_0^{(0)}(r)+\epsilon^2 G_0^{(2)}(r)+\cdots . \tag{14}$$

Since the normalized lengths r and R scale as $r\rightarrow r$ and $R\rightarrow R$, the lowest-order terms in the expansion ϵ of Eq. (13) are

$$\frac{dG_0^{(0)}(r)}{dr}-\frac{1}{r^2}[1+G_0^{(0)}(r)]=0 . \tag{15}$$

It is easy to show the solution of Eq. (15) with the boundary condition $G_0^{(0)}(r)\rightarrow 0$ as $r\rightarrow\infty$:

$$G_0^{(0)}(r)=-1+e^{-1/r} . \tag{16}$$

B. Long-range correlation function

By normalizing all the lengths by the Debye length λ_D, it becomes clear that the long-range correlation function can be expanded as

$$G_f(r)=\epsilon G_f^{(1)}(r)+\epsilon^2 G_f^{(2)}(r)+\cdots . \tag{17}$$

In the lowest order of the expansion for Eq. (11), we find

$$\vec{\nabla}_1 G_f^{(1)}(r_{21})+\vec{\nabla}_1\frac{1}{r_{21}}=-\frac{1}{4\pi}\int d\vec{r}_3 G_f^{(1)}(r_{32})\vec{\nabla}_1\frac{1}{r_{31}} . \tag{18}$$

In the next order of the expansion, we clearly see the difference between Mori's scaling method and the conventional small parameter expansion as follows:

$$\begin{aligned}\frac{dG_f^{(2)}(r)}{dr}-\frac{1}{r^2}G_f^{(1)}(r)&=\tfrac{1}{2}\int_\epsilon^\infty dR\int_{-1}^{1}d\mu\,\mu\{G_f^{(2)}[(R^2+r^2-2Rr\mu)^{1/2}]+H_f^{(2)}(r,R,\mu)\}\\ &\quad+\tfrac{1}{2}\int_0^\epsilon dR_1\int_{-1}^{1}d\mu\,\mu\{G_f^{(1)}[(R_1^{(2)}+r^2-2R_1r\mu)^{1/2}]+H_0^{(1)}(r,R_1,\mu)\} ,\end{aligned} \tag{19}$$

where $H_f^{(2)}$ and $H_0^{(1)}$ are two different three-particle correlation functions. This difference comes from the scaling properties of R and R_1, i.e.,

$$R\rightarrow R,\quad R_1\rightarrow\epsilon R_1 . \tag{20}$$

In other words, unnormalizing the lengths R and R_1, these lengths take the values of the following order of magnitude: $R \sim \lambda_D$ and $R_1 \sim r_0$. The quantities $\int_\epsilon^\infty dR$ and $\int_0^\epsilon dR_1$ in the right-hand side (rhs) of Eq. (19) come from the scaling properties of cutoff length a, i.e., $a \to a$, while $\lambda_D \to \epsilon^{-1}\lambda_D$ so that $a/\lambda_D \to \epsilon$ in the scaling limit.

Three-particle correlation functions $H_f^{(2)}$ and $H_0^{(1)}$ are derived in Appendices A and B, respectively. The second term in the rhs of Eq. (19) is, however, found to vanish as shown in Appendix B. Therefore, Eq. (19), which determines $G_f^{(2)}$, can be rewritten in the following form:

$$\vec{\nabla}_1 G_f^{(2)}(r_{21}) + G_f^{(1)}(r_{21})\vec{\nabla}_1 \frac{1}{r_{21}} = -\frac{1}{4\pi}\int d\vec{r}_3 [G_f^{(2)}(r_{32}) + H_f^{(2)}(\vec{r}_{21},\vec{r}_{31})]\vec{\nabla}_1 \frac{1}{r_{31}} . \tag{21}$$

We now define the Fourier transform of the correlation functions as follows:

$$G(\vec{q}) = \int d\vec{r}_{21} e^{-i\vec{q}\cdot\vec{r}_{21}} G(r_{21}) , \tag{22}$$

$$H_f^{(2)}(\vec{q}_2,\vec{q}_3) = \int d\vec{r}_{21} e^{-i\vec{q}_2\cdot\vec{r}_{21}} \int d\vec{r}_{31} e^{-i\vec{q}_3\cdot\vec{r}_{31}} H_f^{(2)}(\vec{r}_{21},\vec{r}_{31}) . \tag{23}$$

The inversion formulas of the correlation functions are

$$G(\vec{r}_{21}) = \frac{1}{(2\pi)^3}\int d\vec{q}\, e^{i\vec{q}\cdot\vec{r}_{21}} G(\vec{q}) , \tag{24}$$

$$H_f^{(2)}(\vec{r}_{21},\vec{r}_{31}) = \frac{1}{(2\pi)^6}\int d\vec{q}_2 e^{i\vec{q}_2\cdot\vec{r}_{21}} \int d\vec{q}_3 e^{i\vec{q}_3\cdot\vec{r}_{31}} H_f^{(2)}(\vec{q}_2,\vec{q}_3) . \tag{25}$$

It is easy to solve Eq. (18) and we find

$$G_f^{(1)}(q) = -\frac{4\pi}{q^2+1} , \tag{26}$$

$$G_f^{(1)}(r) = -\frac{1}{r}e^{-r} , \tag{27}$$

where the wave number q is normalized by λ_D.

Taking the Fourier transform of Eq. (21), we find

$$G_f^{(2)}(\vec{q}) = \frac{(4\pi)^2}{q^2+1}\int \frac{d\vec{q}\,'}{(2\pi)^3}\frac{\vec{q}\cdot\vec{q}\,'}{q'^2}\frac{1}{(\vec{q}-\vec{q}\,')^2+1} - \frac{1}{q^2+1}\int \frac{d\vec{q}\,'}{(2\pi)^3}\frac{\vec{q}\cdot\vec{q}\,'}{q'^2} H_f^{(2)}(\vec{q},-\vec{q}\,') . \tag{28}$$

Making use of the expressions $H_f^{(2)}(\vec{q}_2,\vec{q}_3)$ derived in Appendix A and simplifying it, we obtain the following:

$$\begin{aligned} G_f^{(2)}(\vec{q}) &= \frac{(4\pi)^2}{(q^2+1)^2}\int \frac{d\vec{q}\,'}{(2\pi)^3}\frac{\vec{q}\cdot\vec{q}\,'}{q'^2+1}\left[\frac{q^2}{1+q^2+q'^2-2\vec{q}\cdot\vec{q}\,'} - \frac{1}{q'^2}\right] \\ &= -\frac{2q^2}{(q^2+1)^2}\int_{-\infty}^{\infty} dq' \frac{q'^2}{1+q'^2}\left[1 + \frac{1+q^2+q'^2}{2qq'}\ln\left|\frac{q'-q+i}{q'+q+i}\right|\right] \\ &= -i\pi\frac{q^3}{(q^2+1)^2}\ln\left|\frac{2i-q}{2i+q}\right| . \end{aligned} \tag{29}$$

This expression of $G_f^{(2)}(q)$ is identical to that derived by O'Neil and Rostoker.[10] The inverse Fourier transform of $G_f^{(2)}(q)$ can be similarly obtained and we find

$$G_f^{(2)}(r) = \frac{1}{2r}\left[\left[\frac{1}{r}+\frac{1}{3}\right]e^{-2r} - \left[\frac{3-r}{4}\ln 3 + \frac{1}{3}\right]e^{-r} + \frac{(3-r)e^{-r}}{4}\int_r^\infty \frac{e^{-t}}{t}dt - \frac{(3+r)e^r}{4}\int_{3r}^\infty \frac{e^{-t}}{t}dt\right] . \tag{30}$$

III. DISCUSSION

In Sec. II, we derived the short-range correlation function $G_0(r)$ to the order ϵ^0 while the long-range correlation function $G_f(r)$ was derived up to the order ϵ^2. In order to extend it to the higher order in ϵ, the following expansions of the correlation functions are required:

$$G_0(r)=\epsilon^0 G_0^{(0)}(r)+\epsilon^2 G_0^{(2)}(r)+\cdots ,$$
$$G_f(r)=\epsilon G_f^{(1)}(r)+\epsilon^2 G_f^{(2)}(r)+\epsilon^3 G_f^{(3)}(r)+\cdots ,$$
$$H_f(\vec{r}_{21},\vec{r}_{31})=\epsilon^2 H_f^{(2)}(\vec{r}_{21},\vec{r}_{31}) +\epsilon^3 H_f^{(3)}(\vec{r}_{21},\vec{r}_{31})+\cdots ,$$

where the last terms in the rhs of the three above equations are not obtained in this work. Besides the above quantities, we have the contribution of the three-particle correlation function $H_0(\vec{r}_{21},\vec{r}_{31})$ to the order ϵ^3, where the scaling properties of two lengths r_{21} and r_{31} are $r_{21}\to\epsilon^{-1}r_{21}$, $r_{31}\to r_{31}$, respectively. To this order ϵ^3, the four-particle correlation function $I_f(\vec{r}_{21},\vec{r}_{31},\vec{r}_{41})$ also has to be calculated, where $r_{21}\to\epsilon^{-1}r_{21}$, $r_{31}\to\epsilon^{-1}r_{31}$, and $r_{41}\to\epsilon^{-1}r_{41}$. The structure of the BBGKY hierarchy equation of $H_f(\vec{r}_{21},\vec{r}_{31})$ and $I_f(\vec{r}_{21},\vec{r}_{31},\vec{r}_{41})$ in our method is similar to that of the recent work by Shima, Yatom, Golden, and Kalman[11] who also reproduced O'Neil-Rostoker structure to the order ϵ^2. However, they do not have the corresponding two- and three-particle correlation function to $G_0^{(2)}(r)$ and $H_0^{(1)}(\vec{r}_{21},\vec{r}_{31})$, although we did not calculate these quantities explicitly since we limited ourselves to the order ϵ^2 in calculating the internal energy.

The internal energy U of the classical electron plasma can be easily calculated by the following formula which is derived by taking account of the contribution from the background ions:

$$\frac{U}{Nk_BT}-\frac{3}{2}=\frac{\rho_0}{2}\int d\vec{r}_{21}\frac{r_0}{r_{21}}G(1,2) , \tag{31}$$

where $\frac{3}{2}$ in the left-hand side (lhs) of Eq. (31) represents the kinetic energy of electrons, and N is the number of electrons. Since the correlation function $G(1,2)$ consist of the short- and long-range correlation functions, Eq. (31) can be rewritten in the following form:

$$\frac{U}{Nk_BT}-\frac{3}{2}=\frac{\epsilon^2}{2}\int_0^1 rG_0^{(0)}(r)dr +\frac{1}{2}\int_\epsilon^\infty r[\epsilon G_f^{(1)}(r)+\epsilon^2 G_f^{(2)}(r)]dr , \tag{32}$$

where all lengths are normalized by r_0 and λ_D in the first and second term of the rhs of Eq. (32), respectively. The quantities $\int_0^1 dr$ come from the scaling properties of the cutoff length a, i.e., $a/r_0\to 1$ in the scaling limit.

Substituting Eqs. (16), (27), and (30) into Eq. (32), we obtain

$$\frac{U}{Nk_BT}-\frac{3}{2}=\frac{\epsilon^2}{4}\left[\int_1^\infty \frac{e^{-t}}{t}dt-1\right]-\frac{\epsilon}{2}e^{-\epsilon} -\frac{\epsilon^2}{4}(\tfrac{1}{6}+\ln 3\epsilon+\gamma) , \tag{33}$$

where γ is Euler's constant, and the first term in the rhs of Eq. (33) comes from the short-range correlation function. Contribution from $G_f^{(1)}(r)$ is proportional to $e^{-\epsilon}$ and, if we expand this term in ϵ, we will have the contribution to the order ϵ^2. This is not allowed in Mori's scaling method since this method enables us to derive the quantities of our interest at each level of the expansion of ϵ and the contribution from $G_f^{(1)}(r)$ is limited to the order of ϵ. The contribution from $G_f^{(1)}(r)$ to the order ϵ^2 is already taken into account as clearly shown in Eq. (19).

We now compare Eq. (33) with that of Abe in the following:

$$\begin{aligned}\frac{U}{Nk_BT}-\frac{3}{2}&=-\frac{\epsilon}{2}+\frac{\epsilon^2}{2}\left[\frac{2}{3}-\gamma-\frac{\ln 3\epsilon}{2}\right] -\epsilon^2\left[\frac{5}{8}-\frac{1}{4}\left[\int_1^\infty\frac{e^{-t}}{t}dt+\gamma\right]\right]\\ &=\left[\frac{U}{Nk_BT}-\frac{3}{2}\right]_{\text{Abe}}-0.4258\epsilon^2 .\end{aligned} \tag{34}$$

In the small plasma parameter limit, Abe's expression of the internal energy has been regarded as correct.[20] In regard to this belief, the author would like to point out that Abe did not insist on it, as is clear from his remark on the diagrams not taken into account: "it is *quite probable* that their contribution are of higher order . . . ."[21] The author does not insist that Eq. (34) is valid at the limit of small values of ϵ. He has simply shown that if one applies Mori's scaling method to the classical electron plasma in thermal equilibrium, the internal energy is expressed by Eq. (34).

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APPENDIX A

Normalizing all lengths which appear in Eq. (12) by λ_D and taking account the contribution from the short-range effect similar to the second term of Eq. (19), we find $H_f(\vec{r}_{21},\vec{r}_{31})$ can be expanded as follows:

$$H_f(\vec{r}_{21},\vec{r}_{31})=\epsilon^2 H_f^{(2)}(\vec{r}_{21},\vec{r}_{31})+\epsilon^3 H_f^{(3)}+\cdots\ , \tag{A1}$$

where the normalized lengths r_{21} and r_{31} scale as $r_{21}\rightarrow r_{21}$ and $r_{31}\rightarrow r_{31}$, respectively. Substituting this equation into the normalized form of Eq. (12), we find the following equation for $H_f^{(2)}$:

$$\begin{aligned}\vec{\nabla}_1 H_f^{(2)}(\vec{r}_{21},\vec{r}_{31})=&-[G_f^{(1)}(r_{31})+G_f^{(1)}(r_{32})]\vec{\nabla}_1\frac{1}{r_{21}}-[G_f^{(1)}(r_{21})+G_f^{(1)}(r_{32})]\vec{\nabla}_1\frac{1}{r_{31}}\\&-\frac{1}{4\pi}\int d\vec{r}_4[G_f^{(1)}(r_{21})G_f^{(1)}(r_{43})+G_f^{(1)}(r_{31})G_f^{(1)}(r_{42})+H_f^{(2)}(\vec{r}_{32},\vec{r}_{42})]\vec{\nabla}_1\frac{1}{r_{41}}\ .\end{aligned} \tag{A2}$$

Taking the Fourier transform of Eq. (A2) and making use of Eq. (26), we obtain the following equation:

$$H_f^{(2)}(\vec{q}_2,\vec{q}_3)+q_3^2 H_f^{(2)}(\vec{q}_1,\vec{q}_2)=A(\vec{q}_1,\vec{q}_2)\ , \tag{A3}$$

$$A(\vec{q}_1,\vec{q}_2)=32\pi^2\frac{(1+q_1^2+q_2^2+\vec{q}_1\cdot\vec{q}_2)}{(q_2^2+1)(q_1^2+1)}\ , \tag{A4}$$

where $\vec{q}_1=-\vec{q}_2-\vec{q}_3$. Changing the variables $\vec{q}_1$, $\vec{q}_2$, and $\vec{q}_3$ of Eq. (A3) in the cyclic order, we have two different equations. Making use of Eq. (A3) and these two equations, we can obtain an expression for $H_f^{(2)}(\vec{q}_1,\vec{q}_2)$ as follows:

$$\begin{aligned}H_f^{(2)}(\vec{q}_1,\vec{q}_2)&=\frac{q_1^2q_2^2A(\vec{q}_1,\vec{q}_2)+A(\vec{q}_3,\vec{q}_1)-q_2^2A(\vec{q}_2,\vec{q}_3)}{1+q_1^2q_2^2q_3^2}\\&=32\pi^2\frac{1+q_1^2+q_2^2+\vec{q}_1\cdot\vec{q}_2}{(1+q_1^2)(1+q_2^2)(1+q_1^2+q_2^2+2\vec{q}_1\cdot\vec{q}_2)}\ .\end{aligned} \tag{A5}$$

APPENDIX B

Three-particle correlation function with $r_{21}=r\sim\lambda_D$ and $r_{31}=R\sim r_0$ is expressed as $H_0(r,R,\mu)$, where μ is defined as $(\vec{r}_{21}\cdot\vec{r}_{31}/r_{21}r_{31})$. By rewriting Eq. (12) in a different form, we have the following equation which determines $H_0(r,R,\mu)$:

$$\begin{aligned}&\left[\frac{\partial H_0}{\partial r}-\frac{r_0}{r_2}\{G_0(R)+G_f[(R^2+r^2-2Rr\mu)^{1/2}]+H_0\}\right.\\&\qquad\left.-\frac{G_0(R)}{2\lambda_D^2}\int dx\int_{-1}^{1}d\mu'\mu' G_f[(x^2+r^2-2xr\mu')^{1/2}]\right]\frac{\vec{r}_{21}}{r_{21}}\\&+\left[\frac{\partial H_0}{\partial R}-\frac{r_0}{R^2}\{G_f(r)+G_f[(R^2+r^2-2Rr\mu)^{1/2}]+H_0\}\right.\\&\qquad\left.-\frac{G_f(r)}{2\lambda_D^2}\int dx\int_{-1}^{1}d\mu'\mu' G[(R^2+x^2-2Rx\mu')^{1/2}]\right]\frac{\vec{r}_{31}}{r_{31}}\\&\qquad=\rho_0\int d\vec{r}_4\frac{r_0}{r_{41}^2}\frac{\vec{r}_{41}}{r_{41}}[H_f(\vec{r}_{32},\vec{r}_{42})+I(\vec{r}_{21},\vec{r}_{31},\vec{r}_{41})]\ ,\end{aligned} \tag{B1}$$

where lengths which appear in Eq. (B1) are not normalized. The scaling properties of r and R are $r\rightarrow\epsilon^{-1}r$ and $R\rightarrow R$, respectively. The limit of the integral $\int dx$ is different depending on the scaling properties of x, i.e., when $x \rightarrow x$, this quantity takes the form $\int_0^a dx$; while $x \rightarrow \epsilon^{-1}x$, this quantity should be understood as $\int_a^\infty dx$. Therefore, two different terms are expressed in the terms associated with $\int dx$. Similar care has to be taken for the rhs of Eq. (B1). From the scaling properties of Eq. (B1), we find the three-particle correlation function H_0 can be expanded as follows:

$$H_0(r,R,\mu)=\epsilon H_0^{(1)}(r,R,\mu)+\epsilon^2 H_0^{(2)}+\cdots \ . \tag{B2}$$

The lowest-order term of Eq. (B1) gives the following equation for $H_0^{(1)}$:

$$\frac{\partial H_0^{(1)}}{\partial R}-\frac{r_0}{R^2}H_0^{(1)}=\frac{r_0}{R^2}\{G_f^{(1)}(r)+G_f^{(1)}[(R^2+r^2-2Rr\mu)^{1/2}]\} \ . \tag{B3}$$

This equation can be easily solved as follows:

$$\begin{aligned} H_0^{(1)}(r,R,\mu)&=e^{-r_0/R}\int_{\lambda_D}^R dx\, e^{r_0/x}\frac{r_0}{x^2}\{G_f^{(1)}(r)+G_f^{(1)}[(x^2+r^2-2xr\mu)^{1/2}]\} \\ &=G_f^{(1)}(r)(e^{-r_0/R}-1)-\lambda_D r_0 e^{-r_0/R}\int_{\lambda_D}^R dx\, e^{r_0/x}\frac{\exp[-(x^2+r^2-2xr\mu)^{1/2}/\lambda_D]}{x^2(x^2+r^2-2xr\mu)^{1/2}} \ , \end{aligned} \tag{B4}$$

where the boundary condition

$$\lim_{R\rightarrow\lambda_D} H_0^{(1)}(r,R,\mu)=0$$

in the scaling limit is used. Making use of Eq. (B4) we calculate the second term of rhs of Eq. (19) as follows:

$$\begin{aligned} &\tfrac{1}{2}\int_0^\epsilon dR_1\int_{-1}^1 d\mu\,\mu\{G_f^{(1)}[(R_1^2+r^2-2R_1r\mu)^{1/2}]+H_0^{(1)}(r,R,\mu)\} \\ &\quad=\frac{1}{2\epsilon}\int_0^\epsilon dR_1\int_{-1}^1 d\mu\,\mu\frac{\exp[-(R_1^2+r^2-2R_1r\mu)^{1/2}]}{(R_1^2+r^2-2R_1r\mu)^{1/2}} \\ &\qquad+\tfrac{1}{2}\int_0^\epsilon dR_1 e^{-\epsilon/R_1}\int_1^R dx\frac{e^{\epsilon/x}}{x^2}\int_{-1}^1 d\mu\,\mu\frac{\exp[-(x^2+r^2-2xr\mu)^{1/2}]}{(x^2+r^2-2xr\mu)^{1/2}} \\ &\quad=\frac{1}{\epsilon}\frac{r+1}{2r^2}e^{-r}\int_0^\epsilon dR_1\left[\frac{e^{R_1}+e^{-R_1}}{R_1}-\frac{e^{R_1}-e^{-R_1}}{R_1^2}\right] \\ &\qquad+\frac{r+1}{2r^2}e^{-r}\int_0^\epsilon dR_1 e^{-\epsilon/R_1}\int_1^R dx\frac{e^{\epsilon/x}}{x^2}\left[\frac{e^x+e^{-x}}{x}-\frac{e^x-e^{-x}}{x^2}\right] \ . \end{aligned} \tag{B5}$$

It is easy to show that the first term in the rhs of Eq. (B5) is $\epsilon[(r+1)/6r^2]e^{-r}$ by expanding exponential term of the integrand. This term does not contribute to $G_f^{(2)}(r)$ of Eq. (19), since this is of the higher order. Similarly, we expand the $e^{\epsilon/x}$ term in the rhs of Eq. (B5) in the power series and change variables $\epsilon/R_1=t$ and we can show by the straightforward but tedious algebra that the second term in the rhs of Eq. (19) is also of the higher order.

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