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Analytical and numerical studies of multiplicative noise

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We consider stochastic differential equations for a variable q with multiplicative white and nonwhite ("colored") noise appropriate for the description of nonequilibrium systems which experience fluctuations which are not "self-originating." We discuss a numerical algorithm for the simulation of these equations, as well as an alternative analytical treatment. In particular, we derive approximate Fokker-Planck equations for the probability density of the process by an analysis of an expansion in powers of the correlation time τ of the noise. We also discuss the stationary solution of these equations. We have applied our numerical and analytical methods to the "Stratonovich model" often used in the literature to study nonequilibrium systems. The numerical analysis corroborates the analytical predictions for the time-independent properties. We show that for large noise intensity D the stationary distribution develops a peak for increasing τ that becomes dominant in the large-τ limit. The correlation time of the process in the steady state has been analyzed numerically. We find a "slowing down" in the sense that the correlation time increases as a function of both D and τ. This result shows the incorrectness of an earlier analysis of Stratonovich.

I. INTRODUCTION

A. Summary

A common approach to the study of nonlinear, nonequilibrium systems and their associated instabilities involves a description in terms of nonlinear stochastic differential equations, as, for example, Langevin equations. In this paper we present analytical and numerical studies of such equations for the case of multiplicative noise, i.e., for the situation in which the noise depends on the state of the system. In particular, we consider the dependence of model systems on the noise parameters D and τ (where D is the noise intensity and τ is the correlation time) for both white and "colored" (nonwhite) noise. Our interest here is primarily in situations involving "external noise." The motivation for our work is that there exist a variety of physical situations in which external noise can be realized and is of interest, as we describe in Sec. I B. Our analytical and numerical results should be useful in obtaining a better understanding of the role of external noise vis-à-vis D and τ for these cases.

In our analytical approach we obtain two approximate Fokker-Planck equations for the probability density of a process driven by Ornstein-Uhlenbeck noise by considering an expansion in terms of the correlation time of the noise. The first equation is valid for small τ and is in fact a particular limit of the second which is valid for small D. This second equation is obtained by summing up an infinite series of terms in the τ expansion. It is important to note that the random term of the stochastic equation is of order $(D/\tau)^{1/2}$, and therefore our approximations do not necessarily imply that the random term is small.

Our numerical simulations have been carried out for several reasons, the first being to check the domain of validity of the theoretical work described above. Another major reason for the numerical study is that it allows us to study the stochastic model in regions of the (D,τ) parameter space for which the theory fails to give accurate results. We also obtain dynamical information which is beyond the scope of the theory.

We have used our theoretical scheme and simulation procedure to study a model often used in the literature to describe nonequilibrium systems. The model was introduced by Stratonovich[1] in the study of an electric circuit. It has been used to describe "external noise" situations for a parametric oscillator[2] and for a liquid-crystal system undergoing an electrohydrodynamic transition.[3] It has also been shown to be a prototype equation in a variety of physical systems in which multiplicative noise appears as the result of an adiabatic elimination of variables.[4] Finally, recent analytical studies of the model exist in the white-noise limit.[4–7] It thus seems to be a good candidate for a systematic study of the features of these stochastic equations. For this model, we have obtained numerically a complete picture of the behavior of the stationary distribution P_{st} and of mean values as a function of the noise parameters. This picture is well described by our analytical approximate calculation. In particular, for $D>1$ a relative maximum of P_{st} appears when increasing τ. This maximum is absent in the white-noise limit and becomes dominant for large τ. The relaxation time of the process has been calculated numerically for both the white- and colored-noise cases. We have found a slowing down phenomenon in the sense that the relaxation time increases monotonically both as a function of D and of τ. For the white-noise case this slowing down shows the incorrectness of earlier results by Stratonovich[1] based on a decoupling ansatz.

The outline of the paper is as follows. In Sec. I B we discuss the concept of external noise and the situations which are thought to be modeled by our stochastic equations. In Sec. II we derive two approximate Fokker-Planck equations for the probability density starting from stochastic differential equations. We obtain the stationary solution for one of these equations. In Sec. III we discuss the Stratonovich model in terms of the results of Sec. II. In Sec. IV we present the results of the simulation and make a comparison with the analytical results of Sec. II. Finally, in Appendix A we describe the algorithm used in the numerical simulations. Appendix B contains details of the derivation of the equation for the probability density.

B. External noise

To discuss the idea of external noise we begin by noting that the random force which occurs in a Langevin equation can have quite different origins. In an ordinary microscopic derivation of a Langevin equation the random term is interpreted as associated with the thermal fluctuations of the system. This "thermal" or "internal" noise scales with the size of the system (except near instability points). A different interpretation of the random term of a Langevin equation is necessary, however, when this is thought to model what we call an "external noise" situation. In these situations one considers a system which experiences fluctuations which are not "self-originating." These fluctuations can be due to a fluctuating environment or can be the result of an externally applied random force. The mathematical modeling of this fluctuation is made by considering a deterministic equation appropriate in the absence of external fluctuations. One then considers the external parameter which undergoes fluctuations to be a stochastic variable. The noise term of the stochastic differential equation obtained in this way is usually of multiplicative character[8]; that is, it depends on the instantaneous value of the variables of this system. It does not scale with system size and is not necessarily small. We can regard the external noise as an external field which drives the system. Several experimental situations in the presence of external noise have been recently considered. These include illuminated chemical reactions,[9] electric circuits,[2] an electrohydrodynamic instability in liquid crystals,[10] Rayleigh-Benard systems,[11] and an electronic oscillator.[12]

It is known[13] that the presence of noise can stabilize a system that is otherwise unstable. On the other hand, one has the intuitive idea that a strong enough noise would destroy the order present in a system. It then seems necessary to study systematically the possible effects of external noise on a system. Indeed, in some of the experimental situations mentioned above, it has been observed that the threshold value for which the system undergoes a nonequilibrium transition depends on the noise parameters, so that it can be modified by varying, say, the noise intensity.

This experimental observation has been interpreted theoretically as being associated with changes that the stationary distribution of the system undergoes when varying the noise parameters.[14] Although the general validity of this interpretation may be questioned, it is clear that such phenomena deserve further theoretical and experimental study. We should also note here that in an external noise situation the noise parameters can be controlled externally. In particular, the correlation time of the external noise is a parameter independent of the noise intensity and, to some extent, possible to control. Therefore, the usual white-noise assumption for thermal noise is not entirely justified for external noise, since a clear-cut separation of time scales may not exist in some cases.

We finally wish to point out that the term external noise has also been used in the literature in connection with dynamical systems showing chaotic behavior.[15] The effect of noise in such models is an interesting open question.[16] In this context noise has a different physical origin than the one considered above.

II. FOKKER-PLANCK EQUATIONS

A. A general equation and the response function

We consider a stochastic process defined by a nonlinear stochastic differential equation of first order in time

$$\dot{q}(t)=v(q(t))+g(q(t))\xi(t)\ , \tag{2.1}$$

where $v(q)$ and $g(q)$ are in general nonlinear functions of q and $\xi(t)$ is the random force which we shall consider to be a zero mean Gaussian process. The particular case in which $g(q(t))$ is constant (independent of q) is commonly referred to as "additive noise," while the general situation with a q dependent $g(q)$ is referred to as "multiplicative noise." Equation (2.1) is both intuitively clear and amenable to study by computer simulation. However, Eq. (2.1) is difficult to deal with analytically because of the (in general) highly nondifferentiable character of a realization of $\xi(t)$. For this reason it is customary to go to a different representation in which one considers an equation for the probability density $P(q,t)$ of the process. This will be a nonstochastic partial differential equation, instead of a stochastic ordinary differential equation like (2.1).

To obtain an equation for the probability density we shall rely on functional methods.[17–20] These methods provide an alternative to the often more used cumulant techniques.[21–24] Functional methods are very useful for obtaining different approximations. A general equation satisfied by the probability density of the process (2.1) is given by

$$\frac{\partial P(q,t)}{\partial t}=-\frac{\partial}{\partial q}v(q)P(q,t)+\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q}\int_0^t dt'\gamma(t,t')\left\langle \frac{\delta q(t)}{\delta\xi(t')}\bigg|_{q(t)=q}\delta(q(t)-q)\right\rangle\ , \tag{2.2}$$

where

$$P(q,t)=\langle\delta(q(t)-q)\rangle \tag{2.3}$$

and $\gamma(t,t')$ is the noise correlation function.

Equation (2.2) is derived in detail in Appendix B. It is an exact, fundamental equation on which our following development is based. Obviously it is not a closed equation for $P(q,t)$ due to the presence of the average in the last term. This is so because $\delta q(t)/\delta\xi(t')\,|_{q(t)=q}$ can not be taken out of the average symbol because it may explicitly depend on $\xi(t)$. Therefore, the problem of deriving an equation for $P(q,t)$ has been reduced to the problem of evaluating the response function $\delta q(t)/\delta\xi(t')$ and of decoupling the average in (2.2). Of course it is in general impossible to obtain an explicit exact result for the response function since this is equivalent to solving the nonlinear equation (2.1). Except for linear equations (see Appendix B) some kind of approximation has to be made to obtain from (2.2) a closed equation for $P(q,t)$. Quite naturally these approximations are approximations for the response function. Quite different approaches exist for dealing with the response function.[25] Here we shall obtain a formal closed expression for the response function which will be used in our approximations. From the formal integration of (2.1) we have

$$q(t)=q(0)+\int_0^t ds\,[v(q(s))+g(q(s))\xi(s)]\ . \tag{2.4}$$

We obtain after functional differentiation[18]

$$\frac{\delta q(t)}{\delta\xi(t')}=g(q(t'))+\int_{t'}^t ds[v'(q(s))+g'(q(s))\xi(s)]\frac{\delta q(s)}{\delta\xi(t')}\ ,\quad t>t' \tag{2.5}$$

where $v'(q)=\partial v(q)/\partial q$ and $g'(q)=\partial g(q)/\partial q$. This is an integral equation for $\delta q(t)/\delta\xi(t')$ that can be solved, since by differentiating with respect to t we obtain a differential equation for $\delta q(t)/\delta\xi(t')$,

$$\frac{\partial}{\partial t}\left[\frac{\delta q(t)}{\delta\xi(t')}\right]=[v'(q(t))+g'(q(t))\xi(t)]\frac{\delta q(t)}{\delta\xi(t')} \tag{2.6}$$

that can be solved subject to the initial condition

$$\frac{\delta q(t)}{\delta\xi(t')}\bigg|_{t'=t}=g(q(t'))\ , \tag{2.7}$$

which follows from (2.5). Therefore[26]

$$\frac{\delta q(t)}{\delta \xi(t')} = g(q(t')) \times \exp \int_{t'}^{t} ds [v'(q(s)) + g'(q(s))\xi(s)] . \tag{2.8}$$

This is, of course, only a formal expression since an explicit solution requires the knowledge of the solution $q(s)$ of the nonlinear equation (2.1).

B. White-noise limit

There exist different approximations that one may imagine for calculating the response function. A sensible choice among these approximations depends on the type of noise $\xi(t)$ that one is considering. We first consider the case of simple Gaussian white noise. This noise is characterized by a single parameter D which measures its intensity, with the correlation function given by

$$\gamma(t-t') = 2D\delta(t-t') . \tag{2.9}$$

In this case, only the response function at equal times $t'=t$ contributes in (2.2). Therefore using (2.3) we have

$$\frac{\partial P}{\partial t}(q,t) = -\frac{\partial}{\partial q} v(q) P(q,t) + D\frac{\partial}{\partial q} g(q) \frac{\partial}{\partial q} g(q) P(q,t) , \tag{2.10}$$

whose stationary solution for natural boundary conditions is

$$P_0(q) = \frac{N}{g(q)} \exp \int^{q} \frac{v(q)}{Dg^2(q)} dq , \tag{2.11}$$

where N is a normalization constant. Equation (2.10) is the well-known Fokker-Planck equation (in the Stratonovich interpretation[27,28]) for the probability density $P(q,t)$. Since for white noise the process (2.1) is known to be Markovian,[28] the fundamental solution (2.10) is also the transition probability for the process. This property is lost in the other cases that we are now going to consider, since the process $q(t)$ will be no longer Markovian.

C. Small-τ approximation for Ornstein-Uhlenbeck noise

The white noise is, of course, only a mathematical idealization of a broadband noise in which one takes the cutoff frequency to infinity (i.e., the limit of vanishing correlation time). In more realistic situations a Gaussian noise is not uniquely characterized by a single parameter D, but also involves the correlation time of the noise (i.e., inverse cutoff frequency) as another important parameter. If we now assume that the noise is Markovian and stationary (in addition to being Gaussian), we know that up to changes in scale[29] the noise has to be of the Ornstein-Uhlenbeck form, characterized by a correlation function

$$\gamma(t-t') = \frac{D}{\tau} e^{-|t-t'|/\tau} , \tag{2.12}$$

where τ is the correlation time. In the limit $\tau \to 0$ (D fixed) we recover (2.9). The strength of the random term in (2.1) is now measured by $(D/\tau)^{1/2}$ but we shall refer to D as the noise intensity. This noise intensity corresponds to the white-noise intensity obtained as a limiting case.

In the following we consider the process (2.1) driven by Ornstein-Uhlenbeck noise. A possible approach would be to write an equation for $\xi(t)$ in terms of white noise, so that we would have a well defined Markovian Fokker-Planck equation for the two-variable process $[\xi(t), q(t)]$. From a practical point of view this is not very useful since, for example, the stationary solution of a two-variable Fokker-Planck equation is not generally known. Therefore, we consider here the $q(t)$ process without introducing additional variables. As a first approximation we consider the vicinity of the white-noise limit in which τ, although not being strictly zero, can be considered as a small parameter. In this case, $\gamma(t-t')$ becomes a strongly decaying exponential function and the main contribution of the response function in (2.2) will come from its value near $t' \simeq t$. Therefore a sensible approximation in this case is to expand $\delta q(t)/\delta\xi(t')$ around $t' \simeq t$. We will obtain a first correction to the white-noise limit by keeping only the first term in this expansion:

$$\frac{\delta q(t)}{\delta \xi(t')} \simeq \left. \frac{\delta q(t)}{\delta \xi(t')} \right|_{t' \to t} + \left. \frac{d}{dt'} \frac{\delta q(t)}{\delta \xi(t')} \right|_{t' \to t} (t'-t) . \tag{2.13}$$

The time derivative of the response function at equal times is easily evaluated from (2.8) and (2.1):

$$\frac{d}{dt'} \frac{\delta q(t)}{\delta \xi(t')} = [g'(q(t'))v(q(t')) - g(q(t'))v'(q(t'))] \exp \int_{t'}^{t} ds [v'(q(s)) + g'(q(s))\xi(s)]$$
$$\xrightarrow[t' \to t]{} [g'(q(t))v(q(t)) - g(q(t))v'(q(t))] = -g^2(q(t)) \left[\frac{v(q(t))}{g(q(t))} \right]' . \tag{2.14}$$

Substituting (2.13), (2.14), and (2.7) in (2.2) and neglecting transient terms by extending the integrals to infinity we have[19]

$$\frac{\partial P(q,t)}{\partial t} = -\frac{\partial}{\partial q} v(q) P(q,t) + D\frac{\partial}{\partial q} g(q) \frac{\partial}{\partial q} h(q) P(q,t) + O(\tau^2, e^{-t/\tau}) , \tag{2.15}$$

where

$$h(q) = g(q)\{1 + \tau g(q)[v(q)/g(q)]'\} . \tag{2.16}$$

Therefore, we see that the correction to the white-noise limit to order τ corresponds to replacing a factor $g(q)$ by $h(q)$.[30]

We wish now to examine the effect of the parameter τ on the statistical properties of the system. In particular, we look for the steady-state properties of the process. We should note from the very beginning that (2.15) has to be handled with care. This equation appears as a truncation of a series which we will discuss later on and represents an approximation which is not uniform for all values of q. This may lead to unphysical artifacts. In particular, while the physical boundaries of the problem appear at $v(q) = \pm\infty$ and $g(q) = 0$, (2.15) has unphysical boundaries at $1 + \tau g(v/g)' = 0$. Therefore, a formal stationary solution of (2.15) (Ref. 19) may be misleading. The correct way to proceed seems to be to look for a stationary solution of (2.15) of the form

$$P_{st}(q) = P_0(q) + \tau P_1(q) + O(\tau^2) , \tag{2.17}$$

where $P_1(q)$ is the first correction to the white-noise distribution P_0. We shall require that

$$\int_{q_1}^{q_2} P_1(q) dq = 0 \tag{2.18}$$

so that $P_{st}(q)$ is normalized independently of τ. Here q_1 and q_2 are the physical boundaries of the problem.[32]

Substituting (2.17) in (2.15) and using the fact that P_0 is a stationary solution of (2.10) we obtain

$$\frac{\partial}{\partial q} J(q) = 0 , \tag{2.19}$$

$$J(q) = -v(q)P_1(q) + Dg(q)\frac{\partial}{\partial q} g(q) P_1(q) + Dg(q)\frac{\partial}{\partial q} g^2(q) \left[\frac{v(q)}{g(q)}\right]' P_0(q) . \tag{2.20}$$

Setting the current $J(q) = 0$, we obtain the following equation for $P_1(q)$:

$$\frac{\partial P_1(q)}{\partial q} - \frac{v(q) - Dg(q)g'(q)}{Dg^2(q)} P_1(q) = -\left[g(q) \left[\frac{v(q)}{g(q)}\right]' + \frac{1}{2D}\left[\frac{v(q)}{g(q)}\right]^2 \right]' P_0(q) , \tag{2.21}$$

whose formal solution is

$$P_1(q) = CP_0(q) - \left[g(q) \left[\frac{v(q)}{g(q)}\right]' + \frac{1}{2D}\left[\frac{v(q)}{g(q)}\right]^2 \right] P_0(q) . \tag{2.22}$$

Therefore

$$P_{st}(q) = P_0(q)\left\{ 1 + \tau \left[C - g(q)\left[\frac{v(q)}{g(q)}\right]' - \frac{1}{2D}\left[\frac{v(q)}{g(q)}\right]^2 \right] \right\} . \tag{2.23}$$

The constant C is here determined by the condition (2.18):

$$C = \int_{q_1}^{q_2} P_0(q) g(q) \left[\frac{v(q)}{g(q)}\right]' dq + \frac{1}{2D}\int_{q_1}^{q_2} P_0(q)\left[\frac{v(q)}{g(q)}\right]^2 dq = -\frac{1}{2D}\int_{q_1}^{q_2} \frac{v^2(q)}{g^2(q)} P_0(q) dq = -\frac{1}{2D}\left\langle \left[\frac{v(q)}{g(q)}\right]^2 \right\rangle_0 , \tag{2.24}$$

where $\langle \cdots \rangle_0$ indicates an average with respect to P_0. Equation (2.23) is a general result for the stationary solution to first order in τ and will be used in Sec. III as the basis of our analysis for the particular model whose simulation is reported in Sec. IV.

D. Small-D approximation for Ornstein-Uhlenbeck noise

We now examine the possibility of including in the equation for the probability density higher-order corrections in τ. In principle the expansion (2.13) can be formally carried out to all orders in $(t'-t)$:

$$\frac{\delta q(t)}{\delta\xi(t')}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\frac{d^n}{dt'^n}\frac{\delta q(t)}{\delta\xi(t')}\right]_{t'\to t}(t-t')^n . \tag{2.25}$$

Substituting this expression in the last term in (2.2), using (2.12) and again neglecting transients for simplicity, we find

$$\int_0^\infty dt'\gamma(t-t')\left\langle\frac{\delta q(t)}{\delta\xi(t')}\delta(q(t)-q)\right\rangle=D\sum_{n=0}^{\infty}(-1)^n\tau^n\left\langle\frac{d^n}{dt'^n}\frac{\delta q(t)}{\delta\xi(t')}\bigg|_{t'\to t}\delta(q(t)-q)\right\rangle . \tag{2.26}$$

To obtain a better understanding of the nature of these series we examine the first term that was neglected in (2.13). From (2.8) we have

$$\frac{d^2}{dt'^2}\frac{\delta q(t)}{\delta\xi(t')}\bigg|_{t'\to t}=v^2(q(t))\left[v(q(t))\left[\frac{g(q(t))}{v(q(t))}\right]'\right]'+g^2(q(t))\left[\frac{v^2(q(t))}{g(q(t))}\left[\frac{g(q(t))}{v(q(t))}\right]'\right]'\xi(t) . \tag{2.27}$$

The first term in (2.27) will contribute a term of order $D\tau^2$ to the equation for $P(q,t)$ and it would be of the Fokker-Planck form of (2.15). The second term in (2.27) is more complicated because it depends explicitly on $\xi(t)$. Therefore, after substitution in (2.26) we will have a term with a factor $\langle\xi(t)\delta(q(t)-q)\rangle$. It follows from Appendix B and (2.12) that the calculation of this average introduces, to lowest order, a factor $D\,\partial/\partial q$. Therefore our equation will be no longer of the Fokker-Planck form and will contain terms of order D^2. It is clear from the above discussion that in general (2.26) will lead to an equation for $P(q,t)$, of the form of a Kramers-Moyal expansion

$$\frac{\partial P(q,t)}{\partial t}=\sum_{i=0}^{\infty}\frac{\partial^i}{\partial q^i}K_i(q)P(q,t) . \tag{2.28}$$

This expansion can be rearranged by classifying the terms by their coefficient $D^m\tau^n$ $(n\geq m)$. Terms nonlinear in D are due to the explicit dependence on $\xi(t)$ of the time derivatives of the response function and, as explained above, this introduces higher-order derivatives with respect to q in the equation. Therefore it is obvious that K_1 and K_2 contain only terms with coefficient $D\tau^n$ and in fact contain all such terms. What we want to show now is that it is formally possible to sum up all terms of (2.26) which are of order $D\tau^n$. This summation represents then the best Fokker-Planck approximation to the process and contains terms to all order in τ. Since the terms that we neglect are nonlinear in the parameter D, such an approximation can be regarded as an approximation for small D, that is, small noise intensity. It is clear that this series of terms is obtained when neglecting all the terms that would have an explicit ξ dependence in (2.25). The approximation thus amounts to calculating the response function $\delta q(t)/\delta\xi(t')$ in the limit of small noise. We therefore calculate the time derivatives of the response function from

$$\frac{\delta q(t)}{\delta\xi(t')}\simeq g(q(t'))\exp\int_{t'}^{t}ds\,v'(q(s)) \tag{2.29}$$

and

$$\dot q(t)\simeq v(q(t)) , \tag{2.30}$$

which are obtained from (2.8) and (2.1) to lowest order in $\xi(t)$. It is obvious that this approximation for the response function is valid for small D only because it is going to be replaced inside the average in (2.2). We have already noted that in this average $\xi(t)$ introduces, to lowest order, a factor D. In general small D does not imply that $\xi(t)$ need to be small pointwise as used in (2.29) and (2.30). From (2.29) and (2.30) it follows that if

$$\frac{d^n}{dt'^n}\frac{\delta q(t)}{\delta\xi(t')}=B_n(q(t'))\exp\int_{t'}^{t}ds\,v'(q(s)) , \tag{2.31}$$

then

$$\frac{d^{n+1}}{dt'^{n+1}}\frac{\delta q(t)}{\delta\xi(t')}=v^2(q(t'))\left[\frac{B_n(q(t'))}{v(q(t'))}\right]'\times\exp\int_{t'}^{t}ds\,v'(q(s)) . \tag{2.32}$$

Therefore, given (2.7) we have

$$\frac{d^n}{dt'^n}\frac{\delta q(t)}{\delta\xi(t')}\bigg|_{t'=t}=v^2(q(t))\underbrace{\left[v(q(t))\left[v(q(t))\left[\cdots\left[v(q(t))\left[\frac{g(q(t))}{v(q(t))}\right]'\right]'\cdots\right]'\right.\right.}_{n-1}, \tag{2.33}$$

where there are $n-1$ factors $v(q(t))$ between $v^2(q(t))$ and $g(q(t))/v(q(t))$ and n derivatives with respect to q. Substituting this result in (2.26) we obtain the following formal sum of the series:

$$D\sum_{n=0}^{\infty}(-1)^n\tau^n\left\langle\frac{d^n}{dt'^n}\frac{\delta q(t)}{\delta\xi(t')}\bigg|_{t'=t}\delta(q(t)-q)\right\rangle$$
$$=D\left[v(q)\left[1+\tau v(q)\frac{\partial}{\partial q}\right]^{-1}\frac{g(q)}{v(q)}\right]P(q,t)\ . \tag{2.34}$$

Thus, defining the function

$$H(q)=v(q)\left[1+\tau v(q)\frac{\partial}{\partial q}\right]^{-1}\frac{g(q)}{v(q)}\ , \tag{2.35}$$

we have in this approximation the following Fokker-Planck equation for $P(q,t)$:

$$\frac{\partial P(q,t)}{\partial t}=-\frac{\partial}{\partial q}v(q)P(q,t)$$
$$+D\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q}H(q)P(q,t)\ . \tag{2.36}$$

The function $H(q)$ can be calculated in practice either by an explicit summation of the series obtained by expanding $[1+\tau v(q)\partial/\partial q]^{-1}$ in (2.35) or by finding a solution of the equation it satisfies. This equation is obtained from (2.35) as

$$H(q)+\tau[H'(q)v(q)-H(q)v'(q)]=g(q)\ . \tag{2.37}$$

Of course to first order in τ the function $H(q)$ coincides with the function $h(q)$ introduced in (2.16). The function $H(q)$ is calculated for a particular example in Sec. III. Equations (2.15) and (2.36) are two approximations to the problem of finding an equation satisfied by the probability density of the process (2.1) driven by Ornstein-Uhlenbeck noise. These approximations are based on considering τ or D as smallness parameters. In our development we have assumed that (2.1) is written in dimensionless form so that D and τ are dimensionless parameters. The physical meaning of the smallness of D and τ depends then on the problem under consideration. In general the smallness of τ is referred to a characteristic time scale of the system. A difficulty that we wish to point out regarding our approximations is that they are not uniform approximations for all values of q. As a consequence the smallness of the neglected terms depends not only on the values of D and τ but also on the q domain in which one is interested. An additional difficulty with the "small-D" approximation is the convergence of the series defining $H(q)$. In general (see Sec. III for an example) the convergence of the series depends on the value of τ, so that the soundness of the approximation not only depends on D but also on τ. In this sense the values of τ for which $H(q)$ exists give an upper bound for the validity of the small-τ approximation. For these reasons it is difficult to give a reliable, practical, and mathematically sound criterion for the validity of the approximation. The validity of these equations is checked in this work by comparison with the results of the simulation. It turns out that, for the model we consider, our equations give a good qualitative picture for reasonable values of D and τ. It is then tempting to conclude that in general they provide one with qualitatively correct information about the major differences between colored noise and the white-noise behavior usually discussed in the literature.

III. A PARTICULAR STOCHASTIC EQUATION: THE STRATONOVICH MODEL

In this section we use the general equations of the preceding section to analyze in detail a particular stochastic model defined by the equation

$$\dot{q}(t)=\alpha q(t)-\beta q^3(t)+q(t)\xi(t)\ . \tag{3.1}$$

This equation represents the simplest Ginzburg-Landau model in which the coefficient of the linear term fluctuates around a mean value α (where we take α to be positive). As pointed out in the Introduction this model has been considered in Refs. 1–7. Since in this paper we are essentially interested in the study of the properties of the process as a function of the noise parameters, we can always rescale the variables in (3.1) so that we have an equation in which all the parameters of the equation are included in the noise, i.e.,

$$\dot{q}(t)=q(t)-q^3(t)+q(t)\xi(t) . \tag{3.2}$$

A. White noise

The stationary-state time-independent properties of this model are completely known in the white-noise limit.[1] They are obtained from the stationary distribution (2.11) which in this case is

$$P_0(q)=2\left[\frac{1}{2D}\right]^{1/2D}\Gamma^{-1}\left[\frac{1}{2D}\right]q^{-1+1/D}\exp\left[-\frac{1}{2D}q^2\right]. \tag{3.3}$$

The moments of this distribution are given by

$$\langle q^n\rangle_0=(2D)^{n/2}\Gamma^{-1}\left[\frac{1}{2D}\right]\Gamma\left[\frac{1}{2D}+\frac{n}{2}\right] . \tag{3.4}$$

These exact results are used in Sec. IV to check the accuracy of our numerical simulation. An interesting property of this stationary distribution is that, while for small-noise intensity it has a single maximum at a value of $q\neq 0$, the position of this most probable value shifts to smaller values of q for increasing D. For large enough D, $P_0(q)\rightarrow_{q\rightarrow 0}\infty$. The crossover between these two qualitatively different forms of $P_0(q)$ [with a single maximum at $q\neq 0$ (Fig. 4) and going to infinity at $q=0$ (Fig. 5)] can be mathematically characterized by the value of the parameter $D=1$ for which the maximum $q_{\max}$ of P_0 first becomes nonzero. In fact we have

$$q_{\max}=\begin{cases}0 & \text{if } D>1\\ (1-D)^{1/2} & \text{if } D<1 .\end{cases} \tag{3.5}$$

This crossover behavior of $P_0(q)$ occurs, of course, in a continuous fashion. Also, although the most probable value is a nonanalytic function of D around $D=1$, no manifestation of this fact is shown in the moments (3.4).[33]

The dynamical properties of this model are not completely known. They can be analyzed in terms of the eigenvalue problem associated with the Fokker-Planck equation (2.10) for this problem. The eigenvalue problem has been discussed in Refs. 4–6. The eigenvalue spectrum consists of a discrete part and a continuous part which are given, respectively, by

$$\lambda_n=4nD\left[\frac{1}{2D}-n\right] ,$$
$$\lambda(s)=\frac{1}{4D}+s . \tag{3.6}$$

The number of discrete eigenvalues depends on D. No discrete spectrum exists for $D>\frac{1}{4}$ (Refs. 4 and 6).

An interesting quantity to calculate is the relaxation time of the correlation function as a function of noise intensity D. This relaxation time is defined by

$$T=\int_0^\infty dt'\phi(t') ,$$
$$\phi(t')=\frac{[\langle q(t+t')q(t)\rangle_0-\langle q(t)\rangle_0^2]}{[\langle q^2(t)\rangle_0-\langle q(t)\rangle_0^2]} , \tag{3.7}$$

where the averages are calculated with the steady-state distribution. At present no explicit analytical form has been given for the correlation function and the dependence of T on D except for the one obtained by a decoupling approximation.[1] The asymptotic decay in time of the correlation function has been obtained by the method of Carlemann imbedding[6] (see also Ref. 7) from which the spectrum (3.6) is reobtained. We shall comment on these two approaches in Sec. IV where we discuss the dependence of T on D as obtained from the numerical calculation of the correlation function.

B. Colored noise

We first consider the results that follow from the small-τ approximation of Sec. II for this particular model. The stationary distribution (2.22) becomes

$$P_{st}(q)=P_0(q)\left[1-\tau\left[\frac{1}{2D}+1-\frac{2D+1}{D}q^2+\frac{q^4}{2D}\right]\right] , \tag{3.8}$$

where $P_0(q)$ is now given by (3.3). From (3.8) it is straightforward to evaluate the different moments of the distribution. For the mean value we obtain

$$\langle q\rangle=\langle q\rangle_0\left[1-\tau\frac{1+2D}{2D}\right]+\tau\frac{2D+1}{D}\langle q^3\rangle_0$$
$$-\frac{\tau}{2D}\langle q^5\rangle_0 , \tag{3.9}$$

where $\langle q^n\rangle_0$ are given in (3.4). The effect of τ can be measured here by the relative deviation of $\langle q\rangle$ from the white-noise value $\langle q\rangle_0$. From (3.9) and

(3.4) one obtains that this relative deviation grows linearly with τ:

$$\frac{\langle q\rangle-\langle q\rangle_0}{\langle q\rangle_0}=\frac{D\tau}{2} . \tag{3.10}$$

For the second moment we obtain

$$\langle q^2\rangle=\langle q^2\rangle_0=1 , \tag{3.11}$$

so that, at least to this approximation, its value is independent of the noise parameters.

In order to find the shape and extrema of $P_{\rm st}(q)$ for different values of D and τ we can extend the result (3.8) to larger values of τ by considering the bracket in (3.8) as a first-order expansion of an exponential:

$$P_{\rm st}(q)=Nq^{1/D-1}\exp\left[-\tau\frac{2D+1}{2D}+\left[\tau\frac{2D+1}{D}-\frac{1}{2D}\right]q^2-\frac{\tau}{2D}q^4\right] . \tag{3.12}$$

This *ad hoc* exponentiation guarantees the positive definition of $P_{\rm st}(q)$. Also, its extrema are easier to determine than those of (3.8). The positivity requirement of $P_{\rm st}(q)$ is not fulfilled by (3.8) for arbitrary values of τ. From (3.12) it follows that

$$P_{\rm st}(q)\underset{q\to\infty}{\longrightarrow}0 \tag{3.13}$$

for all values of D and τ. The extrema of $P_{\rm st}(q)$ are given by

$$q\geq 0 , \tag{3.14}$$

$$2\tau q^4-[2(2D+1)\tau-1]q^2+D-1=0 . \tag{3.15}$$

For $D<1$, $q=0$ is a minimum $[P_{\rm st}(q)\to_{q\to\infty}0]$ and (3.15) gives a unique positive solution for q^2. Therefore for $D<1$ the general form of $P_{\rm st}(q)$ is the same as we had for the white-noise case, although of course the position of the maximum now depends on the value of τ. In fact, it follows from (3.15) that it shifts to larger values of q as τ increases. For $D>1$, $q=0$ becomes a maximum $[P_{\rm st}(q)\to_{q\to 0}\infty]$ and (3.15) gives rise to two possibilities. For small values of τ there is no positive solution for q^2 and therefore we have again the same qualitative form of $P_{\rm st}(q)$ as in the white-noise case. Nevertheless, for large enough τ (3.15) has positive solutions for q^2 which indicate the appearance of a relative maximum and a relative minimum of $P_{\rm st}(q)$. This form of $P_{\rm st}(q)$ is qualitatively different from anything that exists in the white-noise limit. (See Fig. 7.) The line in parameter space (D,τ) for $D>1$ separating these two cases [that is, the line for which the relative extrema of $P_{\rm st}(q)$ first appear] is given by

$$D=-\frac{1}{2}+\frac{1}{2\tau}\pm(3-12\tau)^{1/2}/4\tau . \tag{3.16}$$

The above discussion of $P_{\rm st}(q)$ is based on an extrapolation of the result (3.8) derived for small τ for larger values of τ. In particular, the appearance of relative extrema would occur for relatively large τ. Nevertheless these predictions are qualitatively confirmed by the simulation of Sec. V. In any case, to see in what measure those results are expected to be modified when τ is not very small, we now analyze the "small-D" approximation of Sec. II for this case.

A solution of the differential equation (2.37) is given for a variety of simple models by the ansatz

$$H(q)=ag(q)+6\tau[g'(q)V(q)-v'(q)g(q)] , \tag{3.17}$$

where a and b are determined by substituting in (2.37). In our case, $a=1$, $b=-1/(2\tau+1)$ so that

$$H(q)=q\left[1-\frac{2\tau q^2}{1+2\tau}\right] . \tag{3.18}$$

Alternatively, the expansion of (2.35) becomes in this case

$$H(q)=q+q^3\sum_{n=1}^{\infty}(-1)^n 2^n\tau^n , \tag{3.19}$$

which reproduces (3.18) but shows that the sum only exists for $\tau<\frac{1}{2}$. We showed in general in Sec. II that the "small-D approximation" amounts to replacing $h(q)$ of (2.15) by $H(q)$. Since in this case

$$h(q)=q(1-2\tau q^2) , \tag{3.20}$$

we see that in this model the substitution consists of the replacement of τ by $\tau/(1+2\tau)$. Therefore the summation of the subseries of terms implies in this case the introduction of a "renormalized" correlation times τ_R,

$$\tau_R=\frac{\tau}{1+2\tau} . \tag{3.21}$$

Furthermore, we have shown that the sum only exists for $\tau<\frac{1}{2}$. One may argue then that the substitution of τ by τ_R in the results of the small-τ approximation should be a way of taking into account a small (but not infinitesimally small) correlation time τ. In other words, we conjecture that such a substitution will indicate the general trend of the modifications that would arise when considering in Eq. (2.28) terms beyond linear order in τ. For instance, we conjecture that a better approximation than (3.10) will be

$$\frac{\langle q \rangle - \langle q \rangle_0}{\langle q \rangle_0} = \frac{D\tau_R}{2} = \frac{D\tau}{2(1+2\tau)} . \tag{3.22}$$

The replacement of τ by τ_R is expected to give better results for $\langle q \rangle$ when D and τ are small. On the same grounds we expect that a better approximation than (3.16) for the line in parameter space for which $P_{\text{st}}(q)$ develops relative extrema for $D > 1$ will be given by (3.16) with τ substituted by τ_R.

In order to have a more complete understanding of the behavior of the process it is interesting to consider the opposite limit of white noise, that is $\tau \to \infty$. In this limit the variable q becomes a fast variable as compared with $\xi(t)$ and adjusts instantaneously to the value of $\xi(t)$. The stationary distribution of $q(t)$ then can be obtained[1,34] by expressing $\xi(t)$ in terms of q by setting $\dot{q}=0$ in (3.2) and using this relation to change variables in the stationary distribution of ξ, i.e.,

$$P_{\text{st}}(q) = P_{\text{st}}(\xi(q)) \left| \frac{d\xi}{dq} \right| . \tag{3.23}$$

Since the stationary distribution of ξ is a Gaussian centered at the origin, with variance D/τ, we have

$$P_{\text{st}}(q) \underset{\tau \gg 1}{\approx} \frac{2q}{\sqrt{2\pi D/\tau}} \exp\left[\frac{-(q^2-1)^2}{2D/\tau} \right] . \tag{3.24}$$

This distribution goes to zero at $q \to 0, q \to \infty$ and has a single maximum at

$$q_{\max} = \left[\frac{1}{2} + \left[\frac{1}{4} + \frac{2D}{\tau} \right]^{1/2} \right]^{1/2} \simeq 1 + \frac{D}{4\tau} . \tag{3.25}$$

Therefore we expect that for large enough τ (and independently of D) $P_{\text{st}}(q)$ will have the same general form as for $D < 1$ and small τ. The effect of increasing τ is the same as the one of decreasing D for small τ. In both cases, the strength of the random term in (3.1) diminishes. From this point of view the development of relative extrema for $D > 1$ when increasing τ is a natural transition towards the asymptotic behavior (3.24). For $\tau \to \infty$, an absolute maximum appears at the deterministic steady state $q=1$. For this reason it is not obvious whether there is any physical significance in this model to the line (3.16) in parameter space.

In Fig. 1, we display the different regions in parameter space for which qualitatively different forms of P_{st} are expected from the above analysis. [A corresponds to $P_{\text{st}}(q)$ in Fig. 5, B to Fig. 4, C to Fig. 7, and D to Fig. 8.] The full line between A and C corresponds to (3.16) and the dashed line to (3.16) with τ replaced by τ_R.

Finally we would like to point out that to our knowledge, no analytical results have been reported for the dynamical properties of this colored-noise model. The correlation function and correlation time will be evaluated numerically in Sec. IV.

IV. NUMERICAL SIMULATION. COMPARISON WITH DIFFERENT APPROXIMATIONS

The simulation of Eq. (3.23) has been done using the algorithm described in Appendix A. For the white-noise and colored-noise cases we have used, respectively, Eqs. (A10) and (A25) to first order in the integration step as discussed in Appendix A. The inclusion of higher-order terms has been shown for a different model to be unimportant. In any case, our results seem to be accurate enough for our purposes. The accuracy of our procedure has been checked by comparing the results of the simulation for mean values and for the stationary distribution with the exact values (3.4) and (3.3) which are known for the white-noise case. The details of our

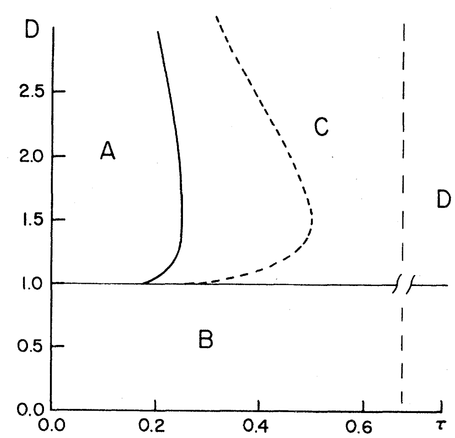

FIG. 1. Regions in parameter space (D,τ) with qualitatively different form of P_{st}.

numerical procedure are as follows. In the stationary state, data from each run were saved at 500 different times, spaced 40 integration steps apart. The size of the integration step, Δ, was equal to 0.005. One thousand independent realizations were obtained; the values for $\langle q \rangle$ and $\langle q^2 \rangle$ were obtained by averaging over both 500 different times and 1000 realizations. In order to obtain P_{st}, a histogram of 1000 runs was constructed at 500 different times. The data quoted for P_{st} are the result of averaging these 500 histograms. Because of this reduction in the number of terms included in the average as well as the finite size of the grid used to construct a histogram, our results for P_{st} are clearly less accurate than our values for $\langle q \rangle$ and $\langle q^2 \rangle$. We can also compute $\phi(t')$ as defined in Eq. (3.7) by averaging over both time and 1000 realizations. The relaxation time T, was then obtained by means of numerical integration.

The mean values obtained from the simulation in the white-noise case differ typically by less than 1% from the exact values given by (3.4). The differences between the stationary distribution obtained from the simulation and (3.3) are also small. In Fig. 2 we show this comparison for $D=0.9$. This is one of the worst cases that we can consider, due to the large fluctuations present in P_{st} for values of q. These fluctuations are due to the important change in $P_{st}(q=0)$ for $q=0$ discussed after (3.4). In fact, the differences between simulation and theory can only be seen in Fig. 2 for such small values of q.

We first discuss our results for the time-independent properties of the colored-noise case. The behavior of the relaxation time for white noise is discussed at the end when we also consider the

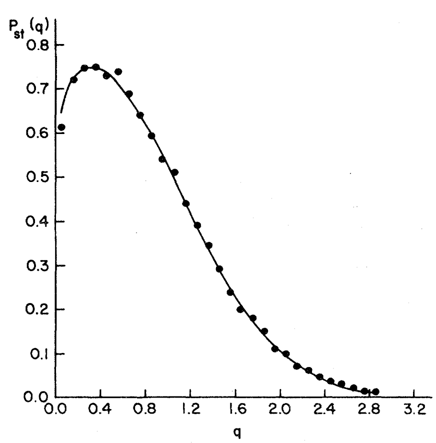

FIG. 2. Comparison of the simulation (full circles) with the exact result (3.3) for the white-noise stationary distribution (full line). $D=0.90$.

same quantity for the colored noise. The mean values obtained for different values of D and τ are listed in Table I where we have also included the corresponding values for white noise (3.4) and the two approximations discussed in Sec. II. In these three cases we obtained $\langle q^2 \rangle = 1$. We obtained values for $\langle q^2 \rangle$ which differ from unity by an amount typically less than 1% for all the explored values of D and τ, in agreement with the theory. On the other hand, $\langle q \rangle$ is seen to be sensitive to changes both in D and in τ. In Fig. 3 we check the validity of the two predictions (3.10) and (3.11). This is quite a stringent test of the theory since it refers to relative deviations from the white-noise value. With respect to the small-τ approximation we can see that it gives a good representation of the data for small enough values of τ $(=1/18)$ where it is expected to be reliable. We also see that the approximation becomes better as D becomes smaller. This is of course expected from the analysis of the τ expansion in Sec. II. One interesting thing to notice about the small-D approximation is that no matter what the value of D, the introduction of τ_R gives a significant improvement with respect to the small-τ approximation. This may be accidental, however, since there is no *a priori* reason why the approximation should work for some of the large values of D considered. On the other hand, for small enough values of D $(D \leq 0.50)$ we see that the introduction of τ_R gives an accurate representation of the data (see points 7–12). We finally remark that the approximations give good results for values of D and τ for which the strength of the random term of order $(D/\tau)^{1/2}$ is not small as for examples in points 1, 2, 6–9.

We now turn to the consideration of the form of the stationary distribution. In Figs. 4–7 we compare our result (3.8) with the simulation results and the white-noise distribution (3.3) for representative points of the different regions in Fig. 1. Figures 4 and 5 show the results for a small value of τ. There clearly exists a good quantitative agreement with the theory. Of course the deviations from white noise in this case are quite small but still noticeable. For larger values of τ (Fig. 6) the deviation from white noise becomes important. Although (3.8) does not give a perfect quantitative fit of the data, it gives a qualitatively correct description of the deviation from the white-noise distribution. In Fig. 7 we consider values of D and τ for which *a priori* (3.8) is not reliable. As expected, the agreement between the theory and the simulation is worse. Nevertheless, our approximation, at least in this example, does predict the main qualitative feature

TABLE I. Mean values for different values of D and τ. $\langle q \rangle$ and $\langle q^2 \rangle$ are the numerical results. $\langle q \rangle_0$ corresponds to (3.4), $\langle q \rangle$ small τ is given by (3.9), and $\langle q \rangle$ small D is given by (3.9) with τ replaced by τ_R [Eq. (3.21)].

Point	D	τ^{-1}	$\langle q^2 \rangle$	$\langle q \rangle_0$	$\langle q \rangle$	$\langle q \rangle_{\text{small}\,\tau}$	$\langle q \rangle_{\text{small}\,D}$
1	0.25	18		0.940	0.946	0.946	0.946
2	1.50	18		0.730	0.754	0.760	0.757
3	1.50	5	1.01	0.730	0.793	0.839	0.808
4	0.75	4	1.00	0.839	0.883	0.918	0.891
5	0.05	3	0.000	0.987	0.992	0.996	0.992
6	0.10	3	0.990	0.975	0.985	0.992	0.985
7	0.15	3	1.00	0.963	0.978	0.987	0.978
8	0.20	3	0.998	0.951	0.969	0.983	0.971
9	0.25	3		0.940	0.963	0.979	0.963
10	0.30	3	1.00	0.929	0.955	0.975	0.959
11	0.40	3	0.999	0.907	0.940	0.967	0.943
12	0.50	3	1.00	0.928	0.928	0.960	0.930
13	0.75	3	1.00	0.894	0.895	0.944	0.902
14	0.90	3	0.999	0.813	0.873	0.935	0.886
15	1.25	3	1.00	0.762	0.836	0.920	0.857
16	1.50	3	1.01	0.730	0.811	0.912	0.839
17	1.75	3	1.00	0.701	0.787	0.906	0.824
18	2.00	3	1.00	0.676	0.767	0.901	0.811
19	2.50	3	1.00	0.632	0.734	0.896	0.790
20	0.90	2.5	1.00	0.813	0.884	0.959	0.894
21	2.00	2.5	1.01	0.676	0.787	0.946	0.826
22	1.50	1.67	0.990	0.730	0.833	1.058	0.879
23	1.50	1	1.00	0.730	0.870	1.277	0.912

displayed when increasing τ for $D>1$, namely, the appearance of relative extrema. This feature is absent in the white-noise case. For larger values of τ, Eq. (3.24) should hold. In Fig. 8 we show a comparison of this prediction and the simulation. Taking into account that (3.24) is an asymptotic formula, the comparison is very good. The dependence of $P_{\text{st}}(q)$ on the parameters D and τ as follows from the simulation is displayed in Figs. 9 and 10 where the stationary distribution is shown for fixed τ and different values of D and for fixed D and different values of τ, respectively. In particular we see in Fig. 10 that the peak that appears for increasing τ for $D>1$ becomes very important. For large enough values of τ a single-peaked distribution ap-

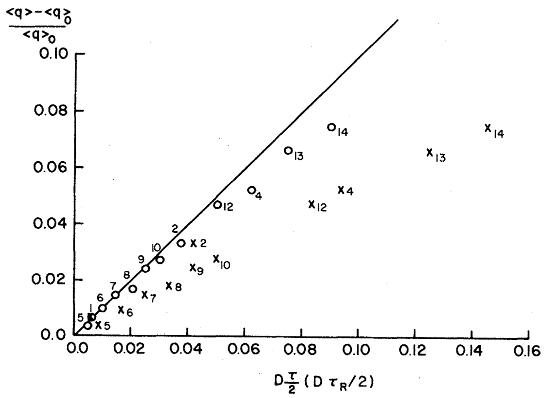

FIG. 3. Comparison of the simulation with Eqs. (3.10) (crosses) and (3.22) (open circles). The straight line corresponds to Eqs. (3.10) and (3.22) when $(\langle q \rangle - \langle q \rangle_0)/\langle q \rangle_0$ is plotted, respectively, vs $D\tau/2$ and $D\tau_R/2$. The numbering of the points corresponds to Table I.

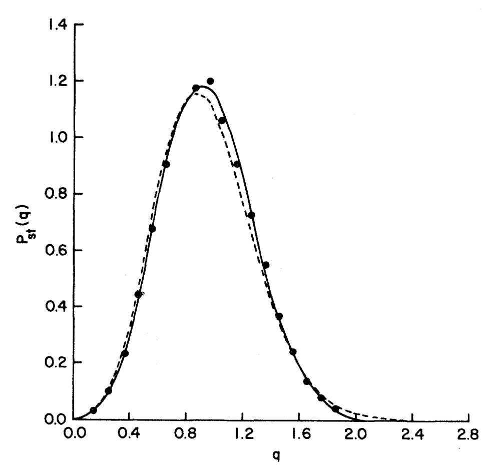

FIG. 4. Stationary distribution for $D=0.25$, $\tau=\frac{1}{18}$. The full circles represent the simulation result, the full line represents the result from Eq. (3.8); the dashed line is the white-noise result [Eq. (3.3)].

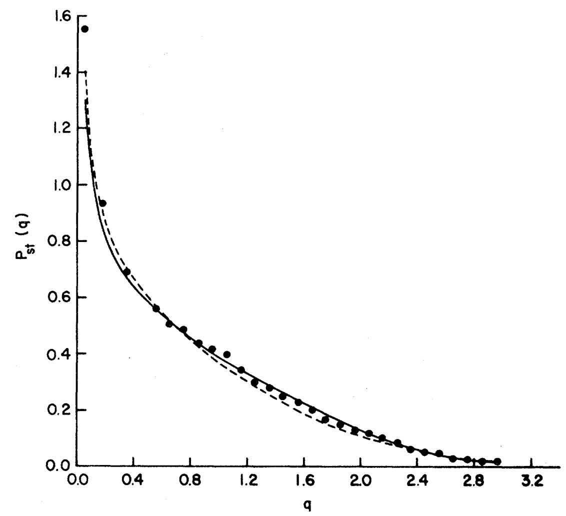

FIG. 5. Stationary distribution for $D=1.5$, $\tau=\frac{1}{18}$. The full circles represent the simulation result, the full line represents the result from Eq. (3.8); the dashed line is the white-noise result [Eq. (3.3)].

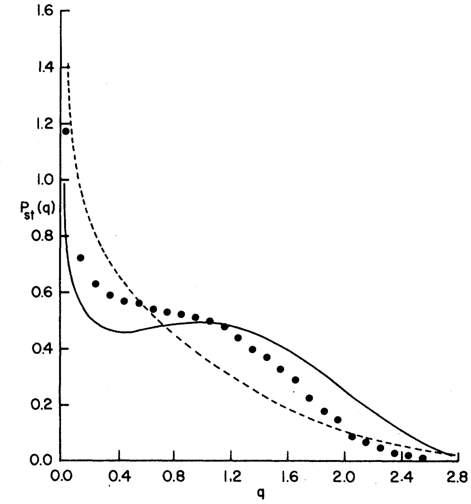

FIG. 7. Stationary distribution for $D=1.5$, $\tau=\frac{1}{3}$. The full circles represent the result of the simulation, the full line represents the result from Eq. (3.8); the dashed line is the white-noise result [Eq. (3.3)].

pears, in agreement with our analysis for $\tau \gg 1$. A detailed investigation of the line in the (D,τ) space for whch this peak appears has not been obtained since the transition occurs very smoothly. Nevertheless, from the results in Fig. 10 it is clear that the line given by (3.16) with τ replaced by τ_R gives a much better estimate than (3.16) itself. In Figs. 6, 9, and 10 it is also clearly seen that when decreasing D at fixed τ or increasing τ at fixed D, the spreading and support of the distribution are reduced, and the distribution becomes more peaked, with the maximum going to $q_{\max}=1$. This is so because in both cases we are reducing the strength of the random term in (3.10).

We now turn to a discussion of our results for the relaxation time defined in (3.7). Figure 11 shows a

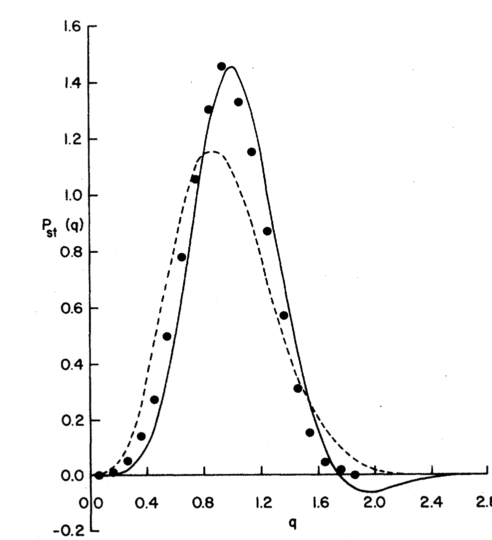

FIG. 6. Stationary distribution for $D=0.5$, $\tau=\frac{1}{3}$. The full circles are the simulation result, the full line represents the result from Eq. (3.8); the dotted line is the white-noise result [Eq. (3.3)].

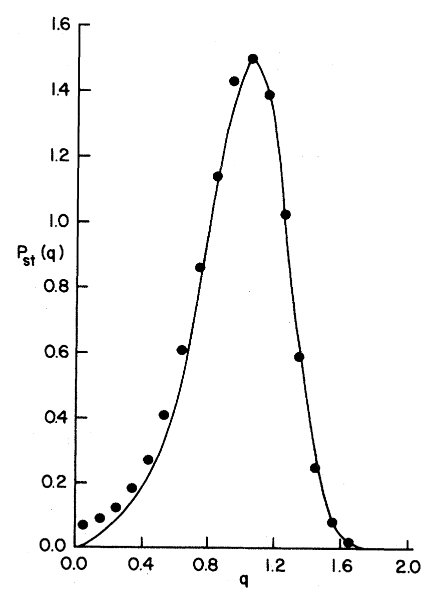

FIG. 8. Stationary distribution for $D=1.5$, $\tau=5$. The full circles are the result of the simulation. The full line is the result from Eq. (3.24).

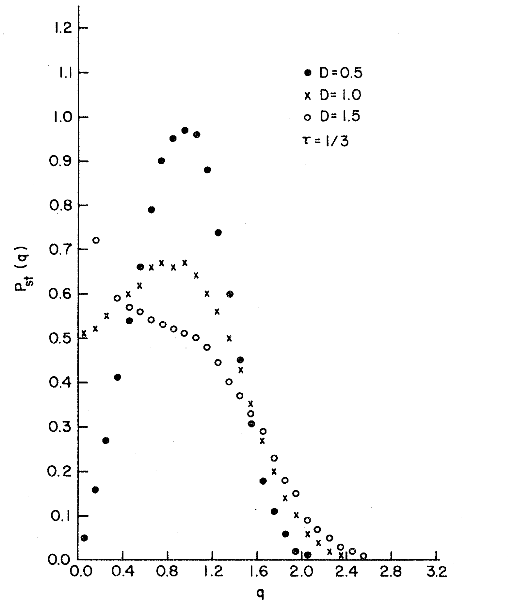

FIG. 9. Changes in the stationary distribution when increasing D for fixed $\tau=\frac{1}{3}$.

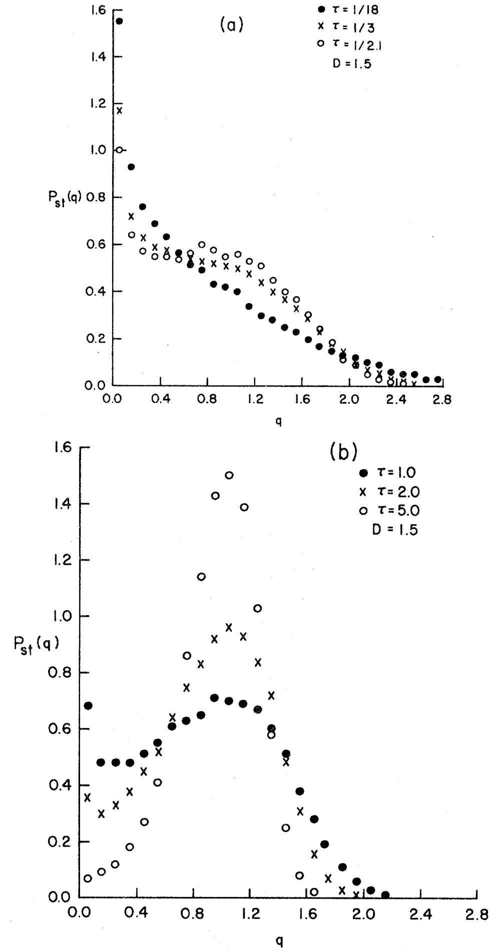

FIG. 10. Changes in the stationary distribution when increasing τ for fixed $d=1.5$.

plot of the normalized correlation function $\phi(t)$ for different values of D in the white-noise case. This shows a monotonic approach to zero which is slower for larger values of D. We therefore expect a slowing down of the process caused by the noise, in the sense that T will increase with increasing noise intensity. Such a behavior of T is explicitly shown in Fig. 12 where it is also seen that T^{-1} as a function of D decreases more rapidly for small D than for larger D. Earlier results by Stratonovich[1] give an opposite result: from a decoupling approximation Stratonovich obtained that T is a monotonically decreasing function of D. We conclude from our numerical simulation that Stratonovich's decoupling ansatz gives in this case a completely misleading result. As a separate fact we mention that Stratonovich's approximation gives a correlation function which decays exponentially for all values of D and τ. This is also in contradiction with the exact long-time dependence obtained in Ref. 6.

The explicit dependence of T on D as displayed in Fig. 11 is very difficult to extract from the eigenvalue spectrum (3.6). Some information can be obtained in the limit of small D. As D goes to zero, the bottom of the continuous band of eigenvalues (3.6) goes to infinity. Therefore only the discrete spectrum contributes in that limit and the correlation function can be written as

$$\langle q(t)q(0)\rangle=\sum_{n=0}^{\infty} C_n e^{-\lambda_n t}\,, \tag{4.1}$$

where λ_n is given in (3.6) and explicit expressions for C_n are given in Ref. 6. From those expressions one finds that $C_n \sim D^n$ for $D \ll 1$. Therefore, to lowest order in D, T^{-1} coincides with λ_1,

$$T^{-1}=2+O(D)\,. \tag{4.2}$$

This value of T^{-1} as $D\to 0$ is in agreement with our simulation and it is also given correctly by Stratonovich. To first order in D the second eigenvalue λ_2 gives a contribution of opposite sign to that of λ_1. This reduces the slope $dT^{-1}/dD\mid_{D=0}$ from the value of 4 (which would correspond to λ_1) but keeps that slope negative. A reliable determination of this slope seems to be beyond our numerical accuracy since for very small D the function $\phi(t)$ in (3.7) becomes the quotient of two very small numbers. Nevertheless, we do obtain a negative slope which is significantly reduced from the value associated with λ_1.

Another interesting aspect of our results is that

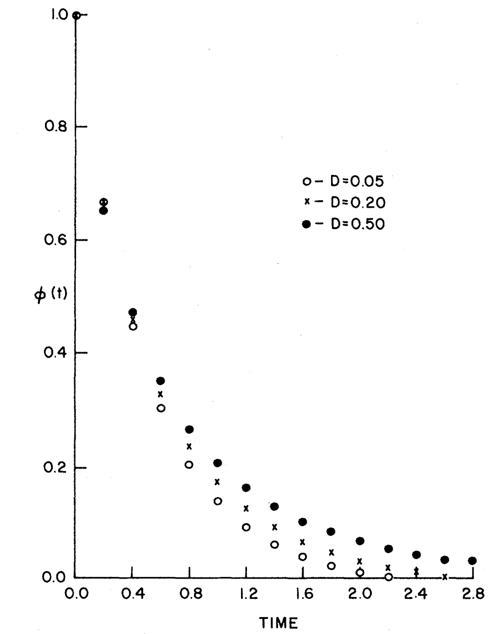

FIG. 11. Normalized correlation function (3.7) for white noise with different values of D.

the change of $P_{\mathrm{st}}(q)$ at $D=1$ is not reflected in the behavior of T, since T does not show any change in behavior around $D=1$. Also, the behavior of T does not reflect the divergence of the decay constants of $\langle q^{-n}(t)\rangle$ at $D=1/n$ pointed out in Ref. 5.

The behavior of T for the colored-noise case is qualitatively similar (Fig. 12) in the sense that for a fixed value of τ we also have a slowing down as a function of D. More interesting is the fact (Fig. 13) that for a fixed value of D, T^{-1} is also a monotonically decreasing function of τ, so that the two parameters of the noise D and τ are capable of pro-

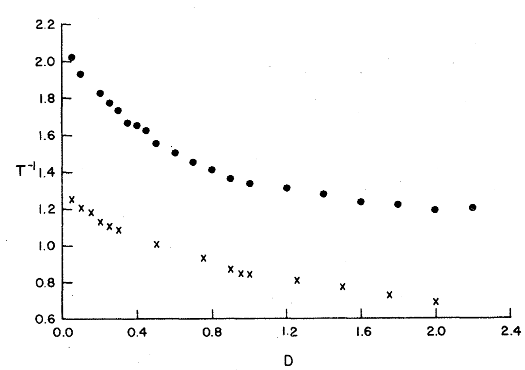

FIG. 12. Results of the simulation showing inverse correlation time vs noise intensity D. The full circles represent the white-noise result and the crosses represent the colored-noise result for $\tau=\frac{1}{3}$.

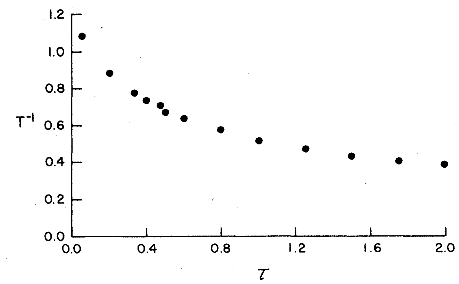

FIG. 13. Inverse correlation time vs τ for $D=1.5$.

ducing the same slowing down phenomena.

Finally we would like to comment on the relevance of our results for T for the experimental results of Kabashima *et al.*[2] for a parametric oscillator under the influence of broadband external noise. The authors report a transition from a state of oscillatory to a state of nonoscillatory output current that occurs when increasing the noise intensity while keeping other parameters fixed. This transition is analyzed in terms of the model (3.1). The authors also discuss two dynamical properties of the system. The first refers to the nonequilibrium relaxation of the system when it is brought (at constant D) from above to below threshold by changing other parameters. It is seen that the decay constant of the oscillation current decreases linearly with noise intensity. The second dynamical property refers to the correlation time of the output current on the steady state which is seen to diverge at the physical point of transition. This last result is in contradiction with our results for T if such a transition is identified with the behavior of $P_{\mathrm{st}}(q)$ for $D=1$ and $\tau \ll 1$. At this stage we can only add some words of caution regarding both the characterization of the transition and the applicability of the model (3.1) for the system of Ref. 2. First we notice that the effective value of the oscillation was determined from the peak of the spectrum of the output current. The identification of this value with the most probable value of $P_{\mathrm{st}}(q)$ is not obvious. Secondly, the model (3.1) was derived in Ref. 2 after a number of simplifications and the variable q represents the amplitude of the oscillation current. The transformation from this variable to the actually observed current is not trivial [see Eq. (2.6) of Ref. 2].

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APPENDIX A: AN ALGORITHM FOR A NUMERICAL SIMULATION OF STOCHASTIC DIFFERENTIAL EQUATIONS

The numerical integration of a stochastic differential equation is intrinsically different from that of an ordinary differential equation. The presence of random terms in the equation makes necessary the simulation of random numbers using Monte Carlo techniques. What one does is simulate a particular trajectory, consistent with the stochastic equation, which corresponds to a particular realization of the stochastic term (a certain sequence of random numbers). The statistical properties in which one is interested are then obtained by taking the average over many of these simulated trajectories.

Several rather different algorithms[35–39] have been proposed recently to integrate Langevin-type equations. We will essentially follow the algorithm proposed in Ref. 35 for general nonlinear equations, keeping in mind the basic idea discussed above. That paper is written in the language of the Itô calculus for stochastic differential equations with a white-noise random term.[28] For a better understanding we present below this algorithm in terms of the more usual Stratonovich calculus.[27,28] We also extend the algorithm to the case of stochastic differential equations driven by an Ornstein-Uhlenbeck process.

We first consider a general stochastic differential equation (2.1) driven by white noise, with a correlation given by (2.9). We interpret (2.1) in the Stratonovich sense. The existence in general of a multiplicative noise complicates the algorithm. When dealing with equations for a single variable q one could always introduce a new variable for which the noise becomes additive. This is the variable defined in (B15). Nevertheless, in practice this transformation is rather involved and it seems better to use the original variables.

To obtain a discretized version of (2.1) we choose a small time interval Δ. The value $q(t+\Delta)$ is generated from the value $q(t)$ by means of the formal integration of (2.1) between t and $t+\Delta$:

$$q(t+\Delta)-q(t)=\int_t^{t+\Delta} v(q(t'))dt' + \int_t^{t+\Delta} g(q(t'))\xi(t')dt' . \tag{A1}$$

In order to expand the right-hand side of (A1) in powers of Δ, we write

$$v(q(t'))=v(q(t))+\left.\frac{dv}{dq}\right|_{q(t)}[q(t')-q(t)]+\cdots , \tag{A2}$$

$$g(q(t'))=g(q(t))+\left.\frac{dg}{dq}\right|_{q(t)}[q(t')-q(t)]+\cdots . \tag{A3}$$

Keeping only the first terms in (A2) and (A3) we obtain

$$q(t+\Delta)-q(t)=v(q(t))\Delta+g(q(t))X_1(t) , \tag{A4}$$

with

$$X_1(t)=\int_t^{t+\Delta}\xi(t')dt'=\sqrt{2D\Delta}\,\gamma_1(t) , \tag{A5}$$

where the random number $X_1(t)$ is Gaussian with zero mean value and variance $\langle X_1^2(t)\rangle=2D\Delta$. It is written in terms of $\gamma_1(t)$ which is defined as a Gaussian random number of zero mean value and variance equal to one. The number $\gamma_1(t)$ can be generated from two independent random numbers η_1,η_2 (which are distributed with equal probability between 0 and 1) by the Box-Mueller formula[40]

$$\gamma_1(t)=(-1\ln\eta_1)^{1/2}\cos 2\pi\eta_2 . \tag{A6}$$

The first term in the rhs of (A4) comes from the deterministic part of (2.1) and is of order Δ. The second term is of order $\Delta^{1/2}$ and therefore dominates for small Δ. An important point here is that although the second term of (A2) gives when substituted in (3.1) a contribution of order higher than Δ, the second term in (A3) gives a contribution of order Δ. It would be inconsistent to keep the term $v(q(t))\Delta$ while neglecting this other one. This other term is obtained as follows. We have

$$\left.\frac{dg}{dq}\right|_{q(t)}\int_t^{t+\Delta}dt'[q(t')-q(t)]\xi(t) \simeq \left.\frac{dg}{dq}\right|_{q(t)}\int_t^{t+\Delta}dt' g(q(t))\int_t^{t'}dt''\xi(t')\xi(t'') , \tag{A7}$$

where we have used (A4) to lowest order in Δ. Since we have that

$$\int_t^{t+\Delta} dt' \int_t^{t'} dt''\xi(t')\xi(t'') = \frac{1}{2}\int_t^{t+\Delta} dt' \int_t^{t+\Delta} dt''\xi(t')\xi(t'') , \quad (A8)$$

the contribution (A7) can be written as

$$\frac{1}{2}g(q(t))\frac{dg(q(t))}{dq(t)}X_1^2(t) , \quad (A9)$$

which is of order Δ. The fact that such a term of order Δ appears is due to the very special character of the white-noise term. It is also important to notice that this term vanished for additive noise. Therefore to first order in Δ, we have

$$q(t+\Delta)=q(t)+v(q(t))\Delta+g(q(t))X_1(t) + \frac{1}{2}g(q(t))\frac{dg(q(t))}{dq(t)}X_1^2(t)+O(\Delta^{3/2}) . \quad (A10)$$

This first-order algorithm is the one used in our calculations. The successive corrections to (A10) are, respectively, of order $\Delta^{3/2},\Delta^2,\Delta^{5/2},\Delta^3$, etc., and can be obtained in a similar way as in Ref. 35, using the above procedure. We have only considered in our simulation the first-order contribution, which we think is sufficient for our purpose. The accuracy of the algorithm seems to be more sensitive to the actual value of Δ and to the number N of realizations (or samplings of the process) used to obtain averages quantities, than to the inclusion of higher-order terms in Δ.[41] The statistical error associated with N seems to be ordinarily larger than the reduction in error which one may obtain by considering higher-order terms. On the other hand, Δ has to be chosen necessarily quite small since it measures the correlation time of the simulated "white noise." It is also important to note that we try to simulate possible trajectories which are erratic in nature and that our interest is in averaged quantities. Higher-order terms could become important if one is interested in trajectories themselves.[42]

We now turn to the case in which $\xi(t)$ in (2.1) is given by the Ornstein-Uhlenbeck process (2.12). The Ornstein-Uhlenbeck process $\xi(t)$ can be generated from the Gaussian white noise ξ_w through the stochastic equation

$$\dot{\xi}(t)=-\tau^{-1}\xi(t)+\tau^{-1}\xi_w(t) . \quad (A11)$$

Therefore we now have to solve the coupled set of equations (2.1) and (A11). By choosing an integration step Δ we rewrite (2.1) as (A1) and (A11) as

$$\xi(t+\Delta)=\xi(t)+\int_t^{t+\Delta}[-\tau^{-1}\xi(t')+\tau^{-1}\xi_w(t')]dt' . \quad (A12)$$

In this case we will make use of the expansions (A2), (A3), and

$$\xi(t')-\xi(t)=\int_t^{t'}[-\tau^{-1}\xi(t'')+\tau^{-1}\xi_w(t'')]dt'' . \quad (A13)$$

The first term on the rhs of (A1) is given by

$$\int_t^{t+\Delta} v(q(t'))dt'=\int_t^{t+\Delta}\left[v(q(t))+v'(q(t))\left[\int_t^{t'} v(q(t''))dt''+\int_t^{t'} g(q(t''))\xi(t'')dt''\right]\right]dt'$$
$$=v(q(t))\Delta+v'(q(t))v(q(t))\frac{\Delta^2}{2}+v'(q(t))g(q(t))\xi(t)\frac{\Delta^2}{2}+O(\Delta^{5/2}) , \quad (A14)$$

where the first equality follows from (A1) and (A2). In the second equality we have approximated $v(q(t''))$ and $g(q(t''))\xi(t'')$ by their values at time t. Using (A3) and (A13) the second term on the rhs of (A1) can be written as

$$\int_t^{t+\Delta} g(q(t'))\xi(t')dt'=g(q(t))\xi(t)\Delta+A+B+C , \quad (A15)$$

where

$$A=\int_t^{t+\Delta} g'(q(t))\xi(t)[q(t')-q(t)]dt' , \quad (A16)$$

$$B=\int_t^{t+\Delta} dt'g(q(t))\int_t^{t'} dt''[-\tau^{-1}\xi(t'') +\tau^{-1}\xi_w(t'')] , \quad (A17)$$

$$C=\int_t^{t+\Delta} dt'g'(q(t))[q(t')-q(t)] \times\int_t^{t'} dt''[-\tau^{-1}\xi(t'')+\tau^{-1}\xi_w(t'')] . \quad (A18)$$

Using (A1) and approximating $v(q(t''))$ and $g(q(t''))\xi(t'')$ by their values at time t, as we did in (A14) we have

$$A = g'(q(t))v(q(t))\xi(t)\frac{\Delta^2}{2} + g'(q(t))g(q(t))\xi^2(t)\frac{\Delta^2}{2} + O(\Delta^{5/2}) . \tag{A19}$$

Approximating $\xi(t'')$ by $\xi(t)$ in (A17), B becomes

$$B = -\tau^{-1}g(q(t))\xi(t)\frac{\Delta^2}{2} + \tau^{-1}g(q(t))X_2(t) + O(\Delta^{5/2}) , \tag{A20}$$

where

$$X_2(t) = \int_t^{t+\Delta} dt' \int_t^{t'} dt''\xi_w(t'') . \tag{A21}$$

The random number $X_2(t)$ is Gaussian because it is a linear transformation of $\xi_w(t)$. It has zero mean value and variance

$$\langle X_2(t) \rangle = \frac{2D\Delta^3}{3} . \tag{A22}$$

$X_2(t)$ is not independent of $X_1(t)$ since

$$\langle X_1(t)X_2(t) \rangle = D\Delta^2 . \tag{A23}$$

Therefore $X_2(t)$ can be written as

$$X_2(t) = D^{1/2}\Delta^{3/2}\left[\frac{1}{\sqrt{2}}\gamma_1(t) + \frac{1}{\sqrt{6}}\gamma_2(t)\right] , \tag{A24}$$

where γ_1 is defined in (A5) and $\gamma_2(t)$ is another Gaussian number of mean value zero, variance equal to one, and statistically independent of $\gamma_1(t)$. The number $\gamma_2(t)$ is generated in the same way as $\gamma_1(t)$. The term C is easily seen to be of order higher than Δ^2. Therefore, from (A1), (A14), (A15), (A19), and (A20) we have

$$\begin{aligned} g(t+\Delta) = {} & q(t) + v(q(t))\Delta + g(q(t))\xi(t)\Delta + g'(q(t))g(q(t))\xi^2(t)\frac{\Delta^2}{2} \\ & + v'(q(t))v(q(t))\frac{\Delta^2}{2} + v'(q(t))g(q(t))\xi(t)\frac{\Delta^2}{2} + g'(q(t))v(q(t))\xi(t)\frac{\Delta^2}{2} \\ & - \tau^{-1}g(q(t))\xi(t)\frac{\Delta^2}{2} + \tau^{-1}g(q(t))X_2(t) + O(\Delta^{5/2}) . \end{aligned} \tag{A25}$$

This equation has to be iterated together with the equation for $\xi(t)$. The equation (A11) for $\xi(t)$ is a white-noise stochastic equation; therefore according to (A10) we have

$$\xi(t+\Delta) = \xi(t) + \tau^{-1}X_1(t) - \tau^{-1}\Delta\xi(t) + O(\Delta^{3/2}) . \tag{A26}$$

We now wish to analyze how (A25) becomes the white-noise algorithm (A10) in the limit $\tau \rightarrow 0$ in which $\xi(t)$ becomes $\xi_w(t)$. To first order in Δ, we have from (A26) that

$$\xi(t)\Delta \underset{\tau\rightarrow 0}{\longrightarrow} X_1(t) . \tag{A27}$$

Using (A27) in (A25) we recover (A10). It is important to notice that the first four terms on the rhs of (A25) contribute to the first order in Δ algorithm given by (A10). The fourth term is of order Δ^2 in (A25) but in the white-noise limit becomes of order Δ. For the same reasons as given in the white-noise case, it would seem sufficient to use the algorithm given by (A25) and (A26), keeping only the terms of order Δ in (A25). Since in some cases we are exploring in our calculations the vicinity of the white-noise limit, we also include the term $g'(q(t))g(q(t))\xi^2(t)\Delta^2/2$. As we have seen, this term is in principle important in this vicinity and will become negligible for large τ. In summary, most of our calculations use (A25) and (A26), keeping only in (A25) the first four terms of the right-hand side. Nevertheless in our particular example it turns out that the fourth term in (A25) is of no importance in the results.

APPENDIX B

In this appendix we give a detailed derivation of (2.2). We also obtain an exact Fokker-Planck equation that follows from (2.2) for special models.

We start by considering an ensemble of systems in q space obeying Eq. (2.1) for a given realization of $\xi(t)$ and different initial conditions. This ensemble is represented by a density $\rho(q,t)$ which evolves in times according to a continuity equation. This equation expresses the conservation of the number of systems in the ensemble:

$$\frac{\partial\rho(q,t)}{\partial t} = -\frac{\partial}{\partial q}[v(q) + g(q)\xi(t)]\rho(q,t) . \tag{B1}$$

In this context this equation is known as the "stochastic Liouville equation."[23] The passage from (2.1) to (B1) is a well-known method of reducing a nonlinear stochastic equation to a linear one.[22] Equation (B1) expresses the variation of ρ with time at a fixed point q; therefore $v(q)$ and $g(q)$ are given functions independent of $\xi(t)$, while $\rho(q,t)$ is a functional of $\xi(t)$ defined through (B1). The nonstochastic equation for the probability density $P(q,t)$ is obtained by averaging (B1) over the realizations of $\xi(t)$,

$$P(q,t)=\langle \rho(q,t)\rangle \ . \tag{B2}$$

This intuitively obvious relation is known as Van Kampen's lemma.[22,43] Since in (B2) q is a fixed point, we have

$$\frac{\partial P(q,t)}{\partial t}=-\frac{\partial}{\partial q}v(q)P(q,t)-\frac{\partial}{\partial q}g(q)\langle \xi(t)\rho(q,t)\rangle \ . \tag{B3}$$

We now consider this average over initial conditions to be independent of that over the realizations of $\xi(t)$ and we note that $\rho(q,t)$ is just the average over the initial conditions of $\delta(q(t)-q)$:

$$\rho(q,t)=\langle \delta(q(t)-q)\rangle \ , \tag{B4}$$

where $q(t)$ is the formal solution of (2.1) for a given realization of $\xi(t)$. We can then rewrite (B3) as

$$\frac{\partial P(q,t)}{\partial t}=-\frac{\partial}{\partial q}v(q)P(q,t) -\frac{\partial}{\partial q}g(q)\langle \xi(t)\delta(q(t)-q)\rangle \ . \tag{B5}$$

This equation can also be obtained taking the time derivative of

$$P(q,t)=\langle \delta(q(t)-q)\rangle \ ,$$

using (2.1) and the identity

$$f(x)\delta(x-y)=f(y)\delta(x-y) \ . \tag{B6}$$

The average which remains in (B5) can be handled by using the assumption that $\xi(t)$ is a Gaussian noise. A Gaussian process is characterized in functional language by a formula due to Noyikov[44] (see also Ref. 17) valid for a process with mean value $\langle \xi(t)\rangle=0$,

$$\langle \xi(t)\phi[\xi(t)]\rangle=\int_0^t dt'\gamma(t,t')\left\langle \frac{\delta\phi[\xi]}{\delta\xi(t')}\right\rangle \ , \tag{B7}$$

where $\phi[\xi]$ is a functional of ξ and $\gamma(t,t')$ is its correlation function

$$\langle \xi(t)\xi(t')\rangle=\gamma(t,t') \ . \tag{B8}$$

Equation (B7) follows by considering a functional Taylor-series expansion of $\phi[\xi]$ and using the Gaussian property of $\xi(t)$ for the factorization of moments. Choosing $\rho(q,t)$ in (B3) or $\delta(q(t)-q)$ in (B5) to be the functional $\phi[\xi]$ and using (B7), we have

$$\frac{\partial P(q,t)}{\partial t}=-\frac{\partial}{\partial q}v(q)P(q,t) -\frac{\partial}{\partial q}g(q)\int_0^t dt'\gamma(t,t') \times\left\langle \frac{\delta[\delta(q(t)-q)]}{\delta\xi(t')}\right\rangle \ . \tag{B9}$$

We now can write

$$\frac{\delta[\delta(q(t)-q)]}{\delta\xi(t')}=\frac{\partial\delta(q(t)-q)}{\partial q(t)}\frac{\delta q(t)}{\delta\xi(t')} =-\frac{\delta q(t)}{\delta\xi(t')}\frac{\partial}{\partial q}\delta(q(t)-q) \ . \tag{B10}$$

Using the identity

$$f(y)\frac{\partial}{\partial x}\delta(y-x)=\frac{\partial}{\partial x}f(x)\delta(y-x) \ , \tag{B11}$$

we obtain

$$\delta\frac{[\delta(q(t)-q)]}{\delta\xi(t')}=-\frac{\partial}{\partial q}\left.\frac{\delta q(t)}{\delta\xi(t')}\right|_{q(t)=q}\delta(q(t)-q) \ . \tag{B12}$$

Substituting (B12) in (B10) we finally obtain (2.2).

Equation (2.2) can be converted into an exact Fokker-Planck equation in the special cases in which (2.1) is either a linear equation, i.e., $v(q)=aq+b, g(q)=1$, or is reducible to a linear equation by a change of variables. The necessary and sufficient condition for this transformation to be possible is[45,46,31]

$$g(q)\frac{\partial}{\partial q}g^{-1}(q)v(q)=A=\text{const} \ . \tag{B13}$$

When (B13) is fulfilled, the stochastic equation can be written as

$$\dot{Q}=AQ+B+\xi(t) \ , \tag{B14}$$

where the new variable Q is defined by

$$Q=\int\frac{dq}{g(q)} \ . \tag{B15}$$

In this case

$$\frac{\delta Q(t)}{\delta\xi(t')}=\exp[A(t-t')] \ , \tag{B16}$$

and from (2.2) one easily arrives at an exact equation for the probability density in Q space. Transforming back to the original variables we obtain

$$\frac{\partial P(q,t)}{\partial t} = -\frac{\partial}{\partial q}[AQ(q)+B]g(q)P(q,t) + D(t)\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q}g(q)P(q,t) \,, \tag{B17}$$

where

$$D(t) = \int_0^t dt'\gamma(t,t')\exp[A(t-t')] \,. \tag{B18}$$

Equation (B17) is an exact Fokker-Planck equation for the process (2.1) when (B13) is satisfied. It is a local equation in time for a process which for general $\xi(t)$ is not Markovian.

[1]R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1967), Vol. II.

[2]S. Kabashima, S. Kogura, T. Kawakubo, and T. Okada, J. Appl. Phys. 50, 6296 (1979).

[3]H. Brand and A. Schenzle, J. Phys. Soc. Jpn. 48, 1382 (1980).

[4]A. Schenzle and H. Brand, Phys. Rev. A 20, 1628 (1979).

[5]M. Suzuki, K. Kaneko, and F. Sasagawa, Prog. Theor. Phys. 65, 828 (1981).

[6]R. Graham and A. Schenzle Phys. Rev. A 25, 1731 (1982).

[7]L. Brenig and N. Banai, Physica D 4, (1981).

[8]This is also usually the case after adiabatic elimination of variables in stochastic differential equations. See Ref. 4.

[9]P. DeKepper and W. Horsthemke, C. R. Acad. Sci. Paris Ser. C 287, 251 (1978).

[10]S. Kai, T. Kai, M. Takata, and K. Hirakawa, J. Phys. Soc. Jpn. 47, 1379 (1979).

[11]J. P. Gollub and J. F. Steinman, Phys. Rev. Lett. 45, 551 (1980).

[12]F. T. Arecchi and A. Politi, Phys. Rev. Lett. 45, 1219 (1980).

[13]L. Arnold, in *Stochastic Systems in Physics, Chemistry and Biology,* edited by L. Arnold and R. Lefever (Springer, Berlin, 1981).

[14]W. Horsthemke, in *Dynamics of Synergetic Systems,* edited by H. Haken (Springer, Berlin, 1981), and references therein.

[15]R. May, Nature 261, 459 (1976); P. Collet and J. P. Eckman, *Iterated Maps on the Interval as Dynamical Systems* (Birkhauser, Boston, 1980).

[16]J. Crutchfield and B. A. Huberman, Phys. Lett. A 77, 407 (1980); J. Crutchfield, M. Nauenberg, and J. Rudnick, Phys. Rev. Lett. 46, 933 (1981); B. Schraiman, C. Wayne, and P. C. Martin, *ibid.* 46, 935 (1981); G. Mayer-Kress and H. Haken, J. Stat. Phys. 26, 149 (1981); A. Zippelius and M. Lucke, J. Stat. Phys. 24, 345 (1981). This last paper reports a simulation similar to ours used to study the effect of small additive noise on the behavior of the Lorenz model of turbulence. Among other things they also find that the threshold value of the Reynolds number for the transition to chaotic behavior depends on the noise intensity: in the presence of noise the system behaves chaotically for $R < R_c$, where R_c is the deterministic threshold value of the Reynolds number.

[17]V. I. Klyatskin and V. I. Tatarskii, Usp. Fiz. Nauk. 110, 499 (1973) [Sov. Phys.—Usp. 16, 494 (1973)].

[18]P. Hanggi, Z. Phys. B 31, 407 (1978).

[19]J. M. Sancho and M. San Miguel, Z. Phys. B 36, 357 (1980).

[20]M. San Miguel and J. M. Sancho, Z. Phys. B 43, 361 (1981).

[21]M. Lax, Rev. Mod. Phys. 38, 359 (1966); 38, 541 (1966).

[22]N. G. Van Kampen, Phys. Rep. 24, 171 (1976).

[23]R. Kubo, J. Math. Phys. 4, 174 (1963).

[24]S. Mukamel, I. Oppenheim, and J. Ross, Phys. Rev. A 17, 1999 (1977).

[25]An interesting possibility is to use the operator formalism for classical statistical dynamics. See Ref. 19.

[26]This result has also been obtained by a different method by M. Suzuki, Prog. Theor. Phys. Suppl. 69, 160 (1980).

[27]N. G. Van Kampen, J. Stat. Phys. 24, 175 (1981).

[28]L. Arnold, *Stochastic Differential Equations* (Wiley, New York, 1979).

[29]J. L. Doob, Ann. Math. 43, 351 (1942).

[30]For an extension of this approximation to N variables see Ref. 31. Equation (2.15) has also been derived by an eigenfunction expansion method by K. Kaneko, Prog. Theor. Phys. 66, 129 (1981).

[31]M. San Miguel and J. M. Sancho, Phys. Lett. A 76, 97 (1980).

[32]This procedure leads to the same result as the more elaborate method given by W. Horsthemke and R. Lefever [Z. Phys. B 40, 241 (1980)]. The latter is based on the τ expansion proposed in Ref. 16 but it is formulated in terms of a two variable Markov process to explicitly avoid the problems mentioned with the boundaries.

[33]In Ref. 4 this behavior of $P_0(q)$ was described by suggesting that $q_{\max}$ might be interpreted as some kind of order parameter and assigning to $D=1$ the character of a threshold value. Stratonovich (Ref. 1) also considered a second change of regime for $D=\frac{1}{2}$ by look-

ing at the form of $P_0(q)$ near $q=0$. The behavior of the most probable value of the stationary state has often been used (Ref. 14) to describe the influence of noise in a system, but the physical implications of such behavior for the particular model (3.1) are at present unclear.

[34]L. Arnold, W. Horsthemke, and R. Lefever, Z. Phys. B 29, 367 (1978).

[35]N. J. Rao, J. D. Borwankar, and D. Ramkrishna, SIAM (Soc. Ind. Appl. Math.) J. Control 12, 124 (1974).

[36]R. Morf and E. Stoll, in *Numerical Analysis,* edited by J. Descloux and J. Marti (Birkhauser, Basel, 1977); ISNM 37, 139; T. Schneider and E. Stoll, Phys. Rev. B 17, 1302 (1978).

[37]E. Helfand, Bell Syst. Tech. J. 58, 2289 (1979); H. S. Greenside and E. Helfand (unpublished).

[38]D. L. Ermak and H. Buckholz, J. Comput. Phys. 35, 169 (1980).

[39]P. Turk, F. Lantelme, and H. L. Friedman, J. Chem. Phys. 66, 3039 (1977).

[40]Equation (A6) is an exact formula for obtaining Gaussian numbers from numbers with a flat distribution. An approximate method followed in Ref. 36 consists in invoking the central limit theorem and obtaining Gaussian numbers by combining a set of 12 random numbers equally distributed between zero and one. Although some computer time may be saved with the last method we prefer to use (A6).

[41]The relevance of the smallness of Δ in the accuracy of the results has also been recognized in Refs. 37 and 42. On the other hand, in Ref. 38 the authors claimed that their method gives good results even for very large values of Δ. In that work only linear equations with additive noise were considered. For such equations, the transition probability is *a priori* known and the simulation is based on choosing a point $q(t')$ from $q(t)$ according to that transition probability. The simulation is therefore based on the knowledge of the transition probability.

[42]F. de Pasquale, P. Tartaglia, and P. Tombesi, Physica A 99, 58 (1979). F. de Pasquale, P. Tartaglia, and P. Tombesi (unpublished).

[43]N. G. Van Kampen, in *Fundamental Problems in Statistical Mechanics III,* edited by Cohen (North-Holland, Amsterdam, 1975).

[44]E. A. Novikov, Zh. Eksp. Teor. Fiz. 47, 1919 (1964) [Sov. Phys.—JETP 20, 1290 (1965)].

[45]P. Hanggi, Z. Phys. B 30, 85 (1978).

[46]M. San Miguel, Z. Phys. B 33, 307 (1979).