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Exact solitary-wave solution of short different-wavelength optical pulses in many-level atomic absorbers

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A new exact result is presented on the lossless propagation of simultaneous different-wavelength optical pulses through an atomic medium with general N energy levels in an extended Λ or V configuration.

I. INTRODUCTION

In this paper we report a new exact result in the theory of coherent light propagation through a certain type of atomic absorbers having several connected dipole transitions among their energy levels.

Coherent effects in many-level atomic systems have received increasing attention in recent years, and a number of exact analytic results concerning special N-level systems have been given.[1] We have recently contributed a set of previously unexpected conservation laws which apply to the dynamics of a general N-level system.[2] Those results all apply to situations in which an atom or molecule is subjected to steady laser excitation.

Also recently, Konopnicki, Drummond, and Eberly have obtained new exact results[3,4] in the theory of lossless propagation of a number of simultaneous different-wavelength optical pulses through an atomic medium with N energy levels in a cascade configuration. These simultaneous solitons are said to constitute a "pulse." Our new result in this paper also concerns the lossless propagation of simultaneous different-wavelength optical pulses through a general N-level atomic medium but with the energy levels in an extended Λ or V configuration. The two results are, as we shall see, different and in a sense complement each other.

In order for pulse propagation to occur, it is true not only that the pulses and the medium have to satisfy certain conditions, but also that the medium has to be initially prepared in a certain manner. It is thus unlike self-induced transparency in a two-level system.[5] The fact that the simultaneous solitary waves can have widely different wavelengths also makes them distinct from pulse trains observed after large-area pulse breakup in two-level systems. A pulse is a true many-level phenomenon.

II. PROPAGATION OF OPTICAL PULSES IN MANY-LEVEL ABSORBERS

We assume a plane-wave incident electric field $\vec{E}(z,t)$ with $N-1$ distinct frequency components:

$$\vec{E}(z,t)=\sum_{j=1}^{N-1}\left\{\vec{\epsilon}_{j,j+1}\mathscr{E}_{j,j+1}(z,t)\times\exp\left[-i\nu_{j,j+1}\left(t-\frac{z}{c}\right)\right]+\text{c.c.}\right\}, \tag{2.1}$$

where $|\nu_{j,j+1}|$ denotes the carrier frequency of the jth component, $\vec{\epsilon}_{j,j+1}$ is its possibly complex polarization vector, and $\mathscr{E}_{j,j+1}(z,t)$ is its complex amplitude, assumed to be a slowly varying function of z and t in the usual sense. The frequencies $|\nu_{j,j+1}|$ are chosen to be nearly resonant with the successive transition frequencies $|\omega_{j,j+1}|$ in a chain of N dipole-connected energy levels in an atomic or molecular system. In Fig. 1 we give an example of a six-level system, showing the convention of numbering along the chain of dipole allowed transition

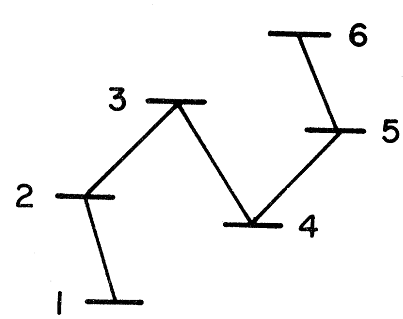

FIG. 1. Possible configurations of energy levels in an atom. Lines connecting levels 1 and 2, 2 and 3, etc., indicate the allowed dipole transitions ($1\leftrightarrow2$) and ($2\leftrightarrow3$), etc.

and different possible configurations. The quantities $\nu_{j,j+1}$ and $\omega_{j,j+1}$ may be considered as being given by

$$\nu_{j,j+1}=\nu_j-\nu_{j+1}\ ,$$
$$\omega_{j,j+1}=\omega_j-\omega_{j+1}=(E_j-E_{j+1})/\hbar\ , \tag{2.2}$$

so that for increasing energies ($E_{j+1}>E_j$), $\nu_{j,j+1}(=-|\nu_{j,j+1}|)$ and $\omega_{j,j+1}(=-|\omega_{j,j+1}|)$ are negative quantities, and for decreasing energies ($E_{j+1}<E_j$), $\nu_{j,j+1}(=|\nu_{j,j+1}|)$ and $\omega_{j,j+1}(=|\omega_{j,j+1}|)$ are positive quantities. This allows the Bloch equations in the rotating-wave approximation to be expressed in a form independent of the energy-level ordering.

We shall express the Bloch equations for an N-level atomic system in the rotating-wave approximation in terms of the time evolution of the (N^2-1)-dimensional generalized Bloch vector

$$\vec{s}(t)=[s_1(t),s_2(t),\ldots,s_{N^2-1}(t)] \tag{2.3}$$

in which the components of $\vec{s}(t)$ are defined as follows.

The first $\frac{1}{2}N(N-1)$ pairs of components of $\vec{s}(t)$ are defined, as were done usually, by

$$u_{jk}=\rho_{jk}+\rho_{kj}\ ,$$
$$v_{jk}=-i(\rho_{jk}-\rho_{kj})\ , \tag{2.4a}$$

where $j,k=1,2,\ldots,N$, $k>j$, and ρ_{jk} are the density-matrix elements in the rotating-wave approximation. The remaining $(N-1)$ diagonal components of $\vec{s}(t)$ are defined by

$$w_j=\left[\frac{2}{j(j-1)}\right]^{1/2}[\rho_{11}+\rho_{22}+\cdots+\rho_{j-1,j-1}-(j-1)\rho_{jj}]$$
$$=\left[\frac{2}{j(j-1)}\right]^{1/2}[w_{12}+2w_{23}+\cdots+(j-1)w_{j-1,j}]\ ,\ j=2,3,\ldots,N \tag{2.4b}$$

where w_{jk} is the usual atomic inversion defined by

$$w_{jk}=\rho_{jj}-\rho_{kk}\ .$$

The generalized Bloch vector so defined has the desirable property that in the absence of decays, the "length" of the vector is constant (of the motion): i.e.,

$$\sum_{j=1}^{N^2-1}s_j(t)^2=u_{12}(t)^2+\cdots+w_N(t)^2=\text{const (of the motion)}\ . \tag{2.5}$$

The equations of motion for the N^2-1 components of the generalized Bloch vector in the rotating-wave approximation are

$$\dot{u}_{j,j+1}=-\Delta_{j,j+1}v_{j,j+1}-\Omega_{j+1,j+2}v_{j,j+2}+\Omega_{j-1,j}v_{j-1,j+1}\ ,$$
$$\dot{v}_{j,j+1}=\Delta_{j,j+1}u_{j,j+1}+\Omega_{j+1,j+2}u_{j,j+2}-\Omega_{j-1,j}u_{j-1,j+1}$$
$$+\left[\frac{2(j+1)}{j}\right]^{1/2}\Omega_{j,j+1}w_{j+1}-\left[\frac{2(j-1)}{j}\right]^{1/2}\Omega_{j,j+1}w_j\ , \tag{2.6a}$$

and for $k>j+1$,

$$\dot{u}_{jk}=-\Delta_{jk}v_{jk}-\Omega_{k-1,k}v_{j,k-1}-\Omega_{k,k+1}v_{j,k+1}+\Omega_{j-1,j}v_{j-1,k}+\Omega_{j,j+1}v_{j+1,k}\ ,$$
$$\dot{v}_{jk}=\Delta_{jk}u_{jk}+\Omega_{k-1,k}u_{j,k-1}+\Omega_{k,k+1}u_{j,k+1}-\Omega_{j-1,j}u_{j-1,k}-\Omega_{j,j+1}u_{j+1,k}\ , \tag{2.6b}$$

and

$$\dot{w}_j = -\left[\frac{2j}{j-1}\right]^{1/2}\Omega_{j-1,j}v_{j-1,j} + \left[\frac{2(j-1)}{j}\right]^{1/2}\Omega_{j,j+1}v_{j,j+1} , \tag{2.6c}$$

where variables with indices <1 or $>N$ should be set equal to zero. Here Δ_{jk} is the cumulative detuning of the $j-k$ transition and $\Omega_{j,j+1}$ is (half of) the appropriate Rabi frequency

$$\Delta_{jk} = \omega_{jk} - \nu_{jk} , \tag{2.7a}$$

$$\Omega_{j,j+1} = \frac{d_{j,j+1}\mathscr{E}_{j,j+1}}{\hbar} , \tag{2.7b}$$

where

$$\omega_{jk} = \left[\frac{E_j - E_k}{\hbar}\right]\left[1 + \frac{v}{c}\right] \tag{2.8}$$

is the usual Doppler-shifted transition frequency between atomic levels j and k (of energies E_j, E_k) and

$$d_{j,j+1} = \langle j \mid \vec{\mathrm{d}} \mid j+1 \rangle \cdot \vec{\epsilon}_{j,j+1}$$

is the appropriate component of the dipole matrix element.

The N^2-1 Bloch equations (2.6) for the atomic variables, combined with $N-1$ reduced Maxwell equations

$$\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial (ct)}\right]\Omega_{j,j+1}(z,t) = C\nu_{j,j+1}d_{j,j+1}^2\langle v_{j,j+1}\rangle , \quad j = 1,2,\ldots,N-1 \tag{2.9}$$

comprise a semiclassical description of $N-1$ electromagnetic pulses propagating in an atomic or molecular medium with pulse lengths short relative to atomic or molecular relaxation times. In Eq. (2.9), C is given by

$$C = 2\pi\mathscr{N}/\hbar c ,$$

where $\mathscr{N}$ is the atomic density, and $\langle \cdots \rangle$ denotes averaging over the Maxwellian velocity distribution of atoms.

Our simultaneous solitary wave (pulse) solution of the coupled Maxwell-Bloch equations (2.6) and (2.9) in the case of the resonant input pulses is obtained as follows.

We first pose the following question. In the Bloch equations (2.6), if we set all the $u(t)$'s and $v(t)$'s equal to zero for all t except the following set of u's and v's:

$$u_{j,j+2}(t) , \quad j = 1,2,\ldots,N-2$$

$$v_{j,j+1}(t) , \quad j = 1,2,\ldots,N-1$$

and if we also leave all the $w(t)$'s generally not equal to zero, can the Bloch equations remain consistent? The answer, as can be easily verified, is affirmative, and we obtain the following set of equations:

$$\begin{aligned}
\dot{u}_{j,j+2} &= -\Omega_{j+1,j+2}v_{j,j+1} + \Omega_{j,j+1}v_{j+1,j+2} , \\
\dot{v}_{j,j+1} &= \Omega_{j+1,j+2}u_{j,j+2} - \Omega_{j-1,j}u_{j-1,j+1} \\
&\quad + \left[\frac{2(j+1)}{j}\right]^{1/2}\Omega_{j,j+1}w_{j+1} - \left[\frac{2(j-1)}{j}\right]^{1/2}\Omega_{j,j+1}w_j , \\
\dot{w}_j &= -\left[\frac{2j}{j-1}\right]^{1/2}\Omega_{j-1,j}v_{j-1,j} + \left[\frac{2(j-1)}{j}\right]^{1/2}\Omega_{j,j+1}v_{j,j+1} .
\end{aligned} \tag{2.10}$$

If we refer to $p = k-j$ in u_{jk} and v_{jk} to be the order of the coherences, our above consideration amounts to restricting ourselves to finding a particular solution of the Bloch equations in which coherences of order three or higher are zero at all times. Although this eliminates a number of variables from our consideration, it still leaves us with a large number of independent variables to consider. We next confine ourselves to an even more special case in which we postulate that the only independent variables are $v_{12}(t)$, $u_{13}(t)$, and $w_2(t)$ and that the remaining variables are dependent on them in a rather simple way. That is, we postulate the following solution:

$$\begin{aligned}
v_{j+1,j+2}(t) &= \beta_{j+1}v_{12}(t) , \\
u_{j+1,j+3}(t) &= \alpha_{j+1}u_{j,j+2}(t) \\
&= \alpha_{j+1}\alpha_j \cdots \alpha_2 u_{13}(t) , \\
w_j(t) &= \gamma_j w_{12}(t) ,
\end{aligned} \tag{2.11}$$

where α_j, β_j, and γ_j must of course be determined in such a way that the Bloch equations are satisfied.

We have found that it is possible to have a solution of the form (2.11) if the pulses are related by

$$\Omega_{j,j+1}(z,t)=\frac{[j(N-j)]^{1/2}}{(N-1)^{1/2}}\Omega_{12}(z,t)\ , \quad j=2,3,\ldots,N \tag{2.12}$$

which gives

$$\alpha_j=\left[\frac{(j+1)(N-j-1)}{(j-1)(N-j-1)}\right]^{1/2},$$
$$\beta_j=\frac{(N-2j)[j(N-j)]^{1/2}}{(N-2)(N-1)^{1/2}}\ , \tag{2.13}$$
$$\gamma_j=\frac{(3N+2-4j)[2j(j-1)]^{1/2}}{6(N-2)}\ ,$$

and hence

$$\begin{aligned}
u_{j,j+2}(t)&=\left[\frac{j(N-j)(j+1)(N-j-1)}{2(N-1)(N-2)}\right]^{1/2}u_{13}(t)\ ,\\
v_{j,j+1}(t)&=\frac{(N-2j)[j(N-j)]^{1/2}}{(N-2)(N-1)^{1/2}}v_{12}(t)\ ,\\
w_j(t)&=\frac{(3N+2-4j)[2j(j-1)]^{1/2}}{6(N-2)}w_{12}(t)\ .
\end{aligned} \tag{2.14}$$

Pulses of the form (2.12) were first used by Cook and Shore.[6] It is easy to verify that Eqs. (2.12) and (2.14) reduce the Bloch equations (2.10) into the following three coupled equations for the three independent variables $u_{13}(t)$, $v_{12}(t)$, and $w_{12}(t)$:

$$\begin{aligned}
\dot{u}_{13}&=-\left[\frac{8}{(N-1)(N-2)}\right]^{1/2}\Omega_{12}v_{12}\ ,\\
\dot{v}_{12}&=\left[\frac{2(N-2)}{N-1}\right]^{1/2}\Omega_{12}u_{13}\\
&\quad+2\Omega_{12}w_{12}\ ,\\
\dot{w}_{12}&=-\frac{6}{N-1}\Omega_{12}v_{12}\ .
\end{aligned} \tag{2.15}$$

We have thus reduced the Bloch equations for the N-level case to a familiar form encountered in the two-level case. This was done, as we have seen, by requiring the remaining atomic variables to depend on u_{13}, v_{12}, and w_{12} in a specific way given by Eq. (2.14), and the pulses to be given by Eq. (2.12). It should be emphasized, however, that the solution remains truly N level, even though special.

It is seen from Eq. (2.15) that the time evolutions of $u_{13}(t)$ and $w_{12}(t)$ are also related, namely, they are proportional to each other save for a constant term. Thus if we let

$$u_{13}(t)=\left[\frac{2(N-1)}{9(N-2)}\right]^{1/2}[w_{12}(t)-w_{12}(-\infty)] \tag{2.16a}$$

and

$$\bar{w}_{12}(t)=\frac{2(N-1)^{1/2}}{3}[w_{12}(t)-\tfrac{1}{4}w_{12}(-\infty)]\ , \tag{2.16b}$$

Eqs. (2.15) reduce to the familiar form

$$\begin{bmatrix}\dot{v}_{12}\\ \dot{\bar{w}}_{12}\end{bmatrix}=\begin{bmatrix}0 & \dfrac{4}{(N-1)^{1/2}}\Omega_{12}\\ -\dfrac{4}{(N-1)^{1/2}}\Omega_{12} & 0\end{bmatrix}\begin{bmatrix}v_{12}\\ \bar{w}_{12}\end{bmatrix}, \tag{2.17}$$

which can be readily solved.

We note from Eqs. (2.14) that if v_{12} is positive, $v_{j,j+1}$ is positive for $j<N/2$ but becomes negative for $j>N/2$. We also note from Eq. (2.12) that $\Omega_{j,j+1}$ is positive for all $j=1,2,\ldots,N-1$. Thus in order to be consistent when combining our Bloch equation result with the reduced Maxwell equations (2.9), we now need to assume an energy-level structure such that $\nu_{j,j+1}$ changes sign when $j>N/2$ to cancel out the effect of the sign change in $v_{j,j+1}$

when $j > N/2$. Thus we assume that N, the number of energy levels, is an odd number ($=2n+1$, $n=1,2,\ldots$) and that the level configuration is an "extended" Λ and V configuration shown in Fig. 2, which has n levels (excluding the top or the bottom level) in the cascade configuration on each side. The levels need not be evenly spaced, but the dipole moments and the frequencies of the incident $N-1$ simultaneous pulses are assumed to satisfy the following conditions:

$$\frac{N-2j}{N-2}\nu_{j,j+1}d_{j,j+1}^2=\nu_{12}d_{12}^2\ ,\qquad j=2,3,\ldots,N-1\ . \tag{2.18}$$

Furthermore, the initial level populations are assumed to satisfy the following relations:

$$\frac{N-2}{N-2j}w_{j,j+1}(0)=w_{12}(0)\ ,\qquad j=2,3,\ldots,N-1 \tag{2.19}$$

where $w_{12}(0)$ is assumed positive.

Assuming the energy-level configuration shown in Fig. 2 for the atomic medium, and assuming that the conditions (2.18) and (2.19) are satisfied, then the existence of the pulse solution follows from Eqs. (2.17) and (2.9).[5] The set of $N-1$ simultaneous solitary pulses, which can propagate through the N-level medium without losses, is given from Eq. (2.17) by

$$\Omega_{j,j+1}(\xi)=\frac{[j(N-j)]^{1/2}}{2\tau}\operatorname{sech}\frac{\xi-\xi_0}{\tau}\ ,\qquad j=1,2,\ldots,N-1 \tag{2.20}$$

where $\xi=t-z/V$, V is the velocity of the pulse, and τ is the pulse length. The "area" of the pulse is

$$\Theta(z,t)=\int_{-\infty}^{t}\frac{4}{(N-1)^{1/2}}\Omega_{12}(z,t')dt' =4\tan^{-1}\left[\exp\frac{\xi-\xi_0}{\tau}\right]\ . \tag{2.21}$$

The evolutions of the atomic variables are given by

$$\begin{aligned}
v_{j,j+1}(\xi)&=-w_{12}(0)\frac{(N-2j)[j(N-j)]^{1/2}}{N-2}\operatorname{sech}\frac{\xi-\xi_0}{\tau}\tanh\frac{\xi-\xi_0}{\tau}\ ,\quad j=1,2,\ldots,N-1\\
u_{j,j+2}(\xi)&=-w_{12}(0)\frac{[j(N-j)(j+1)(N-j-1)]^{1/2}}{2(N-2)}\operatorname{sech}^2\frac{\xi-\xi_0}{\tau}\ ,\quad j=1,2,\ldots,N-2\\
w_j(\xi)&=-w_{12}(0)\frac{(3N+2-4j)[2j(j-1)]^{1/2}}{6(N-2)}\left[-1+\tfrac{3}{2}\operatorname{sech}^2\frac{\xi-\xi_0}{\tau}\right]\ ,\quad j=2,3,\ldots,N\ .
\end{aligned} \tag{2.22}$$

All other u's and v's are equal to zero. We note that we have nonzero two-photon coherences in this solution. The square of the length of the generalized Bloch vector is

$$|\vec{s}(t)|^2=\sum_{j=1}^{N^2-1}s_j^2(t)=u_{12}^2+\cdots+w_N^2 =\frac{(N-1)N(N+1)(N+2)}{90(N-2)}w_{12}(0)^2\ , \tag{2.23}$$

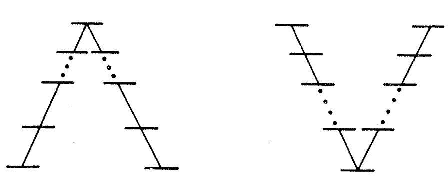

FIG. 2. "Extended Λ and V" configurations of energy levels.

which as we already mentioned is a constant of the motion. There are $N-1$ other independent constants of the motion.[1]

The pulse solution (2.20)–(2.22) for the special $N=3$ case was first given by Konopnicki.[4] For completeness and easy comparison, we write here also the $N-1$ pulse solution which Konopnicki, Drummond, and Eberly[3] recently found for an N-level medium with the *cascade* level configuration as shown in Fig. 3. First the conditions which must be satisfied for the existence of their pulse solutions, i.e., the analog of Eq. (2.18), are

$$\nu_{j,j+1}d_{j,j+1}^2=\nu_{12}d_{12}^2\ ,\quad j=2,3,\ldots,N-1 \tag{2.24}$$

and the initial populations must satisfy

$$w_{j,j+1}(0)=w_{12}(0)>0\ ,\quad j=2,3,\ldots,N-1 \tag{2.25}$$

which is the analog of Eq. (2.19). When Eqs. (2.24)

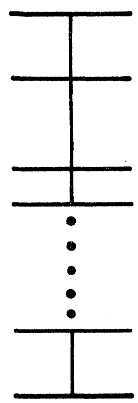

FIG. 3. "Cascade" configurations of energy levels.

and (2.25) are satisfied, their pulses are given by

$$\Omega_{j,j+1}(\xi)=\frac{[j(N-j)]^{1/2}}{\tau}\operatorname{sech}\frac{\xi-\xi_0}{\tau}\,,\qquad j=1,2,\ldots,N-1 \tag{2.26}$$

and the "area" is

$$\Theta(z,t)=\int_{-\infty}^{t}\frac{2}{(N-1)^{1/2}}\,\Omega_{12}(z,t')dt' =4\tan^{-1}\left[\exp\frac{\xi-\xi_0}{\tau}\right]. \tag{2.27}$$

The evolutions of the atomic variables are given by

$$v_{j,j+1}(\xi)=-w_{12}(0)2[j(N-j)]^{1/2}\times\operatorname{sech}\frac{\xi-\xi_0}{\tau}\tanh\frac{\xi-\xi_0}{\tau}\,,\qquad j=1,2,\ldots,N-1$$

$$w_j(\xi)=-w_{12}(0)\frac{[2j(j-1)]^{1/2}}{2}\times\left[-1+2\operatorname{sech}^2\frac{\xi-\xi_0}{\tau}\right],\qquad j=2,3,\ldots,N\,. \tag{2.28}$$

All other v's and all the u's are zero. In contrast to Eqs. (2.22), there is no nonzero two-photon coherence in this case. We find that the square of the length of the generalized Bloch vector for this case is

$$|\vec{s}(t)|^2=\sum_{j=1}^{N^2-1}s_j(t)^2=u_{12}^2+\cdots+w_N^2 =\tfrac{1}{6}N(N^2-1)w_{12}(0)^2\,. \tag{2.29}$$

The two pulse solutions Eqs. (2.18)–(2.25) and Eqs. (2.24)–(2.29) given above illustrate the fact that pulse propagation requires three conditions: The $N-1$ pulses must be simultaneous; the absorbing medium must be, in general, partially excited out of its ground state in accordance with appropriate initial conditions; and the pulse amplitudes and the dipole moments of the atomic medium must satisfy appropriate relations.

We have seen that in obtaining the above pulse solutions, the initial conditions play an important role. In the Appendix, we discuss a special type of symmetry consideration which enables us to predict the symmetry properties of the solutions.

III. SUMMARY

The principal new result in this paper is the analytic solution (2.18)–(2.23) for $N-1$ simultaneous solitary waves propagating through an N-level atomic medium, which has an extended Λ or V energy-level configuration as shown in Fig. 2 under the on-resonance condition. We have included, for completeness, the other analytic pulse solution (2.24)– (2.29) for the case when the N energy levels of the atomic medium have a cascade configuration as shown in Fig. 3, which was found earlier by Konopnicki, Drummond, and Eberly.[3] In both cases, we have seen that pulse propagation requires several conditions: the $N-1$ pulse amplitudes, frequencies, and the dipole moments of the atomic medium must satisfy appropriate relations; the $N-1$ pulse must be simultaneous or nearly so; and the populations of the energy levels of the atomic medium must satisfy certain conditions.

Instead of thinking that the conditions for pulse propagation are difficult to satisfy, one may think in terms of how to use these conditions to control the propagations of laser light pulses, for we now know the optimum conditions for the light pulses to propagate through the medium without losses. This appears to open new possibilities to study ways in which light can be used to control light. The "light switch" would consist of a set of coherent pulses which, when turned on, would prepare the level populations of the atomic medium in appropriate proportions [so to satisfy Eqs. (2.18) or (2.25), for example], and the propagating pulses would consist of a set of simultaneous pulses of the appropriate form and frequencies [e.g., of the form (2.20) or (2.26) and frequencies given by (2.18) or (2.24)]. The use of such light switch may provide an exceedingly rapid response mechanism which could have many useful applications.

ACKNOWLEDGMENTS

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APPENDIX: SYMMETRIES OF BLOCH EQUATION SOLUTIONS UNDER SPECIAL INITIAL CONDITIONS

Suppose the laser-induced N-level atomic system (or any level configuration) has the following "reflection" symmetry:

$$\begin{aligned}&\Omega_{j,j+1}=\Omega_{N-j,N-j+1}\ ,\\&\Delta_{j,j+k}=-\Delta_{N-j-k+1,N-j+1},\\&\qquad j,k=1,2,\ldots\end{aligned}\tag{A1}$$

where Ω denotes the generally time-dependent Rabi frequency and Δ the detuning. If initially at $t=0$ we have

$$\begin{aligned}&u_{j,j+k}(t)=\pm u_{N-j-k+1,N-j+1}(t)\ ,\\&v_{j,j+k}(t)=\mp v_{N-j-k+1,N-j+1}(t)\ ,\\&w_{j,j+1}(t)=\mp w_{N-j,N-j+1}(t)\ ,\end{aligned}\tag{A2}$$

then (A2) remains true for all later time t (where the upper and lower signs refer to two separate conditions).

Conversely, we can make the two separate substitutions given by (A2) into the Bloch equations (2.6) and thereby reduce the N^2-1 coupled Bloch equations into two reduced sets of equations with $\frac{1}{2}(N^2-1)$ equations each if N is an odd number, or if N is an even number, into two reduced sets with $\frac{1}{2}N^2-1$ and $\frac{1}{2}N^2$ equations each. The two substitutions can be shown to be associated with the symmetric and antisymmetric representations of the S_2 symmetric group. We note that solutions (2.22) and (2.28) are special classes of solutions which exhibit the respective symmetries given by (A2).

The substitutions (A2) under the special conditions (A1) reduce the number of Bloch equations to be considered in half. This is a considerable simplification especially when N is not too large.

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