PHYSICAL REVIEW A VOLUME 26, NUMBER 3 SEPTEMBER 1982

Modified effective-range function

H. van Haeringen
Department of Mathematics, Delft University of Technology, Delft, The Netherlands

L. P. Kok
Institute for Theoretical Physics, P.O. Box 800, Groningen, The Netherlands
(Received 19 December 1980)

We derive a general formula for a modified effective-range function (MERF), $K_l^M(k^2)$, for all partial waves, $l=0,1,\ldots$. This is a generalization of the effective-range function associated with a short-range potential, $K_l(k^2)=k^{2l+1}\cot\delta_l(k)$. Here k^2 is the energy variable and $\delta_l(k)$ the phase shift. The MERF $K_l^M(k^2)$ can be associated with a potential $V(r)$ that allows a decomposition into a long-range and a short-range component. It is a complex real-meromorphic function of k^2 in the complex k plane in a domain containing the origin. This (large) domain is determined by the short-range part of the potential. We give a simple formula for $K_l^M(k^2)$, valid for all $l=0,1,\ldots$. It can be used if the long-range part of the potential is analytic at $r=0$. For $l=0$ we have the simple expression

$$K_0^M(k^2)=|f_0(k)|^{-2}k[\cot\delta_0^M(k)-i]+f_0'(k,0)/f_0(k) .$$

Here $f_0(k)$ and $f_0(k,r)$ are the Jost function and Jost solution, respectively, associated with the long-range part of the potential, and $\delta_0^M(k)$ is the difference between the s-wave phase shift associated with the total potential and that of the long-range potential. The prime in $f_0'(k,0)$ denotes differentiation with respect to r. The extension to the case of the Coulomb potential which violates the condition of analyticity at $r=0$ is briefly discussed.

I. INTRODUCTION

Effective-range theory has been very successful in the analysis and interpretation of low-energy two-body scattering data. The basic idea of this theory in the early days of quantum scattering theory was that at low energy the (rotationally invariant) interaction and the s-wave phase shift δ caused by the interaction can be parametrized, in good approximation, by two parameters, called the scattering length a and the effective range r_0.

In this section we shall restrict ourselves to s waves, i.e., angular-momentum quantum number $l=0$. Throughout in this paper we shall use units such that $\hbar=1=2m$, where m is the reduced mass, and we denote the energy variable by k^2. Then we can write the effective-range (ER) expansion as

$$k\cot\delta(k)=-a^{-1}+\tfrac{1}{2}r_0k^2+\ldots , \qquad (1.1)$$

where the ellipses represent higher-order terms in k^2, which may be neglected when the scattering energy is sufficiently small. The so-called shape-independent approximation is obtained by retaining only the first two terms of the series on the right-hand side (rhs) of Eq. (1.1). A simple derivation of Eq. (1.1) was given by Blatt and Jackson[1] and Bethe[2] in 1949.

The validity of Eq. (1.1) depends on the asymptotic behavior of the potential. We restrict ourselves to local rotationally invariant potentials V with the following properties:

(i) $V(r)\to 0$ for $r\to\infty$;
(ii) V is integrable on (s,t) for all positive s and t; and
(iii) near the origin V satisfies

$$V(r)=O(r^{-\beta}),\quad r\to 0,\quad \beta<2 .$$

Furthermore, we assume that a proper Jost solution $f(k,r)$ can be associated with V (see the discussion in the second paragraph of Sec. III). In this paper we shall call V a *short-range* potential if

$$\int_R^\infty |V(r)|e^{\mu r}dr<\infty , \qquad (1.2)$$

from some $\mu>0$, $R>0$. When the integral in Eq. (1.2) equals ∞ for all $\mu>0$ and $R>0$, we shall call

V a *long-range* potential.[3]

There are many systems in physics where the interaction can be decomposed in a natural way into long- and short-range parts. The interaction between two charged hadrons consisting of a Coulomb and a nuclear part exemplifies such a two-range potential. Since the ER expansion (1.1) is not valid for such a potential one is interested in finding suitable modifications. The theory of modified ER expansions was initiated by Breit and co-workers[4] for the Coulomb plus nuclear potential. Subsequently many investigators[1,2,5–25] have studied modifications of Eq. (1.1), not only for the Coulomb plus nuclear potential, but also for other two-range potentials.

In atomic and molecular physics long-range potentials of the type

$$V(r)=cr^{-\alpha}, \quad r>R>0\,, \tag{1.3}$$

where α is a positive integer, play an important role. For such potentials the ER expansion (1.1) breaks down. The term at which it breaks down depends on α (and on l). Modified ER expansions for such potentials, especially for $\alpha=4$ and 6, have been studied extensively by Spruch and co-workers in a series of papers,[7–11] and by others. Shakeshaft[13] has investigated the r^{-3} potential. The case of real $\alpha>3$ has been studied extensively by Levy and Keller.[6] O'Malley *et al.*[8] have exploited the fact that the exact solution of the radial Schrödinger equation with the r^{-4} potential can be expressed in terms of modified Mathieu functions. (In this connection see also Refs. 19 and 26.) Afterwards Hinckelmann and Spruch[11] showed that these (complicated) functions need not be introduced into the analysis if one is interested in first-order effects (in the potential strength) only. These authors give, for linear combinations of the r^{-4} and the r^{-6} potentials, expansions that can be expressed as

$$\begin{aligned} k^{-1}\tan\eta_0(k) = {} & c_1+c_2k+c_3k^2\ln k \\ & +c_4k^2+c_5k^3+c_6k^4\ln k \\ & +\dots\,, \end{aligned} \tag{1.4}$$

where η_0 is the s-wave phase shift. The explicitly calculated coefficients c_i are not relevant for the discussion here. The most interesting point in Eq. (1.4) is the occurrence of $\ln k$ which shows that $k\cot\eta_0(k)$ is not an analytic function of k at $k=0$. Note also the occurrence of odd powers of k.

It is instructive for the discussion in this paper to consider also in some detail the two-range potential

$$V(r)=V_c(r)+V_s(r)\,, \tag{1.5}$$

where

$$V_c(r)\equiv Ze^2/r\equiv 2k\gamma/r \tag{1.6}$$

is the pure Coulomb potential, γ is Sommerfeld's parameter, and V_s is a short-range potential. For such a potential it has become customary to modify the left-hand side (lhs) of Eq. (1.1) in such a manner that the rhs again is an expansion in powers of k^2. In the case of a *repulsive* Coulomb potential the following modification of Eq. (1.1) has been found:

$$2k\gamma g(\gamma)+C_0^2k\cot\delta^c(k)=-1/a^c+\tfrac{1}{2}r^ck^2+\dots\,. \tag{1.7}$$

Here k is assumed to be positive,

$$C_0^2\equiv 2\pi\gamma(e^{2\pi\gamma}-1)^{-1}$$

and $g(\gamma)$ are *pure Coulomb* quantities,

$$\begin{aligned} g(\gamma) &= -\ln\gamma+\mathrm{Re}\psi(i\gamma) \\ &= -\ln\gamma - C+\gamma^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+\gamma^2)}\,, \\ & \qquad k>0,\ \gamma>0\,, \end{aligned} \tag{1.8}$$

where $\psi(z)\equiv\Gamma'(z)/\Gamma(z)$ is the digamma function and $C\approx 0.5772$ is Euler's constant. In Eq. (1.7), $\delta^c(k)$ is defined by

$$\delta^c(k)=\delta^v(k)-\sigma(k)\,, \tag{1.9}$$

where

$$\sigma(k)=\arg\Gamma(1+i\gamma)$$

is the pure Coulomb phase shift and $\delta^v(k)$ is the total phase shift caused by V [Eq. (1.5)].

It is relatively easy to modify Eq. (1.7) such that it is also valid for an *attractive* Coulomb potential. More complicated is the relaxation of the condition $k>0$. We are interested especially in *Taylor-series* expansions. Although other expansions such as those given by Eq. (1.4) can be quite useful, a Taylor series has many advantages. Indeed, by analytic continuation into the complex k plane we then get a function which is *analytic* in a certain domain of the k plane, which contains the origin [cf. Eq. (1.4) where analytic continuation does *not* lead to a function analytic at $k=0$].

Since the function on the lhs of Eq. (1.7) is of great importance, we shall call its analytic continuation the Coulomb-MERF and denote it by $K^c(k^2)$. The analytic continuation of $g(\gamma)$ leads to a function that is often denoted by $h(\gamma)$. Its generalization, valid for Coulomb *repulsion and attraction,* is denoted by $H(\gamma)$, cf. Ref. 21,

$$H(\gamma)=\psi(i\gamma)+\tfrac{1}{2}(i\gamma)^{-1}-\ln[i\gamma\,\mathrm{sgn}(Z)]\ , \tag{1.10}$$

where sgn is the signum function, $\mathrm{sgn}(Z)=1$ for $Z>0$ and $\mathrm{sgn}(Z)=-1$ for $Z<0$. The Coulomb-MERF then is

$$K^c(k^2)=2k\gamma H(\gamma)+C_0^2k[\cot\delta^c(k)-i]\ . \tag{1.11}$$

Let ρ be the radius of convergence of the Taylor expansion (in powers of k) in Eq. (1.7), then K^c is analytic in $|k|<\rho$. An interesting problem concerns the relation between ρ and the short-range potential V_s. For some special forms for $V_s(r)$ this problem has been solved by Cornille and Martin,[14] Lambert,[17] Hamilton *et al.*,[20] and others. In Ref. 21 the Coulomb-MERF for Coulomb plus (nonlocal) separable potentials with rational form factors has been studied.

We would like to make the following two remarks about the analyticity of K^c.

(i) Since K^c is real when k^2 is real (the coefficients $-1/a^c, \frac{1}{2}r^c, \ldots$, are real), it is a *real-analytic function.*

(ii) K^c *can* have poles near $k=0$, whose position depends on the coupling constant of the short-range potential V_s, but not on the form of the function $V_s(r)$.

Therefore we prefer to call K^c a *real-meromorphic* function (meromorphic means analytic except for poles). It is instructive to consider an example.

Let V_s be the Yukawa potential

$$V_s(r)=\lambda r^{-1}\exp(-\mu r)\ ,$$

with λ real and $\mu>0$. Then K^c has branch points at $k=\frac{1}{2}i\mu$ and $k=-\frac{1}{2}i\mu$. The position of these branch-point singularities is independent of the coupling constant λ. In addition K^c can have one or more poles near $k=0$, whose position depends upon λ. For a certain value of λ the Coulomb-modified scattering length a^c can be zero so that K^c has a pole at the origin. In this example K^c is real meromorphic in the domain $|k|<\frac{1}{2}\mu$.

We note that Oppenheim Berger *et al.*[9,10] and Oppenheim Berger and Spruch[10] have studied modified effective-range *theory* (MERT) for long-range-plus-short-range potentials. These authors have already pointed out the possibility of constructing a function that has the same structure as the MERF, $K_l^M(k^2)$. Our expression for $K_l^M(k^2)$ [see Eqs. (4.10) and (4.11)] is given in a simple and more explicit form, in terms of the Jost solutions.

The organization of this paper is as follows. In Sec. II we give the definition of the MERF in terms of integrals involving so-called regular and irregular solutions of Schrödinger's equation. These solutions are required to be entire real-analytic functions of k^2. Regular solutions having this property are easily determined, in contrast to irregular solutions with this property. In Sec. III we reduce the formula for the MERF by using the Jost solution associated with the long-range part of the potential, V_L. In Sec. IV we determine the proper irregular solution $\chi_l(k,r)$ associated with V_L, such that the MERF is a generalization of the ordinary ER function

$$K_l(k^2)=k^{2l+1}\cot\delta_l(k)\ .$$

Our main result consists of Eqs. (4.10) and (4.11) which give a simple formula for the MERF. This formula is applicable if $V_L(r)$ is analytic at $r=0$. In Sec. V we briefly discuss the extension to the case of the Coulomb potential which violates this condition of analyticity. Preliminary and related results on MERF have been reported in Ref. 27.

II. DEFINITION OF MERF

The purpose of this paper is to construct a general formula for the MERF which is valid for any *two-range potential*

$$V(r)=V_L(r)+V_s(r)\ , \tag{2.1}$$

and for all $l=0,1,2,\ldots$. Here V_L is a *long-range potential* whose range is (much) larger than the range of V_s. It may or may not satisfy the short-range condition given by Eq. (1.2). It is instructive to consider an example. Let

$$V_L(r)=\lambda_1 e^{-\mu r},\quad V_s(r)=\lambda_2 e^{-\nu r}\ ,$$
$$0<\mu<<\nu\ . \tag{2.2}$$

Then $k\cot\delta_0(k)$ is real meromorphic in the region $|k|<\frac{1}{2}\mu$. This follows from analyticity properties of the Jost function, cf. Refs. 28–31. We want to construct a MERF that is real meromorphic in the much larger region $|k|<\frac{1}{2}\nu$. This *large* region is associated with the *short*-range part of the potential. (Note that the splitting into V_L and V_s is not unique, since the small-r behavior of the potentials is irrelevant. The range of a potential is connected with the region in which the Jost function is analytic.)

The MERF should be a generalization of the Coulomb-MERF. *A fortiori,* it should be a generalization of the (ordinary) ER function[32]

$$K_l(k^2)=k^{2l+1}\cot\delta_l(k),\quad l=0,1,2,\dots\ . \tag{2.3}$$

In analogy with and as a generalization of the work on the Coulomb-MERF by Cornille and Martin[14] and by Lambert,[17] we define the following MERF K_l^M:

$$K_l^M(k^2)=-(W+I_1)/I_2\ , \tag{2.4}$$

where W is the Wronskian

$$W=W(\chi_l,\phi_l)=\chi_l(k,r)\frac{d}{dr}\phi_l(k,r)-\phi_l(k,r)\frac{d}{dr}\chi_l(k,r)\ ,$$

and

$$I_1=\int_0^\infty \chi_l(k,r)V_s(r)\phi_{Vl}(k,r)dr\ ,$$
$$I_2=\int_0^\infty \phi_l(k,r)V_s(r)\phi_{Vl}(k,r)dr\ .$$

The functions ϕ_l and ϕ_{Vl} are the so-called *regular* solutions of Schrödinger's equation with V_L and V, respectively, determined by[30]

$$\lim_{r\to 0} r^{-l-1}\phi_l(k,r)=1\ ,$$
$$\lim_{r\to 0} r^{-l-1}\phi_{Vl}(k,r)=1\ .$$

χ_l is an *irregular* solution of Schrödinger's equation with the long-range potential V_L. In Sec. IV we shall give the precise definition of χ_l.

The functions ϕ_l, ϕ_{Vl}, and χ_l are (required to be) *entire* analytic functions of k^2, and they are real when k is real. As a consequence, K_l^M is real meromorphic at $k=0$.

The definition of K_l^M in the form given by Eq. (2.4) serves to deduce the domain in which K_l^M is real meromorphic. This can be performed by means of estimates in the same way as Cornille and Martin have done in the Coulomb case.[14] However, Eq. (2.4) gives no explicit information about the behavior of the (modified) phase shift at zero energy. We shall reduce the rhs of Eq. (2.4) to a simpler, more explicit form that has an apparent resemblance with the well-known Coulomb-MERF.

III. REDUCTION OF THE MERF FORMULA

In this section we shall reduce Eq. (2.4), obtaining thereby a simpler expression for the MERF K_l^M. For convenience we shall assume in this section that k is real positive.

Let $f_l(k,r)$ be the Jost solution for V_L, and $f_{Vl}(k,r)$ the Jost solution for $V=V_L+V_s$. If

$$\int^\infty |V_L(r)|\,dr<\infty$$

(cf. Ref. 3), these Jost solutions have a simple asymptotic behavior, as follows from[30]:

$$\lim_{r\to\infty} e^{-ikr}f_l(k,r)=1\ ,$$
$$\lim_{r\to\infty} e^{-ikr}f_{Vl}(k,r)=1\ . \tag{3.1}$$

However, if

$$\int^\infty |V_L(r)|\,dr=\infty\ ,$$

the Jost solutions have a different and more complicated asymptotic behavior. In this case a modified version of Eq. (3.1) holds. For example, for the Coulomb-Jost solution one has

$$\lim_{r\to\infty}\exp[-ikr+i\gamma\ln(2kr)]f_l(k,r)=1\ , \tag{3.2}$$

as is well known. For other potentials tending (more) slowly to zero for $r\to\infty$, modifications of Eq. (3.1) have been given in the literature, see, e.g., Matveev and Skriganov,[33] and Reed and Simon.[34] We shall not discuss these modifications here, but we simply assume that proper Jost solutions associated with V_L and V exist.

Following Newton (Ref. 30, Chap. 12), we define the Jost *function* for V_L by

$$f_l(k)=\lim_{r\to 0}[l!/(2l)!](-2ikr)^l f_l(k,r)\ . \tag{3.3}$$

The Jost function for V, $f_{Vl}(k)$, is similarly defined by replacing on the rhs of Eq. (3.3), $f_l(k,r)$ by $f_{Vl}(k,r)$.

We define a double-factorial function by putting, for all integers n,

$$(2n-1)!!=\pi^{-1/2}2^n\Gamma(n+\tfrac{1}{2}) .$$

Then we have for $l\in\mathcal{N}$, $l=0,1,\ldots,$

$$(2l+1)!!=1.3.5\ldots(2l+1) .$$

We introduce an auxiliary solution β_l, defined by

$$\beta_l(k,r)=(2l+1)!!(-ik)^l f_l(k,r)/f_l(k) . \quad (3.4)$$

From Eq. (3.3) we have

$$\lim_{r\to 0} r^l\beta_l(k,r)=d_l , \quad (3.5)$$

where

$$d_l=(2l-1)!!(2l+1)!! . \quad (3.6)$$

It is not difficult to show that (note $k>0$)

$$\phi_l(k,r)=k^{-2l-1}|f_l(k)|^{-2}\text{Im}\beta_l(k,r) . \quad (3.7)$$

Any irregular solution for V_L can be expressed as a linear combination of the regular solution ϕ_l and the Jost solution. For the irregular solution χ_l we choose the following combination:

$$\chi_l(k,r)=\text{Re}\beta_l(k,r)-A_l(k)\phi_l(k,r) . \quad (3.8)$$

Here A_l is a real function of k (note $k>0$) which still has to be determined. One easily verifies

$$\lim_{r\to 0} r^l\chi_l(k,r)=d_l , \quad (3.9)$$

$$W(\chi_l,\phi_l)=(2l+1)d_l=[(2l+1)!!]^2 . \quad (3.10)$$

Note that the function A_l plays no role in these expressions.

In order to reduce Eq. (2.4) we use

$$f_{Vl}(k)=f_l(k)+(-ik)^l[(2l+1)!!]^{-1}\int_0^\infty f_l(k,r)V_s(r)\phi_{Vl}(k,r)dr . \quad (3.11)$$

This important, nontrivial relation can be derived with the help of the two-potential formalism, as has been proved in Ref. 35. (It holds also for nonlocal V_s, provided the long-range potential V_L is local.)

Substitution of Eq. (3.4) into Eq. (3.11) gives

$$f_{Vl}(k)/f_l(k)=1+[(2l+1)!!]^{-2}\int_0^\infty \beta_l(k,r)V_s(r)\phi_{Vl}(k,r)dr . \quad (3.12)$$

By inserting Eqs. (3.7), (3.8), and (3.10) into Eq. (2.4), and using Eq. (3.12) we easily obtain

$$K_l^M(k^2)=A_l(k)-k^{2l+1}|f_l(k)|^{-2}\frac{\text{Re}[f_{Vl}(k)/f_l(k)]}{\text{Im}[f_{Vl}(k)/f_l(k)]} . \quad (3.13)$$

Let the phase shifts τ_l and δ_l^M be defined by

$$f_l(k)=\exp[-i\tau_l(k)]\,|f_l(k)| , \quad (3.14)$$
$$f_{Vl}(k)=\exp\{-i[\tau_l(k)+\delta_l^M(k)]\}\,|f_{Vl}(k)| .$$

Then we have

$$f_{Vl}(k)/f_l(k)\pm\text{c.c.}=\{\exp[-i\delta_l^M(k)]\pm\exp[i\delta_l^M(k)]\}\,|f_{Vl}(k)/f_l(k)| ,$$

where c.c. stands for complex conjugate. By using this expression we obtain from Eq. (3.13),

$$K_l^M(k^2)=A_l(k)+|f_l(k)|^{-2}k^{2l+1}\cot\delta_l^M(k) . \quad (3.15)$$

In the next section we shall study the irregular solution χ_l, and express A_l in terms of the Jost solution associated with the long-range potential V_L.

IV. UNIQUE IRREGULAR SOLUTION χ_l AND A SIMPLE FORMULA FOR THE MERF

In this section we shall determine the function χ_l and derive a simple expression for A_l, and thereby for the MERF K_l^M, cf. Eqs. (3.8) and (3.15).

We assume that the large-range part of the potential, V_L, is analytic at $r=0$. Examples are given by

$$V_L(r)=(r+d)^{-\alpha} \quad (d,\alpha>0)$$

and

$$V_L(r)=e^{-\mu r} \quad (\mu>0)\ .$$

Then any irregular solution associated with V_L can be expanded in a Laurent series at $r=0$. In view of Eq. (3.9) we have for χ_l,

$$\chi_l(k,r)=d_l r^{-l}+c_1 r^{1-l}+\ldots+c_{2l}r^l +c_{2l+1}r^{l+1}+O(r^{l+2}),\quad r\to 0 \tag{4.1}$$

where the (unknown) coefficients c_n are functions of k. Similarly,

$$\phi_l(k,r)=r^{l+1}+O(r^{l+2}),\quad r\to 0\ . \tag{4.2}$$

Since the class of irregular solutions consists of linear combinations of χ_l and ϕ_l, it follows from Eqs. (4.1) and (4.2) that the coefficients d_l and c_{2l+1} determine χ_l completely. We have chosen d_l already, which leaves the freedom to determine c_{2l+1}. We use this freedom to ensure that the following two requirements are met:

(i) For vanishing V_L the ER function is retrieved, i.e., when

$$V_L\to 0,\quad K_l^M(k^2)\to k^{2l+1}\cot\delta_l(k);$$

(ii) $\chi_l(k,r)$ is an entire analytic function of k^2.

One easily verifies that $r^l\chi_l(k,r)$ is an analytic function of r^2 when $V_L\equiv 0$. So in this case the coefficient c_{2l+1} in Eq. (4.1) vanishes. Therefore we impose this condition on χ_l,

$$c_{2l+1}=0\ . \tag{4.3a}$$

It is more convenient to use the equivalent condition

$$D^{2l+1}r^l\chi_l(k,r)=0\ , \tag{4.3b}$$

where we have introduced the operation D, defined by

$$D^{2l+1}=\lim_{r\to 0}\left[\frac{d}{dr}\right]^{2l+1}\ . \tag{4.4}$$

It is interesting to note that for $l=0$ we obtain from Eqs. (4.1)–(4.3) the familiar boundary conditions

$$\chi_0(k,0)=1,\quad \chi_0'(k,0)=0\ ,$$

$$\phi_0(k,0)=0,\quad \phi_0'(k,0)=1\ .$$

The function χ_l is completely determined by the boundary conditions Eqs. (3.6) and (4.3):

$$d_l=(2l+1)!!(2l-1)!!$$

and $c_{2l+1}=0$. Since the variable k plays no role in these boundary conditions, our requirement (ii) is also fulfilled: According to a theorem by Poincaré, a solution of Schrödinger's equation that is determined by k-independent boundary conditions is an entire analytic function of k^2 (cf. De Alfaro and Regge, Ref. 29, p. 9).

The derivation of a simple expression for A_l is now easy. From Eq. (4.2) we have

$$D^{2l+1}r^l\phi_l(k,r)=(2l+1)!\ , \tag{4.5}$$

hence, by using Eqs. (3.8) and (4.3b),

$$A_l(k)=D^{2l+1}r^l\mathrm{Re}\beta_l(k,r)/(2l+1)!,\quad k>0\ . \tag{4.6}$$

The condition $k>0$ can easily be relaxed. Defining

$$M_l(k)=D^{2l+1}r^l\beta_l(k,r)/(2l+1)!\ , \tag{4.7}$$

we have

$$A_l(k)=\mathrm{Re}M_l(k),\quad k>0\ . \tag{4.8}$$

Furthermore, from Eqs. (3.7) and (4.5) we obtain

$$\mathrm{Im}M_l(k)=k^{2l+1}\,|f_l(k)|^{-2},\quad k>0\ . \tag{4.9}$$

By combining Eqs. (3.15) and (4.7)–(4.9) we finally obtain

$$K_l^M(k^2)=M_l(k)+|f_l(k)|^{-2}k^{2l+1}[cot\delta_l^M(k)-i]\ , \tag{4.10}$$

where

$$M_l(k)=(-\tfrac{1}{2}ik)^l(l!)^{-1}D^{2l+1}r^lf_l(k,r)/f_l(k)\ . \tag{4.11}$$

These equations (4.10) and (4.11), valid for *complex* k, give the desired formula for the MERF. It is interesting to note the similarity with the formula for the s-wave Coulomb-MERF, $K_0^c(k^2)$, given by Eq. (1.11). [For $K_l^c(k^2)$, $l=0,1,2,\ldots$, a similar expression holds; see Ref. 21.]

For $l=0$ Eqs. (4.9)–(4.11) reduce to

$$\mathrm{Im}M_0(k)=k\,|f_0(k)|^{-2},\quad k>0\,,$$
$$K_0^M(k^2)=M_0(k)+|f_0(k)|^{-2}k[\cot\delta_0^M(k)-i]\,,$$
$$M_0(k)=f_0'(k,0)/f_0(k,0)\,. \qquad (4.12)$$

From these expressions we can easily verify that the ordinary ER function $K_0(k^2)=k\,\cot\delta_0(k)$ is retrieved when $V_L=0$. Indeed, in this case we have $f_0(k,r)=e^{ikr}$, hence $f_0(k)=f_0(k;0)=1$, and $f_0'(k,0)=ik=M_0(k)$.

V. DISCUSSION

The principal result of this paper consists of a simple formula for the MERF K_l^M, $l=0,1,\ldots$, which we associate with a two-range potential V consisting of a long- and a short-range component $V=V_L+V_s$. This formula is given by Eqs. (4.10) and (4.11),

$$\begin{aligned}K_l^M(k^2)&=M_l(k)\\&\quad+|f_l(k)|^{-2}k^{2l+1}[\cot\delta_l^M(k)-i]\,,\\M_l(k)&=(-\tfrac{1}{2}ik)^l(l!)^{-1}\\&\quad\times\lim_{r\to 0}\left[\frac{d}{dr}\right]^{2l+1}r^lf_l(k,r)/f_l(k)\,.\end{aligned} \qquad (5.1)$$

Here $f_l(k,r)$ is the Jost solution and $f_l(k)$ the Jost function[30] associated with V_L. This is an important result in the following sense.

The MERF K_l^M is a real-meromorphic function of k^2 in a (relatively) *large* region containing the origin $k=0$, determined by the *short*-range component V_s of the two-range potential. In contrast, the ordinary ER function

$$K_l(k^2)=k^{2l+1}\cot\delta_l(k)$$

is real meromorphic in a *small* region determined by the *long*-range component V_L. When $V_L(r)$ behaves asymptotically as a power of r, $K_l(k^2)$ is even not analytic at $k=0$.

It is worthwhile to observe that both terms on the rhs of Eq. (4.10) separately may be strongly varying functions near $k=0$, but that their sum K_l^M is smooth (meromorphic) near $k=0$.

The function M_l is well defined when the long-range part of the potential, V_L, is analytic at $r=0$. However, the definition of M_l given by Eq. (5.1) has to be modified when V_L has a first-order pole at $r=0$ (e.g., when V_L is the Coulomb, Hulthén, or Yukawa potential.) In this paper we shall not give all the details of this modification. Instead we briefly discuss the Coulomb case and refer to Ref. 27 for more details.

First we consider the $l=0$ case. The irregular solution $\chi_0(k,r)$ of Schrödinger's equation with the Coulomb potential, $V_c(r)=2k\gamma/r$, contains a logarithmic term $\ln r$ which means that the boundary condition given by Eq. (4.3b) (for $l=0$) cannot be imposed. In this case we determine $\chi_0(k,r)$ by the modified boundary condition

$$\lim_{r\to 0}\left[\frac{d}{dr}\chi_0(k,r)-B_0(k^2,r)\right]=0\,, \qquad (5.2)$$

where

$$B_0(k^2,r)=2k\gamma[\ln|2k\gamma r|+2C] \qquad (5.3)$$

and C is Euler's constant as before. Note that B_0 is k independent since $k\gamma$ is constant. With this boundary condition (5.2) we get the following expressions for the modified functions A_0 and M_0:

$$\begin{aligned}A_0(k)&=2k\gamma\,\mathrm{Re}H(\gamma),\quad k>0\\M(k)&=2k\gamma H(\gamma)\,,\end{aligned} \qquad (5.4)$$

where $H(\gamma)$ is given by Eq. (1.10). Consequently we retrieve for K_0^M just the well-known expression given by Eq. (1.11) for the Coulomb-MERF K_0^c.

For $l>0$ one can define functions $B_l(k^2,r)$ to determine $\chi_l(k,r)$ by imposing boundary conditions similar to Eq. (5.2). These functions *do* depend on k^2. However, they are real *polynomials* in k^2 and their r dependence is simple,

$$B_l(k^2,r)=P_l^{(1)}(k^2)\ln r+P_l^{(2)}(k^2)\,.$$

The polynomials $P_l^{(1)}$ and $P_l^{(2)}$ may be chosen such that the well-known expression for the Coulomb-modified effective-range function $K_l^c(k^2)$ is retrieved.

[1]J. M. Blatt and J. D. Jackson, Phys. Rev. 76, 18 (1949).

[2]H. A. Bethe, Phys. Rev. 76, 38 (1949).

[3]In a different context, a *short-range* potential is defined by requiring that the T matrix be continuous on the energy shell. In this definition $V(r)$ has a short range if

$$\int_R^\infty |V(r)|\,dr < \infty \; .$$

See, e.g., Ref. 34.

[4]G. Breit, E. Condon, and R. Present, Phys. Rev. 50, 825 (1936); G. Breit, B. Thaxton, and L. Eisenbud, *ibid.* 55, 1018 (1939).

[5]L. D. Landau and Ya. Smorodinsky, Zh. Eksp. Teor. Fiz. 14, 269 (1944); G. F. Chew and M. L. Goldberger, Phys. Rev 75, 1637 (1949); J. D. Jackson and J. M. Blatt, Rev. Mod. Phys. 22, 77 (1950); T. Teichman, Phys. Rev. 83, 141 (1951); T. L. Trueman, Nucl. Phys. 26, 57 (1961); A. Partensky and M. Ericson, *ibid.* B 1, 382 (1967).

[6]B. R. Levy and J. B. Keller, J. Math. Phys. 4, 54 (1963).

[7]L. Spruch, T. F. O'Malley, and L. Rosenberg, Phys. Rev. Lett. 5, 375 (1960).

[8]T. F. O'Malley, L. Spruch, and L. Rosenberg, J. Math. Phys. 2, 491 (1961).

[9]T. F. O'Malley, L. Rosenberg, and L. Spruch, Phys. Rev. 125, 1300 (1962); T. F. O'Malley, *ibid* 130, 1020 (1963); R. Oppenheim Berger, H. B. Snodgrass, and L. Spruch, *ibid.* 185, 113 (1969).

[10]R. Oppenheim Berger, T. F. O'Malley, and L. Spruch, Phys. Rev. 137, A1068 (1965); R. Oppenheim Berger and L. Spruch, *ibid.* 138, B1106 (1965).

[11]O. Hinckelmann and L. Spruch, Phys. Rev. A 3, 642 (1971).

[12]P. S. Ganas, Phys. Rev. A 5, 1684 (1972); 4, 2224 (1971).

[13]R. Shakeshaft, J. Phys. B 5, L115 (1972).

[14]H. Cornille and A. Martin, Nuovo Cimento 26, 298 (1962).

[15]Yu. L. Mentkovsky, Nucl. Phys. 65, 673 (1965).

[16]S. Klarsfeld, Nuovo Cimento A 48, 1059 (1967).

[17]E. Lambert, Helv. Phys. Acta 42, 667 (1969).

[18]A. M. Badalyan and Yu. A. Simonov, Yad. Fiz. 11, 1112 (1970) [Sov. J. Nucl. Phys. 11, 618 (1970)].

[19]N. A. W. Holzwarth, J. Math. Phys. 14, 191 (1973).

[20]J. Hamilton, I. Øverbö, and B. Tromborg, Nucl. Phys. B 60, 443 (1973).

[21]H. van Haeringen, J. Math. Phys. 18, 927 (1977).

[22]E. S. Chang, Phys. Rev. 9, 1644 (1974); J. Phys. B 14, 893 (1981).

[23]Vishwamittar, J. Phys. B 11, 2313 (1978); Vishwamittar and Harminder, *ibid.* 13, 141 (1980).

[24]M. J. Seaton, Proc. Phys. Soc. London A 70, 620 (1970); 88, 801 (1966); 88, 815 (1966).

[25]P. G. Burke, *Potential Scattering in Atomic Physics* (Plenum, New York, 1977).

[26]E. Vogt and G. H. Wannier, Phys. Rev. 95, 1190 (1954).

[27]H. van Haeringen and L. P. Kok, Report 80 10 Delft, 1980 (unpublished); L. P. Kok, Nucl. Phys. A 353, 171c (1981); H. van Haeringen and L. P. Kok, Czech. J. Phys (in press); A. M. Badalyan, L. P. Kok, M. I. Polikarpov, and Yu. A. Simonov, Phys. Rep. C (in press).

[28]R. Jost, Helv. Phys. Acta 20, 256 (1947).

[29]V. de Alfaro and T. Regge, *Potential Scattering* (North-Holland, Amsterdam, 1965).

[30]R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).

[31]J. R. Taylor, *Scattering Theory* (Wiley, New York, 1972).

[32]Compare Refs. 20 and 21, and Ref. 29, p. 40. It may be noted that Lambert defines the ER function differently, by

$$k^{2l+1}\cot\delta_l(k)/[(2l+1)!!]^2 \, ,$$

according to Eq. (4.8) of Ref. 17.

[33]V. B. Matveev and M. M. Skriganov, Theor. Math. Phys. 10, 156 (1972).

[34]M. Read and B. Simon, *Methods of Modern Mathematical Physics III: Scattering Theory* (Academic, New York, 1979), Chap. XI. 9.

[35]H. van Haeringen, J. Math. Phys. 20, 2520 (1979). See, H. van Haeringen, Ph.D. thesis, Free University, Amsterdam, 1978 (unpublished).