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Wave-function equations for photons and electrons

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The wave equations for photons and electrons are presented. The wave functions must include positrons as particles as well as electrons and photons. These differential equations are shown to be equivalent to the unrenormalized perturbation series generated by Feynman rules. The procedure for replacing the bare mass and charge by the experimental values (renormalization) is illustrated for the case of the dressed electron. The introduction of atomic nuclei is discussed.

I. INTRODUCTION

Considerable progress has been made in recent years with the aid of computers in developing numerical techniques for computing the properties of atoms, molecules, and more complicated systems using the nonrelativistic wave equations. Requirements of more accuracy and the need to include photons have, however, created problems to be solved in the domain of relativistic quantum electrodynamics. Since, however, quantum electrodynamics is presented in terms of quantum-field operators and Green's functions instead of wave functions, techniques cannot readily be transferred. There is no identification of photon-electron wave functions satisfying local differential equations in the published literature. To the contrary, it is often stated that there is no wave function for the photon.

Landau and Pierls[1] attempted to find these wave functions and the equations they satisfy. Unfortunately, the equations satisfied by their wave functions were differential integral in form. They were of such complexity that few if any applications have been made of them. Their difficulties can be traced to the unfortunate choice of the electromagnetic fields in the form of $\vec{E}+i\vec{B}$ as the classical limit of the photon wave function.

The Green's-function formulations[2] of quantum electrodynamics suggest strongly that wave functions exist. In particular, Feynman[2] has developed a set of rules for drawing diagrams and interpreting them for the purpose of constructing the perturbation expansions of the Green's functions. We shall use these rules to verify that we have found the correct equations for the wave functions for photons and electrons.

In order to identify the photon wave function, a thought experiment capable of measuring photon coordinates has been studied.[3] This analysis indicates that the classical analog of the photon wave function is the electromagnetic four-vector potential A^{μ}. It is this identification that has allowed us to write relatively simple differential equations satisfied by the photon-electron wave functions.

The wave functions are in Fock space, that is, they can be thought of as an infinite column matrix consisting of all numbers of electrons, positrons, and photons consistent with the total charge and quantum numbers of the system. For example, the Fock-space wave function for the dressed electron consists of a one-electron wave function, a one-electron—one-photon function, a one-electron—one-pair function, a one-electron—two-photon function, etc.

These wave functions satisfy an infinite set of differential equations that link them together. We shall begin in Sec. II by writing a few of these equations and then presenting sets of rules based on diagrams that describe the construction of any member of the infinite hierarchy.

To verify that these differential equations are correct, we shall in Sec. III show how to write the integral equations corresponding to these differential equations. Iterating these integral equations leads to the rules for writing the perturbation series. We shall note that these rules are equivalent to those of Feynman.[2]

In order to identify the expressions for the renormalized mass and charge, we shall, in Sec. V, consider the two lowest-order wave functions in the dressed-electron Fock-space wave function. Finally, in Sec. VI, we shall have a few words to write on

the addition of atomic nuclei to the equations for photons and electrons.

II. DIFFERENTIAL EQUATION HIERARCHY

The Fock-space wave functions for photons and electrons include, in addition, a third set of coordinates for the positrons. We can write a component of the infinite column matrix wave function in configuration space in the form

$$\underline{\psi}^{\mu_{1''},\mu_{2''},\cdots}(1',2',\ldots N';1''_a,2''_a,\ldots N''_a;\bar{1}''',\bar{2}''',\ldots\bar{N}''') \,. \tag{2.1}$$

This is the wave function for N' electrons, N'' photons, and N''' positrons. It has arguments that include the space-time coordinates of point $1'$ for an electron, point $2'$ for an electron, etc., point $1''$ for a photon, $2''$ for a photon, etc., and point $1'''$ for a positron, $2'''$ for a positron, etc. Photon coordinates are identified by a subscript "a", positron coordinates by a bar overhead, and electron coordinates have no additional identifying marks. The photon polarizations are indicated by the superscripts μ_1, μ_2, etc. There is a set of electron-spin quantum numbers corresponding to each electron space-time point and a set of positron-spin quantum numbers for each positron point. These quantum numbers label the four elements of a Dirac wave function in the case of a single electron or positron wave function. We shall think of the wave function in (2.1) as being the matrix that would be constructed by taking the direct products of Dirac column matrices corresponding to the electron points and the Dirac row matrices for the position points. Thus each element of the matrix in (2.1) can be labeled with subscripts made up of the spin quantum numbers for the electrons and positrons. Of course (2.1) is more general than a direct product of one-electron and positron wave matrices.

The one-particle wave functions are coupled to two-particle functions by the three equations,

$$(i\partial_{1'}-m)\underline{\psi}(1')=-e\underline{\gamma}_\eta\underline{\psi}^\eta(1'_a,1') \,, \tag{2.2}$$

$$\underline{\psi}(\bar{1}''')(-i\overleftarrow{\partial}_{1'''}-m)=-e\underline{\psi}^\eta(\bar{1}''',1_a''')\underline{\gamma}_\eta \,, \tag{2.3}$$

and

$$\partial_{\eta 1''}\partial^{\eta}_{1''}\psi^\mu(1''_a)=\tfrac{1}{2}e\,\mathrm{Tr}[\underline{\psi}(1'',\bar{1}''^{\pm})\underline{\gamma}^\mu] \,, \tag{2.4}$$

where

$$\underline{\psi}(1'',\bar{1}''^{\pm})=\tfrac{1}{2}\lim_{\vec{x}''_2\to\vec{x}''_1}\Big[\lim_{\substack{x_2^{0''}\to x_1^{0''}\\ x_2^{0''}>x_1^{0''}}}\underline{\psi}(1'',\bar{2}'') + \lim_{\substack{x_2^{0''}\to x_1^{0''}\\ x_2^{0''}<x_1^{0''}}}\underline{\psi}(1'',\bar{2}'')\Big] \,, \tag{2.5}$$

and we are using units where $\hbar=c=1$. The symbols m and e represent the bare electron mass and charge. The limiting processes in the last two equations are necessary to avoid singularities and are equivalent to the replacement of the field-operator combination $\bar{\Psi}A^\mu\Psi$ by the commutator $[\bar{\Psi},A^\mu\Psi]$ in quantum-field developments.[4]

We see that a two-particle wave function acts as a source for each one-particle function. Thus, for example, an electron wave function is generated by an electron-photon wave function when the electron and photon coordinates are placed at the same space-time point.

Higher-order equations can be written by taking a kind of direct product of Eqs. (2.2)–(2.4) except that additional terms may have to be added to the right-hand side. From Eq. (2.2), we can see that the equation with two differential electron operators on the left is given by

$$(i\partial_{1'}-m)(i\partial_{2'}-m)\underline{\psi}(1',2') = e^2\underline{\gamma}_{\eta_{1'}}\underline{\gamma}_{\eta_{2'}}\underline{\psi}^{\eta_{1'}\eta_{2'}}(1',2',1'_a,2'_a) \,. \tag{2.6}$$

"Multiplying" Eqs. (2.2) and (2.4) and adding a term gives

$$(i\partial_{1'}-m)\partial_{\eta 1''}\partial^{\eta}_{1''}\underline{\psi}^\mu(1',1''_a)=-ie\delta(1',1'')\underline{\gamma}^\mu\underline{\psi}(1'')-\tfrac{1}{2}e^2\underline{\gamma}_\eta\mathrm{Tr}_{1''}[\underline{\psi}^\eta(1',1'',1'_a,\bar{1}''^{\pm})\underline{\gamma}^\mu] \,. \tag{2.7}$$

The trace $\mathrm{Tr}_{1''}$ is applied only to the spin quantum numbers $1''$ and $\bar{1}''$.

Finally, an example of an equation with three differential operators can be derived by "multiplying" equations (2.2), (2.3), and (2.4) and adding terms to obtain

$$(i\partial_{1'}-m)\partial_{\eta 1''}\partial^{\eta}_{1''}\underline{\psi}^{\mu}(1',1''_a,\bar{1}''')(-i\overleftarrow{\partial}_{1'''}-m)$$
$$=ie\,\mathrm{Tr}[\underline{\psi}(1'',\bar{1}''^{\pm})\underline{\gamma}^{\mu}](i\partial_{1'}-m)\delta(1',1''')$$
$$+e\underline{\gamma}^{\mu}_{1'1'''}\delta(1',1''')\delta(1'',1''')+ie^2\underline{\gamma}_{\eta}\underline{\psi}^{\eta}(1',1'_a,\bar{1}'')\underline{\gamma}^{\mu}\delta(1'',1''')$$
$$-ie^2\underline{\gamma}_{\eta}\mathrm{Tr}[\underline{\psi}^{\eta}(1'',1'''_a,\bar{1}''^{\pm})\underline{\gamma}^{\mu}]\delta(1',1''')+ie^2\underline{\gamma}^{\mu}\underline{\psi}^{\eta}(1'',1'''_a,\bar{1}''')\underline{\gamma}_{\eta}\delta(1',1'')$$
$$+e^3\underline{\gamma}_{\eta_{1'}}\mathrm{Tr}_{1''}[\psi^{\eta_{1'}\eta_{1'''}}(1',1'',1'_a,1'''_a,\bar{1}''^{\pm},\bar{1}''')\underline{\gamma}^{\mu}_{1''}]\underline{\gamma}_{\eta_{1'''}}\,. \tag{2.8}$$

Just as Eq. (2.3) can be derived from Eq. (2.2) by taking its adjoint, so can higher-order equations be generated by taking the adjoints of all the Fermions in an equation.

The differential equations that we have written as well as any other equation in the hierarchy can be constructed with the aid of two sets of rules. We shall first list the rules for drawing the diagrams corresponding to an arbitrary member of the hierarchy and then write the rules for converting these diagrams into algebraic expressions. The diagrams corresponding to the equation where the differential operators are applied to a wave function containing N electron coordinates, N' photon coordinates, and N'' adjoint electron coordinates are drawn as follows.

A. Rules for differential equation diagrams

We have the following.

(1) Each diagram has a row of $N'+N''+N'''$ points.

(2) The points are numbered from the left as 1', 2', up to N' (electron points) then continuing, 1'', 2'', up to N'' with a subscript a on each (photon points), and finally followed by 1''', 2''', up to N''' with bars over the numbers (positron points).

(3) All possible distinct diagrams are drawn so that their points are unconnected or are connected in one or more of the following ways: (a) A solid line above or below the row may connect a point labeled with a triple-primed, barred integer to one with a single-prime label. (b) A dashed line may pass above the row to connect two points numbered with double-primed integers with subscripts a. (c) A dashed line passing under the row may connect a triple-primed, barred point to a point labeled by a double-primed integer with a subscript a. (d) A solid line below the row may connect a single-primed labeled point to a point with a double-primed label with subscript a. (e) Not more than one line may be attached to a point by the above rules except that a triple-primed, barred point may have one solid and one dashed line below the row (and no others) attached to it. It is convenient to add an arrowhead to each solid line pointing toward the single-primed point to which it is connected.

B. Translation of differential equation diagrams

We have the following.

1. The left-hand side of the equation

(a) A wave function $\underline{\psi}^{\mu\cdots}$ appears on the left-hand side of the equation. Its arguments are those represented by the points in the diagrams and they appear in the same order as the points. The points without subscripts or bars represent electron space-time coordinates and column matrix element indices. The points with subscripts a represent photon space-time coordinates and polarization indices. The points numbered with integers with bars over them imply positron space-time coordinates and row matrix indices.

(b) To each electron coordinate and matrix index set is applied an operation $(i\partial-m)$ from the left. To each photon coordinate set is applied the operator $\partial_{\eta}\partial^{\eta}$. Finally, to each positron coordinate and matrix index set is applied an operator $(-i\overleftarrow{\partial}-m)$ from the right.

2. The overall sign of a term on the right

Each diagram corresponds to a term on the right side of the equation. The sign of the term is given by a factor $(-1)^p$, where p is the number of fermion points that would be crossed in the following rearrangement. Move each point, connected to a second point to its right by a solid line, to the adjacent position on the second point's left.

3. The contribution of a set of connected points to a term on the right

(a) Each connected set of points contributes a factor to the term represented by the diagram and may

insert coordinates into wave functions in the term. Coordinates of points, joined by a line above the row of points and of three points connected together by lines below the row, do not appear as arguments in wave functions. On the other hand, the space-time coordinates of an isolated diagram point appear twice in the wave functions together with the corresponding polarization and matrix indices for (1) a photon and an electron if the point label is once primed but unbarred and unsubscripted, (2) an electron and a positron for a point whose label is double primed and "a" subscripted, and (3) a positron and a photon if the point label is triple primed and barred. If a connected set consists of two points joined by a line below the row, only the space-time coordinates represented by one of them will appear (once) in the wave functions. This single set of coordinates is associated with (1) electron matrix row indices for a pair of connected unbarred points, (2) a photon polarization index for a pair of unsubscripted points, and (3) positron matrix row indices for a barred point connected to a subscripted point. The order of the coordinates appearing in the wave functions is the same as that of the diagram points with the understanding that a pair of connected points is assigned the position of the right-hand point.

(b) A point with a double-primed label with subscript a with no connections to it requires that the positron coordinates represented by it be inserted into the wave function to the left of the electron coordinates. It further requires that the wave functions in the term represented by the diagram appear in two averaged sets, one set evaluated so that the electron coordinates represented by the point approach the positron coordinates from later times, the other from earlier times. The limit must be taken so that the times of the two coordinate sets must approach each other before the spatial coordinates. Thus, if the diagram has n_a isolated, subscripted points, 2^{n_a} wave functions appear in the term and it is multiplied by 2^{-n_a}.

(c) A factor $-e\underline{\gamma}^{\mu}$ is introduced for each set of connected points. If the point set has contributed electron coordinates to the wave functions on the right side of the equation, the associated matrix indices are matrix multiplied with the right matrix indices of $\underline{\gamma}^{\mu}$. If the point set has contributed photon coordinates to these wave functions, the associated polarization index is scalar-product summed with the index of $\underline{\gamma}^{\mu}$. If the point set has contributed positron coordinates to the wave functions, the associated matrix indices are matrix multiplied with the left matrix indices of $\underline{\gamma}^{\mu}$. Those indices of $\underline{\gamma}^{\mu}$, not summed out in the above steps, are identified with those corresponding to this point set introduced into the wave function on the left-hand side of the equation.

TABLE I. Algebraic expressions on the right are represented by the diagram elements on the left. These expressions appear on the right-hand side of an equation. The plus and minus superscripts indicate limits from the positive and negative time sides.

$\bullet\ j'$	$-e\underline{\gamma}_{\eta}\underline{\psi}^{\eta}(\cdots j'_a, j'\cdots)$
$\bullet\ j''_a$	$\frac{1}{2}e\{\mathrm{Tr}_{j''}[\underline{\psi}(\cdots j'', \bar{j}''^{+}\cdots)\underline{\gamma}^{\mu}j'']$ $+\mathrm{Tr}_{j''}[\underline{\psi}(\cdots j'', \bar{j}''^{-}\cdots)\underline{\gamma}^{\mu}j'']\}$
$\bullet\ \bar{j}'''$	$-e\underline{\psi}^{\eta}(\cdots j'''_a, \bar{j}'''\cdots)\underline{\gamma}_{\eta}$
$j''_a\ \bar{k}'''$	$-ie\underline{\psi}(\cdots\bar{k}'''\cdots)\underline{\gamma}^{\mu}_{k'''j''}\delta(j'',k''')$
$j'\ \bar{k}'''$	$-ie\psi_{\eta}(\cdots j'_a\cdots)\underline{\gamma}^{\eta}_{j'k'''}\delta(j',k''')$
$j'\ k''_a$	$-ie\underline{\gamma}^{\mu k''}_{j'}\underline{\psi}(\cdots j'\cdots)\delta(j',k'')$
$j'\ k''_a\ \bar{l}'''$	$e\underline{\gamma}^{\mu k''}_{j'l'''}\delta(j',l''')\delta(k'',l''')$
$j'\ \bar{k}'''$	$i(i\not{\partial}_j-m)\delta(j',k''')$
$j''_a\ k''_a$	$i\,\partial_{\eta}\partial^{\eta}\delta(j'',k'')\,g^{\mu_{j''}\mu_{k''}}$

(d) A line joining two points requires a factor i times a delta function whose arguments are the space-time coordinates represented by the two points. A dashed line implies, in addition, a factor $g_{\eta\nu}$ whose polarization indices are the same as those associated with the two connected points. The delta function for a dashed line passing above the row has one of its coordinate sets operated upon by $\partial_{\eta}\partial^{\eta}$. A solid line above the row requires the application of the operator $(i\not{\partial}-m)$ to its electron coordinates.

The contribution of connected point sets to terms on the right-hand side of the differential equation are listed in Table I. The diagrams for Eqs. (2.7) and (2.8) are presented in Figs. 1 and 2.

Of course the solutions to these differential equations are not determined until we specify the boundary conditions. Since the equations are of the first order in fermion coordinates, the electron and adjoint electrons need only to be specified over half

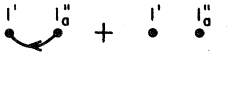

FIG. 1. Diagrams corresponding to Eq. (2.7).

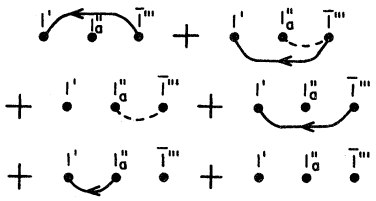

FIG. 2. Diagrams corresponding to Eq. (2.8).

the boundary. The photon differential operator is, however, of second order so that photon wave functions must be given over the entire boundary, or each function and its derivative must be given over half of it. It is often convenient to take a hypercylindrical boundary with a flat constant time top and bottom extending to infinity in the spatial directions. Then the timelike cylindrical sides are at spatial infinity. If we use wave packets, the wave functions vanish on the timelike boundary.

In most physics problems, the boundary values are expressed in terms of wave functions on a constant time surface extending to spatial infinity. These wave functions must have the symmetries prescribed by the Pauli principle, namely, they must be antisymmetric for fermion exchange and symmetric for bosons (photons in this case). In addition, these boundary wave functions must be made up of eigenfunctions of the total Hamiltonian belonging to positive energies if the surface composes the initial time boundary. This assures that positive energies propagate forward in time. If these initial condition wave functions involve isolated packets for individual particles, then the electron packets are made up of positive-energy eigenfunctions of the Dirac equation while the positron packets consist of adjoints of the negative-energy eigenfunctions. The packets and their time derivatives must be prescribed for photons. Again they must consist of positive-energy plane waves.

III. INTEGRAL EQUATIONS

The integral equations corresponding to the differential equations in the last section are easily written in terms of the Green's functions that satisfy

$$\partial_{\eta 2}\partial_2^{\eta} J^{\mu\nu}(2,1)=ig^{\mu\nu}\delta(2,1)\ , \tag{3.1}$$

$$(i\partial_2-m)\underline{K}(2,1)=i\delta(2,1)\ , \tag{3.2}$$

and

$$\underline{K}(2,1)(-i\overleftarrow{\partial}_1-m)=i\delta(2,1)\ . \tag{3.3}$$

With these Green's functions, Eqs. (2.2)–(2.4) may readily be converted into the integral equations

$$\underline{\psi}(1')=\oint \underline{K}(1',1)d\sigma_1\underline{\psi}(1) +ie\int \underline{K}(1',1)\underline{\gamma}_{\eta}\underline{\psi}^{\eta}(1,1_a)d^4x_1\ , \tag{3.4}$$

$$\underline{\psi}(\bar{1}''')=-\oint \underline{\psi}(\bar{1})d\sigma_1\underline{K}(1,1''') +ie\int \underline{\psi}^{\eta}(1_a,\bar{1})\underline{\gamma}_{\eta}\underline{K}(1,1''')d^4x_1\ , \tag{3.5}$$

and

$$\psi^{\mu}(1_a'')=\oint J^{\mu}_{\cdot\nu}(1'',1)d\overleftrightarrow{\sigma}_1\psi^{\nu}(1_a) -ie\int J^{\mu}_{\cdot\nu}(1'',1) \times \mathrm{Tr}[\underline{\psi}(1,\bar{1}^{\pm})\underline{\gamma}^{\nu}]d^4x_1\ . \tag{3.6}$$

The surface integral is over the surface on which the boundary values for the differential equations were given. Applications of the differential operators to these integral equations reduces them back to the differential equations (2.2)–(2.4).

Higher-order integral equations can be written by taking a kind of "direct product" of Eqs. (3.4)–(3.6) with an additional volume integral term on the right generated from each of the extra terms that were added to the corresponding differential equation beyond those generated by direct products. In the case of the equation for $\underline{\psi}(1',2')$, there are no such extra terms so that "multiplying" Eq. (3.4) by self gives

$$\begin{aligned}\underline{\psi}(1',2')=&\oint\oint \underline{K}(1',1)d\sigma_1\underline{K}(2',2)d\sigma_2\underline{\psi}(1,2)\\ &+ie\int\oint \underline{K}(1',1)\underline{\gamma}_{\eta_1}\underline{K}(2',2)d\sigma_2\underline{\psi}^{\eta_1}(1_a,1,2)d^4x_1\\ &+ie\oint\int \underline{K}(1',1)d\sigma_1\underline{K}(2',2)\underline{\gamma}_{\eta_2}\underline{\psi}^{\eta_2}(1,2,2_a)d^4x_2\\ &+(ie)^2\int\int \underline{K}(1',1)\gamma_{\eta_1}\underline{K}(2',2)\underline{\gamma}_{\eta_2}\underline{\psi}^{\eta_1\eta_2}(1_a,1,2,2_a)d^4x_1d^4x_2\ .\end{aligned} \tag{3.7}$$

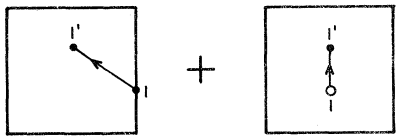

FIG. 3. Diagrams corresponding to Eq. (3.4).

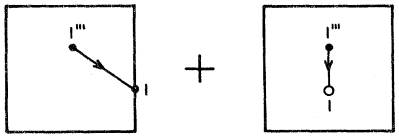

FIG. 4. Diagrams corresponding to Eq. (3.5).

However, the differential equation where $\underline{\psi}^{\mu}(1',1''_a)$ appears on the left [Eq. (2.7)], has a corresponding integral equation that can be generated by "multiplying" Eqs. (3.4) and (3.6) together and then adding the extra terms to obtain

$$\begin{aligned}\underline{\psi}^{\mu}(1',1''_a)=&\oint\oint \underline{K}(1',1)d\sigma_1 J^{\mu}_{\cdot\nu}(1'',2)d^{\leftrightarrow}\sigma_2\underline{\psi}^{\nu}(1,2_a)\\&+ie\int\oint \underline{K}(1',1)\underline{\gamma}_{\eta}J^{\mu}_{\cdot\nu}(1'',2)d^{\leftrightarrow}\sigma_2\underline{\psi}^{\eta\nu}(1_a,1,2_a)d^4x_1\\&-ie\oint\int \underline{K}(1',1)d\sigma_1 J^{\mu}_{\cdot\nu}(1'',2)\mathrm{Tr}_2[\underline{\psi}(1,2,\bar{2}^{\pm})\underline{\gamma}^{\nu}_2]d^4x_2\\&+ie\int \underline{K}(1',1)\underline{\gamma}^{\nu}J^{\mu}_{\cdot\nu}(1'',1)\underline{\psi}(1)d^4x_1\\&-(ie)^2\int\int \underline{K}(1',1)\underline{\gamma}_{\eta}J^{\mu}_{\cdot\nu}(1'',2)\mathrm{Tr}_2[\underline{\psi}^{\eta}(1_a,1,2,\bar{2}^{\pm})\underline{\gamma}^{\nu}_2]d^4x_1d^4x_2\ .\end{aligned}\tag{3.8}$$

The term containing $\underline{\psi}(1)$ corresponds to first term in Eq. (2.7). It is the "extra" term that does not arise from the "multiplication" of the two lower-order integral equations. Of course, the higher-order integral equations can be checked by applying the proper derivative operator to recapture the differential equations.

We can easily draw diagrams to represent the integral equations. For example, Fig. 3 corresponds to Eq. (3.4). The solid lines with arrows indicate the electron propagators $\underline{K}(1',1)$, and the open circle stands for $ie\underline{\gamma}_{\eta}$ just as in Feynman diagrams. The point on the boundary tells us that we must insert its coordinates into a wave function following a surface element matrix $d\sigma$ and then integrate its space-time coordinates over the boundary. If the circle represented a vertex in a Feynman diagram, an electron line and a photon line would be attached to it in addition to the outgoing electron line. Thus it is not surprising that the circle requires us to insert electron and photon coordinates representing the same space-time points into the wave function following the $\underline{\gamma}_{\eta}$ in the last term in Eq. (3.4) and to integrate the circle space-time coordinates over the volume. In a similar way, we can draw the diagrams in Figs 4 and 5 and interpret them to give Eqs. (3.5) and (3.6).

In the same way, we can construct Figs. 6 and 7 to represent Eqs. (3.7) and (3.8). If we compare Figs. 1 and 7, we see that we may think of the points in the interior of the integral equations as being the same as those for the differential equation. The second diagram in Fig. 1 leads to the first, second, third, and fifth diagrams of Fig. 7 by attaching a solid line to the electron point and a dashed line to the photon point. The other ends of the lines are attached in all possible combinations to interior circles on surface points. The fourth diagram in Fig. 7 is the extra one that does not result from multiplying Figs. 3 and 5 together. The line

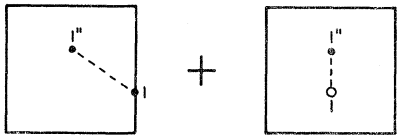

FIG. 5. Diagrams corresponding to Eq. (3.6).

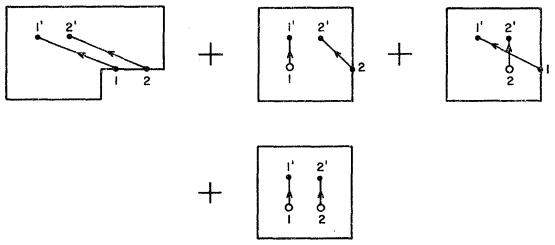

FIG. 6. Diagrams corresponding to Eq. (3.7).

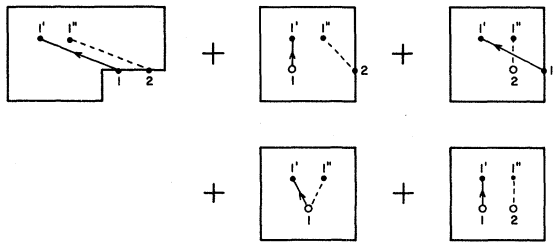

FIG. 7. Diagrams corresponding to Eq. (3.8).

joining the points in the first term in Fig. 1 has had a circle inserted in it to form the fourth term in Fig. 7. Again a dashed segment must be connected to the photon point.

The diagrams for the integral equation for $\underline{\psi}(1',\bar{1}''')$ are shown in Fig. 8. We note there that the second diagram is, except for the boundary, exactly the same as the next to the last diagram in Table I for differential equations. Thus those lines arcing above the points are carried over directly from the differential equation diagrams to the integral equation diagrams. The next-to-the-last diagram in Table I represents a factor $\underline{K}(j',k''')$ for integral equations while the last diagram represents a factor $J^{\mu\nu}(j'',k'')$. As a consequence of this, $\underline{\psi}(1',\bar{1}'')$ includes a term $\underline{K}(1',1'')$ which is independent of the boundary conditions. Similarly $\psi^{\mu\nu}(1'',2'')$ includes a term $J^{\mu\nu}(1'',2'')$.

IV. PERTURBATION EXPANSION

The integral equations in the last section can be iterated to generate a series for any given wave function. This series will involve powers of e. For example, if we want the series for $\underline{\psi}(1')$, we can start with Eq. (3.4). Equation (3.8) provides an expression for $\underline{\psi}^{\eta}(1,1_a)$. Substituting this expression into Eq. (3.4) provides an integral equation for $\underline{\psi}(1')$ with six terms on the right-hand side with powers of e up

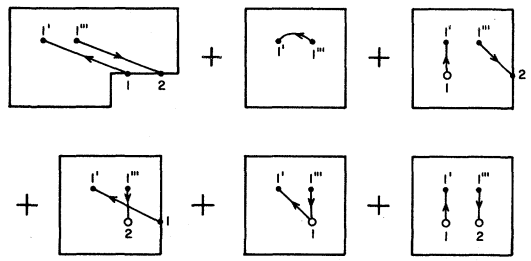

FIG. 8. Diagrams for the integral equation for $\underline{\psi}(1',\bar{1}''')$.

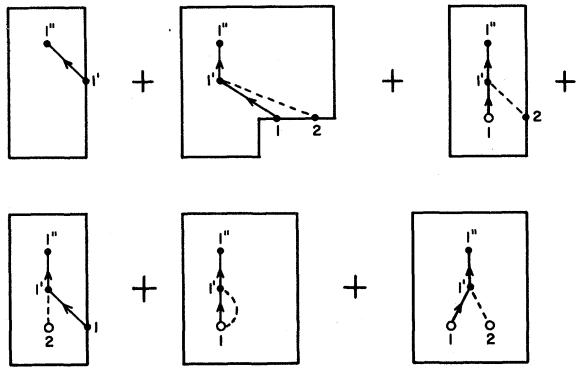

FIG. 9. Diagrams for $\underline{\psi}(1'')$ after one iteration. Lines represent propagators while internal points below the top row represent vertices that supply a factor $ie\gamma^{\nu}$.

to e^3. Although they involve $\underline{\psi}(1')$ for which we are solving, they also require a knowledge of $\psi^{\eta\nu}(1',1_a'',2_a'')$, $\underline{\psi}(1',1',\bar{1}'')$, and $\underline{\psi}^{\eta}(1',2',1_a'',\bar{1}''')$ over the volume. These last three wave functions appear, however, in terms multiplied by e^2 and e^3. If we include only powers up through that of e, we need know only $\underline{\psi}(1')$ and $\underline{\psi}(1',1_a'')$ on the surface to obtain an approximation to $\underline{\psi}(1')$.

We can continue to iterate the hierarchy of integral equations to generate higher and higher-order

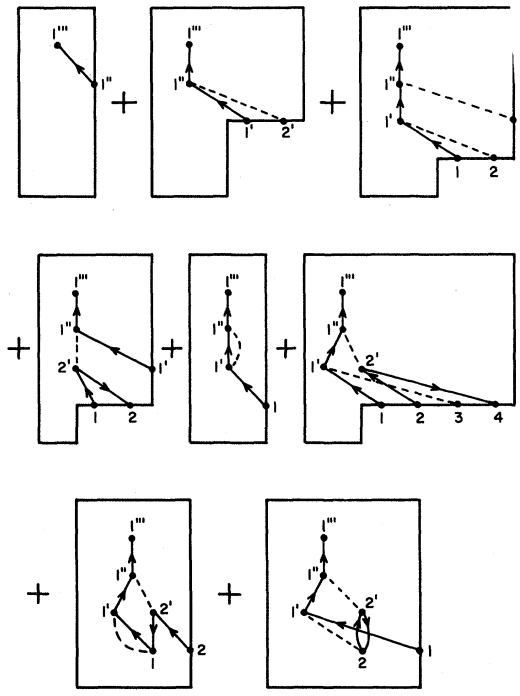

FIG. 10. Some of the diagrams that result from three iterations for $\underline{\psi}(1''')$. Only terminal diagrams are shown.

terms depending only upon wave functions given at the boundary. The resulting series will be the (unrenormalized) one that can be written with the aid of Feynman diagrams.[2] The expressions we get for the series for $\underline{\psi}(1')$ are rather complicated, but we can use the diagrams for integral equations to represent them more simply. The first iteration that we have just described to compute $\underline{\psi}(1)$ can be diagrammed by replacing the second diagram in Fig. 3 by diagrams constructed by attaching points $1'$ and $1''$ in Fig. 7 to the circle in Fig. 3. The result is shown in Fig. 9. We see, as already noted, that the first two terms require a knowledge of $\underline{\psi}(1')$ and $\underline{\psi}(1,2_a)$ only on the boundary. A factor e appears for each internal point or circle so that the third and fifth diagrams represent terms of order e^2 and the last of order e^3.

To continue this diagram iteration for $\underline{\psi}(1')$, we must construct the diagrams for $\underline{\psi}(1',1_a'',2_a'')$, $\underline{\psi}(1',2',\bar{1}'')$, and $\underline{\psi}^{\eta}(1',2',1_a'',\bar{1}''')$ and then connect them in the appropriate fashion to circles in Fig. 9. A large number of diagrams result and some of them that have no circles are illustrated in Fig. 10. We see examples of Feynman electron self-energy diagrams in the fifth and seventh terms and a vacuum polarization diagram in the last term. It is useful to note that these diagrams are drawn so that the points on the first row represent the arguments of the wave function for which the series is generated while each succeeding row represents an iteration.

From the way in which we have generated these diagrams, it is clear that all of those that Feynman would write will appear and that those with no circles may be converted into algebraic expressions with the aid of the following rules.

C. Rules for interpreting perturbation diagrams

We have the following.

(1) The points in the top row represent space-time coordinates and spin and polarization indices in the order in which they are to be written in the wave function represented by the perturbation series. A point with a dashed line attached represents photon coordinates and a point with a solid line attached represents electron coordinates if the arrow head points toward the point or positron coordinates if the arrow head points away.

(2) The points on the surface represent the following contributions to the term represented by the diagram. (a) A point numbered j with a solid line leaving it introduces a surface integral $\oint d\sigma_j$ ahead of a wave function including the argument j and electron-spin column matrix indices multiplied with the right-hand matrix indices of $d\sigma_j$. (b) A point numbered j with a dashed line attached introduces a surface integral over the coordinates j, $\oint d\overleftrightarrow{\sigma}_j$, where

$$d\overleftrightarrow{\sigma}_j = -id\sigma_j^{\mu}\overleftrightarrow{\partial}_{j\mu} ,$$

and these coordinates and the photon polarization index are inserted in a wave function lying to the right. (c) A point numbered j with a solid line entering it introduces a surface integral $-\oint d\sigma_j$ to the right of a wave function containing j as an argument and positron-spin row matrix indices multiplied with the left-hand matrix indices of $d\sigma_j$. The order in which these coordinates appear in the wave function for the term is the same as the order of their corresponding points from left to right.

(3) The term represented by a diagram must be multiplied by $(-1)^p$, where p is the number of times the solid lines in the diagram cross each other plus the total number of fermion points lying between pairs of solid lines above the row plus the number of pairs of such points where the arrowhead points from left to right.

As we have already noted, the wave functions specified on the boundary must be antisymmetric to fermion coordinate exchange and symmetric to all other exchanges. The last rule provides for the Pauli principle. These rules are equivalent to those in standard texts for unrenormalized Feynman diagrams. The procedures for rearranging these series to convert the bare mass m and the bare charge e into their dressed (or renormalized) counterparts are well known and need not be considered here. We shall, instead, in the next section consider how to make this renormalization when the differential equation hierarchy is used.

V. RENORMALIZED MASS AND CHARGE

The quantities m and e that appear in the previous sections are the bare mass and charge and, of course, are unknown. To see how the renormalized (experimental) values can be introduced into the equations, we shall consider next the solutions of the differential equations for the dressed photon and electron.

To find the Fock-space wave function for an isolated photon, we can start with Eq. (2.4). Although we cannot determine $\underline{\psi}(1'',\bar{1}''^{\pm})$ without solving the other equations in the hierarchy, we can determine its form from its properties under the transforma-

tions of the Lorentz group. In particular, if we seek the total momentum solution with four-momentum $\vec{q}$, then

$$\underline{\psi}(1',\bar{1}''')=\underline{\zeta}_q(\bar{x})e^{-i\bar{q}\cdot\bar{x}_{1'}}, \tag{5.1}$$

where

$$\bar{x}=\bar{x}_{1'''}-\bar{x}_{1'}. \tag{5.2}$$

Substituting this into Eq. (2.4) gives

$$\partial_{\eta 1''}\partial^{\eta}_{1''}\psi^{\mu}(1''_a)=C^{\mu}e^{-i\bar{q}\cdot\bar{x}_{1''}}, \tag{5.3}$$

where

$$C^{\mu}=\tfrac{1}{2}e\,\mathrm{Tr}\{[\underline{\zeta}_q(0^+)+\underline{\zeta}_q(0^-)]\underline{\gamma}^{\mu}\}. \tag{5.4}$$

Equation (5.3) is solved by

$$\psi^{\mu}(1''_a)=C^{\mu}e^{-i\bar{q}\cdot\bar{x}_{1''}} \tag{5.5}$$

provided that

$$q_{\eta}q^{\eta}=C^{\mu}. \tag{5.6}$$

We know from experiment that

$$C^{\mu}=0, \tag{5.7}$$

that is, the experimental mass of the photon vanishes. Thus the one-photon component of the photon Fock-space wave function satisfies

$$\partial_{\eta 1''}\partial^{\eta}_{1''}\psi^{\mu}(1''_a)=0. \tag{5.8}$$

We shall consider next the solution of the differential equations for the dressed electron. We will start with the lowest member of the hierarchy involved in this problem, namely, Eq. (2.2). To solve this equation, we must first determine $\underline{\psi}^{\mu}(1',1'_a)$. This requires us to attack Eq. (2.7) which, in turn, introduces a more complicated wave function. To solve the entire problem, we must employ an infinite hierarchy of equations. However, as we shall see, the form of the solution of Eq. (2.2) can be determined with the aid of the transformation properties of the wave functions under the homogeneous Lorentz group.

Since the hierarchy of the differential equations is covariant for transformations of the Poincaré group, the electron as well as each of the other components of the Fock-space wave function can be required to transform like a momentum eigenstate (total spin one-half) solution of the noninteracting Dirac equation. Thus, if we make the transformation

$$\bar{x}_{1'},\bar{x}_{1''}\rightarrow\bar{x}=\bar{x}_{1''}-\bar{x}_{1'},\bar{x}_{1'}, \tag{5.9}$$

then

$$\underline{\psi}^{\mu}(1',1''_a)=\underline{f}^{\mu}(\bar{x})e^{-i\bar{p}\cdot\bar{x}_{1'}}. \tag{5.10}$$

Since $\underline{\psi}^{\mu}$ transforms like the direct product of a photon four-vector representation vector and an electron-spin one-half vector, it can be written in terms of a spin one-half wave function plus a spin three-halves wave function. As we have noted above, it is the spin one-half component that we need for the dressed electron Fock-space wave function. A straightforward application of Clebsch-Gordan coefficients allows us to write for the required spin one-half component,

$$\underline{\psi}^{\mu}(1',1''_a)=\tfrac{1}{4}\underline{\gamma}^{\mu}\underline{u}e^{-i\bar{p}\cdot\bar{x}_{1'}}g(p^2,\bar{p}\cdot\bar{x},\bar{x}^2), \tag{5.11}$$

where $\underline{u}$ is the Dirac bispinor that combines with the exponential to form a solution of the Dirac equation with no interactions.

Substituting this wave function into Eq. (2.2) now gives

$$(i\partial_{1'}-m)\underline{\psi}(1')=eg(p^2,0,0)\underline{u}e^{-i\bar{p}\cdot\bar{x}_{1'}}. \tag{5.12}$$

It is immediately clear that this equation is solved by

$$\underline{\psi}(1')=\underline{u}e^{-i\bar{p}\cdot\bar{x}_{1'}} \tag{5.13}$$

provided

$$p_{\mu}p^{\mu}=m_r^2, \tag{5.14}$$

where

$$m_r=m+\delta m \tag{5.15}$$

and

$$\delta m=eg(p^2,0,0). \tag{5.16}$$

This means that $\underline{\psi}(1')$ satisfies

$$(i\partial_{1'}-m-\delta m)\underline{\psi}(1')=0. \tag{5.17}$$

Of course, m_r is the renormalized mass and, from the above equations,

$$\underline{\psi}^{\mu}(1',1'_a)=\tfrac{1}{4}\underline{\gamma}^{\mu}\underline{u}e^{-i\bar{p}\cdot\bar{x}_{1'}}\delta m/e. \tag{5.18}$$

Thus δm can be determined once $\underline{\psi}^{\mu}(1',1''_a)$ is known from a solution of the entire hierarchy.

We must next solve Eq. (2.7) in order to learn something more about the dressed electron component $\underline{\psi}^{\mu}(1',1''_a)$. Since we do not know the bare mass m from experiment, we introduce δm on both sides and use Eq. (5.15) to obtain

$$(i\partial\!\!\!/_{1'}-m_r)\partial_{\eta 1''}\partial^{\eta}_{1''}\underline{\psi}^{\mu}(1',1''_a)=-ie\delta(1',1'')\underline{\gamma}^{\mu}\underline{\psi}(1'')-\tfrac{1}{2}e^2\underline{\gamma}_{\eta}\mathrm{Tr}_{1''}[\underline{\psi}^{\eta}(1',1'',1'_a\overline{1}''^{\pm})\underline{\gamma}^{\mu}]$$
$$-\delta m\partial_{\eta 1''}\partial^{\eta}_{1''}\underline{\psi}^{\mu}(1',1''_a)\ . \tag{5.19}$$

We have already noted in Eq. (5.11) the form of $\underline{\psi}^{\mu}(1',1''_a)$. In addition we observe that, in order for the photon to leave the electron by a large distance, it must behave like a dressed photon and satisfy Eq. (5.8). Thus the last term in Eq. (5.19) must vanish except when the photon is near to the electron in space and time. This suggests that we should use Eq. (5.11) and make the approximation

$$\partial_{\eta 1''}\partial^{\eta}_{1''}\underline{\psi}^{\mu}(1',1''_a)\approx\tfrac{1}{4}\delta(1',1'')\underline{\gamma}^{\mu}\underline{\psi}(1')\times\int g\,d^4x\ . \tag{5.20}$$

From arguments similar to those used in deriving Eq. (5.11), we can write the electron-photon-pair wave function in such a form that

$$\underline{\gamma}_{\eta}\mathrm{Tr}_{1''}[\underline{\psi}^{\eta}(1',1'',1'_a,\overline{1}''^{\pm})\underline{\gamma}^{\mu}_{1''}]=\underline{F}^{\mu}(\bar{x},\bar{p})\underline{u}e^{-i\bar{p}\cdot\bar{x}_{1'}}\ . \tag{5.21}$$

We can say something about the range of $\underline{F}^{\mu}$ as a function of $\bar{x}$ if we note that the wave function on the left side is the amplitude for finding a positron-electron pair at $1''$ when the electron is at $1'$. Since an energy of at least $2m_rc^2$ is required to create the pair, this amplitude must decrease exponentially because the pair tunnels into the $2m_rc^2$ barrier as it departs from the electron. Thus $\underline{F}^{\mu}(\bar{x},\bar{p})$ has a range the order of a Compton wavelength in $|\vec{x}|$. In the time direction, the excess energy of the $2m_rc^2$ substituted into the energy-time uncertainty relation suggests that the pair cannot exist for $|x^0|$ much greater than a Compton wavelength. From these considerations, we conclude that, for values of $|x^{\mu}|$ much greater than a Compton wavelength

$$\underline{F}^{\mu}\simeq\int\underline{F}^{\mu}d^4x\delta(\bar{x})\ . \tag{5.22}$$

When this approximation is substituted into the previous equation and this, in turn, into Eq. (5.19) together with Eq. (5.20), we can use Eq. (5.12) and write

$$(i\partial\!\!\!/_{1'}-m_r)\partial_{\eta 1''}\partial^{\eta}_{1''}\underline{\psi}^{\mu}(1',1''_a)\approx-i\delta(1',1'')\left[e\underline{\gamma}^{\mu}-ie^2\int\underline{F}^{\mu}d^4x-\tfrac{1}{4}i\delta m\underline{\gamma}^{\mu}\int g\,d^4x\right]\underline{\psi}(1')\ . \tag{5.23}$$

According to Eq. (5.11), the left-hand side of the last equation would have Dirac matrices of the form $\underline{\gamma}^{\mu}$ and, because of $\partial\!\!\!/_{1'}$, $\underline{\gamma}^{\nu}\underline{\gamma}^{\mu}$ multiplying $\underline{\psi}(1)$. The same must be true for the term involving $\underline{F}^{\mu}$ on the right. Thus we can write

$$\underline{F}^{\mu}=\tfrac{1}{4}\underline{\gamma}^{\mu}\mathrm{Tr}(\underline{\gamma}^{\mu}\underline{F}^{\mu})+\underline{\sigma}^{\mu\nu}G_{\nu}\ . \tag{5.24}$$

The second term gives rise to a magnetic dipole that produces a component of $\underline{\psi}^{\mu}(1',1''_a)$ that falls off more rapidly than the first term with larger separations between points $1'$ and $1''$. Thus, asymptotically, Eq. (5.23) goes over to

$$(i\partial\!\!\!/_{1'}-m_r)\partial_{\eta 1''}\partial^{\eta}_{1''}\underline{\psi}^{\mu}(1',1''_a)\sim-i\delta(1',1'')e_r\underline{\gamma}^{\mu}\underline{\psi}(1')\ , \tag{5.25}$$

where the renormalized charge is given by

$$e_r=e-\tfrac{1}{4}e^2i\int\mathrm{Tr}(\underline{\gamma}^{\mu}\underline{F}^{\mu})d^4x-\tfrac{1}{4}i\delta m\int g\,d^4x\ . \tag{5.26}$$

Thus we see that Eq. (2.7) reduces asymptotically to the equation for an electron surrounded by the expected photon field that is coupled by a renormalized charge e_r.

The above equations identify the experimental mass and charge of the electron. In order to solve other differential equations, it is necessary to isolate the expressions for these quantities and then to substitute the experimental numbers.

VI. INTRODUCING NUCLEAR FIELDS

In order to describe atoms, molecules, and more complicated forms of matter with the mathematics of quantum electrodynamics, we must introduce coordinates for nuclei. We can treat these nuclei as space-time points in the same sense as we have electrons provided we can ignore other than electromagnetic interactions and the fact that the electric charge is distributed over a volume with dimensions the order of 10^{-13} cm.

For spin one-half nuclei such as the proton, the elements introduced into the diagrams are the same as for the electron with the understanding that no

lines are drawn connecting different kinds of fermions. The interpretation of the diagrams requires that the electron masses and charges appearing in Table I and in derivative operators must be replaced by the corresponding quantities belonging to the nuclei.

For nuclei of different spins than one-half, the heavy nuclear mass probably will make it profitable to use nonrelativistic forms of derivative operators and of interactions with photons. Corrections can be made for relativistic effects for these nuclei.

VII. DISCUSSION

The infinite hierarchy of differential equations presented in Sec. II involves three kinds of particles, electrons, positrons, and photons. By specifying that only positive-energy wave functions appear on the boundary, we have assured that there are nonnegative-energy states into which any of the three particles may drop. There is neither the negative-energy sea required to be filled by Dirac nor the forward propagation in time required by Feynman.[2] However, if the boundary conditions are prescribed only on the early-time boundary, the integral equation formulation will require Feynman propagators.

Although we have identified the renormalized mass and charge in Eqs. (5.7), (5.8), and (5.15), we have not proved that the differences between the bare and renormalized quantities are finite. This is a question about which many experts disagree. There have been papers[5] by Gell-Mann and Low, Wiley, Baker, and Johnson, and others that suggest that these quantities may be finite. Whether they are finite or not, the expressions in the equations can be replaced by the experimental mass and charge to allow the solution of practical problems.

We have seen that the Fock-space wave function for a dressed electron has an infinite number of elements. The same is true of a "dressed" photon. The bare vacuum state has no wave functions since no particles are present. The "physical" vacuum, however, again consists of a Fock-space wave function with an infinite number of components. These components allow for the presence of virtual positron-electron pairs and photons.

In every practical problem solved with the aid of these differential equations, it will be necessary to truncate the hierarchy by using some approximation. It may be helpful at times to make use of the Hartree-Fock and other approximations in current use in the treatment of atoms, molecules, and other many-body systems.

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