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Reformulation of the statistical equations for turbulent shear flow

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Recent experimental results on the intermittent generation of shear turbulence raise questions about the significance of Reynolds averaging. In this paper, statistical equations are derived by progressively averaging the Navier-Stokes equation over a series of increasing time periods. Averaging over the shortest time smooths out part of the field (which corresponds to the highest-frequency fluctuations). The mean effect of these fluctuations may be calculated from the time-averaged equation of motion, and so eliminated from the equation describing the rest of the velocity field (that is, the unaveraged part). An iterative process leads to equations for the mean and covariance of the fluctuating field, in which the Reynolds stresses do not appear explicitly. They are represented in each cycle of the iteration by the sum of the following: (1) a constitutive relation, expressing the mean-square fluctuation in terms of the mean rate of strain, and (2) the unaveraged portion of the nonlinear term. The method resembles the renormalization group (it differs insofar as the averaging process is the defining operation). A renormalization-group analysis is used to investigate the iteration process. With some simplifying assumptions (e.g., the fluctuations are taken to be isotropic and the spectrum to be a power law), a recursion relation for the viscosity is found to reach a fixed point. In the limit of long averaging times, the mean-field equation reduces to the Reynolds equation, with the turbulent stresses replaced by an effective viscosity.

I. INTRODUCTION

Statistical equations for shear turbulence are traditionally derived by averaging the Navier-Stokes equation with respect to time. The averaging period must, in principle, be long enough to smooth out the rapid fluctuations of the turbulent cascade but short enough not to eliminate any slower variations due to (for example) changes in external conditions. In practice, however, attention is normally restricted to stationary flows and the averaging period is taken to be infinite. As this includes a wide class of important practical flows, the restriction may not seem too severe. Also the averaging procedure has the merit of corresponding unambiguously to the way experimental measurements are usually taken in the laboratory. The end result is the well-known Reynolds equations, for the mean and covariance of the random velocity field.

Although these equations have long provided almost the entire basis for phenomenological theories, and the semiempirical analysis of engineering problems, they are open to at least two serious criticisms. First, they are theoretically intractable. This is due to the moment-closure problem. For this reason theorists generally work with isotropic turbulence, where the mean rate of shear is zero and the closure problem may be studied in isolation. The consequent reduction is complexity is held to outweigh both the artificialities and the problems involved in direct experimental comparison. Second, even at a phenomenological level the equations are inflexible. In general, it is difficult to treat nonstationary flows. In particular, recent experimental results call the basic averaging procedure into question even for steady flows. This is because of the intermittent character of turbulent generation. Measurements of statistical properties show, in some cases, a nearly bimodal probability distribution. Correlation coefficients are found to be zero for stretches of time, interspersed with periods when they are unity. This behavior may be compared to the conventional assumption of a long-time mean value of about 0.4.

In this paper we propose a new approach to this problem. The Navier-Stokes equation is progressively averaged over a series of increasing time periods. Averaging over the shortest time smooths out part of the fluctuating field which corresponds to the highest-turbulent frequencies. The mean effect of these fluctuations may be calculated and hence eliminated from the equation describing the rest of the velocity field (i.e., the lower frequencies). An iterative process leads to equations for the mean and covariance of the fluctuating field, in which the Reynolds stresses (as such) do not appear. They are represented in each cycle of the iteration by both a constitutive relation expressing the mean effect of fluctuations in terms of the mean rate of strain, and the unaveraged portion of the nonlinear term.

This method resembles that of the renormalization group.[1,2] In Sec. IV we make a renormalization-group analysis of the iteration, on the basis of simplifying assumptions about the fluctuating field. The main assumptions are that the fluctuations may be treated as isotropic and the spectrum represented by a power law. On this basis, as the averaging time is increased, a recursion relation for the effective viscosity is found to reach a fixed point. Renormalization-group methods have previously been applied to turbulence problems by Forster *et al.*,[3,4] who study low-frequency correlations; and by Rose,[5] who considers subgrid modeling of passive scalar convection.

We have noted above, that the intermittent generation of turbulence supplies one of the motivations for a different statistical treatment of the equation of motion. As intermittency is currently of interest to theorists we shall briefly develop the point here.

Intermittency effects in turbulence seem to fall roughly into one of three classifications. First there is intermittency of the small scales. This is due to the inability of the small eddies to fill space and is characterized by local fluctuations in the dissipation rate. This effect has been the most studied theoretically.[6,7] Second, there is the intermittency found in free turbulence. This is a large-scale effect and is associated with the instability of the boundary between the turbulent fluid and the surrounding nonturbulent fluid. Developments in anemometry and signal processing have allowed experimentalists to take account of the effect of this intermittency in their measurements. But, apart from this, it has received relatively little attention. Third, there is the "bursting process". This occurs in shear flows and is also a large-scale process. In essence, it consists of an intermittent cycle of inrush and ejection of fluid, in the turbulent boundary layer. From both visual studies and measurements[8-10] it is clear that the process is remarkably regular. For instance, velocity autocorrelations, when measured with short sample times, have shown an oscillatory behavior which corresponds to the bursting process observed visually. It had been established that most of the turbulence production takes place during bursts. In view of its immense practical importance, it is not surprising that the bursting process has been the subject of much experimental work. But there has been little theoretical work, and formidable difficulties stand in the way.[11]

It is, of course, this bursting process which concerns us (even, if only in the sense of providing part of the motivation for the work) in the present paper. It is instructive to consider how the present approach may be relevant. If we consider turbulent shear flow in a straight pipe then the size of the largest eddies will be of the order of the radius a. And, if the mean velocity is U, then the time scale associated with the largest eddies will be $\tau = a/U$. Plausibly, we may take τ to be the longest single time scale associated with the energy cascade. From experiment, we know that the bursting process introduces a longer time scale into the problem in that T_B, the mean time between bursts, is about $5a/U$. That is $T_B \simeq 5\tau$. So, if we average over τ, we may eliminate the cascade but still have some time-dependent behavior. In this sense, even flows that are conventionally described as stationary, are really only quasistationary. We shall develop this point further in Sec. VI, in the light of the theory to be presented.

Finally, we should note that in work of this kind it is difficult to make progress without making some approximations. Several approximations of a straightforward kind will be introduced and discussed as appropriate in the main body of the paper. However, one particular step requires a special mention. In Sec. III we solve for the high-frequency part of the second moment, by neglecting that part of the triple moment which is *entirely* made up of high-frequency components. This is a rigorous step initially (because the averaging time can be taken as arbitrarily small) but there is a possibility of a finite error accumulating during the iteration process. Hopefully, further investigation may provide numerical support, but for the moment this particular step (although plausible), should be regarded as somewhat imponderable.

II. STATISTICAL EQUATIONS

A. Equations of motion

We shall restrict our attention to incompressible flows. Accordingly, we may take the velocity field

$U_\alpha(\vec{x},t)$ as satisfying the Navier-Stokes equations in the form

$$\frac{\partial U_\alpha}{\partial t}-\nu_0\nabla^2 U_\alpha=\frac{\partial p}{\partial x_\alpha}-U_\beta\frac{\partial}{\partial x_\beta}U_\alpha \tag{2.1}$$

and

$$\frac{\partial}{\partial x_\alpha}U_\alpha(\vec{x},t)=0\ , \tag{2.2}$$

where ν_0 is the kinematic viscosity.

Equation (2.1) may be specialized to the case of shear flow by the addition of an externally applied pressure gradient which would be needed to sustain the flow. To do this, we assume that part of the boundary surface is at infinity so that one may add $\partial p_{\text{ext}}/\partial x_\alpha$ to the right-hand side of (2.1) with p_{ext} satisfying

$$\nabla^2 p_{\text{ext}}=0\ .$$

Then we may eliminate the internal pressure by taking the divergence of all terms in (2.1) and invoking (2.2), thus,

$$\nabla^2 p=-\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}U_\alpha U_\beta\ . \tag{2.3}$$

In other words, the external pressure satisfies Laplace's equation whereas the internal pressure satisfies Poisson's equation and is, in fact, a Lagrange multiplier of the velocity field.

We can proceed by introducing the Green function $H(\vec{x},\vec{x}\,')$ such that

$$\nabla^2 H(\vec{x},\vec{x}')=\delta(\vec{x}-\vec{x}') \tag{2.4}$$

and taking (2.1) on the boundary surface in order to establish the boundary conditions on the pressure. A fuller treatment of this procedure will be found elsewhere.[12–14] As we do not aim at practical calculations in this work, we shall simplify matters by taking all boundary surfaces to be at infinity. This only excludes the surface integral which represents the boundary condition on the normal derivative of the pressure.

It follows therefore that we may write (2.1) in the form

$$\left[\frac{\partial}{\partial t}-\nu_0\nabla^2\right]U_\alpha(\vec{x},t)=-\frac{\partial p_{\text{ext}}}{\partial x_\alpha}+M_{\alpha\beta\gamma}U_\beta(\vec{x},t)U_\gamma(\vec{x},t)\ , \tag{2.5}$$

where

$$M_{\alpha\beta\gamma}=-\frac{1}{2}\left[D_{\alpha\beta}\frac{\partial}{\partial x_\gamma}+D_{\alpha\gamma}\frac{\partial}{\partial x_\beta}\right] \tag{2.6}$$

and $D_{\alpha\beta}$ is defined in terms of its effect on an arbitrary function $f(\vec{x})$,

$$D_{\alpha\beta}f(\vec{x})=\delta_{\alpha\beta}f(\vec{x})-\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\int d^3x'H(\vec{x},\vec{x}')f(\vec{x}')\ . \tag{2.7}$$

At a later stage we shall find it necessary to work with the Fourier components of the velocity field. Therefore, it will be convenient to summarize the basic equations at this point.

We may consider the fluid to occupy a cubical box of side L. Then the Fourier components of the velocity field are defined by

$$U_\alpha(\vec{x},t)=\sum_{\vec{k}}U_\alpha(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}\ . \tag{2.8}$$

At a later stage we may take the limit $L\to\infty$ and summations may then be replaced by integrations. With the substitution of (2.8), Eqs. (2.5)–(2.7) become

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]U_\alpha(\vec{k},t)=\Pi_\alpha(k)+\sum_{\vec{j}}M_{\alpha\beta\gamma}(\vec{k})U_\beta(\vec{j},t)\times U_\gamma(\vec{k}-\vec{j},t)\ , \tag{2.9}$$

$$M_{\alpha\beta\gamma}(\vec{k})=\tfrac{1}{2}i[k_\beta D_{\alpha\gamma}(\vec{k})+k_\gamma D_{\alpha\beta}(\vec{k})]\ , \tag{2.10}$$

and

$$D_{\alpha\beta}(\vec{k})=\delta_{\alpha\beta}-k_\alpha k_\beta\,|\,\vec{k}\,|^{-2}\ , \tag{2.11}$$

where $\Pi_\alpha(\vec{k})$ stands for the Fourier transform of the external pressure gradient in Eq. (2.5). We shall not be manipulating this quantity and accordingly need no more specific form than that.

B. Averaging and the Reynolds equations

For quasisteady flows, the average with respect to time, is defined as

$$\bar{U}_\alpha(\vec{x},t)=\frac{1}{2T}\int_{-T}^{T}U_\alpha(\vec{x},t+s)ds\ . \tag{2.12}$$

The averaging period $2T$ must satisfy two conditions. First, it should be long enough to smooth out the fluctuations associated with the turbulent cascade. Second, $2T$ must be shorter than the time-scale associated with any external time dependence that we might wish to study. A flow in which both these conditions can be satisfied is, by definition, quasisteady.

This is conventional wisdom and nowadays we should probably class the bursting process (as discussed in Sec. I) with the external time disturbances. (This does not imply that the bursting process is an external disturbance. It is simply a recognition of the fact that the time between bursts may be long enough for the above criteria to be satisfied. The implication would then be that *all* turbulent flows are no better than quasisteady.)

Theorists normally use the ensemble average. And, of course, in certain circumstances the difference between that and the time average as defined by (2.12) is of some importance. However, this is not the case here. In the well-established analysis leading to the Reynolds equations (which we shall presently summarize), the distinction would be purely formal. The analysis is the same for either form of average.

In this paper, we shall use the integral time average almost exclusively. This will turn out to be central to our approach, which is to assess the effect of repeated smoothing with increasing values of the averaging period. Further, it will be seen as a not entirely trivial point that this operation corresponds exactly to the way in which the experimentalist measures mean quantities in turbulence.

Let us now consider the problem of deriving equations for the mean and covariance of the turbulent velocity. Irrespective of how the average is defined, we shall represent the mean velocity by an overscore and all other averages over the fluctuating field by Dirac brackets, thus, $\langle\ \rangle$.

The analysis is well established and proceeds as follows. Let us represent the total velocity field as the sum of the mean and the fluctuation from the mean, thus,

$$U_\alpha(\vec{x},t)=\overline{U}_\alpha(\vec{x},t)+u_\alpha(\vec{x},t)\ , \tag{2.13}$$

where $u_\alpha(\vec{x},t)$ is the fluctuating velocity field. Evidently $u_\alpha(\vec{x},t)$ itself has zero mean and the statistical description of the velocity field may be completed by the infinite sequence of moments

$$Q_{\alpha\beta}(\vec{x},\vec{x}\,';t,t')=\langle u_\alpha(\vec{x},t)u_\beta(\vec{x}\,',t')\rangle\ , \tag{2.14}$$

$$Q_{\alpha\beta\gamma}(\vec{x},\vec{x}\,',\vec{x}\,'';t,t',t'')=\langle u_\alpha(\vec{x},t)u_\beta(\vec{x}\,',t')u_\gamma(\vec{x}\,'',t'')\rangle\ , \tag{2.15}$$

and so on.

The equation for the mean velocity is obtained by substituting (2.13) into Eq. (2.5) and averaging, thus,

$$\left[\frac{\partial}{\partial t}-\nu_0\nabla^2\right]\overline{U}_\alpha(\vec{x},t)=-\frac{\partial p_{\text{ext}}}{\partial x_\alpha}+M_{\alpha\beta\gamma}[\overline{U}_\beta(\vec{x},t)\overline{U}_\gamma(\vec{x},t)+Q_{\beta\gamma}(\vec{x},t)]\ . \tag{2.16}$$

Also, subtracting Eq. (2.16) from (2.5), multiplying by $u_{\alpha'}(\vec{x}\,',t')$, and averaging, yields the general equation for the second moment of the fluctuating field:

$$\left[\frac{\partial}{\partial t}-\nu_0\nabla^2\right]Q_{\alpha\alpha'}(\vec{x},\vec{x}\,';t,t')=2M_{\alpha\beta\gamma}\overline{U}_\gamma(\vec{x},t)Q_{\beta\alpha'}(\vec{x},\vec{x}\,';t,t')+M_{\alpha\beta\gamma}Q_{\beta\gamma\alpha'}(\vec{x},\vec{x},\vec{x}\,';t,t,t')\ . \tag{2.17}$$

Clearly, an equation for the unknown third-order moment can be formed the same way, and will contain the fourth-order moment, and so on. This is the well-known closure problem. In the engineering literature it has traditionally been evaded by making assumptions about the relationship between the mean velocity and the Reynolds stress; and working only with Eq. (2.16). [Although, recently, this approach has been extended to single-point forms of (2.17).] In contrast, theorists have tended to work with flows that are translationally invariant. Hence, $\overline{U}_\alpha$ and Eq. (2.16) are irrelevant, with full attention being given to the moment hierarchy.

Equation (2.16) is the Reynolds equation. As it stands, it is not in its most familiar guise. However, specializing to a steady, two-dimensional mean flow, we may put $U_\alpha(\vec{x},t)=U_1(x_2)$, for example, and so reduce (2.16) to

$$-\nu\frac{d^2U_1}{dx_2^2}+\frac{d}{dx_2}Q_{12}(x_2)=-\frac{\partial p_{\text{ext}}}{\partial x_1}\ , \tag{2.18}$$

where Q_{12} is the Reynolds stress. An analogy between the turbulent dissipation and molecular dissipation may be adopted, with the introduction of an eddy viscosity ϵ, such that

$$Q_{12}=\epsilon\frac{dU_1}{dx_2} . \tag{2.19}$$

In practice this analogy has generally been developed by rather primitive arguments in which the energy and momentum transfers in turbulence are taken to be like those in the kinetic theory of dilute gases. The end result can be relatively simple forms of Eq. (2.18) for engineering applications. But even when reduced from (2.16) to (2.18), the Reynolds equation still presents a formidable challenge to theorists.

Finally, for completeness we note that the Fourier-transformed Reynolds equation may be derived directly from (2.9), using exactly the same procedure, to obtain

$$\left[\frac{\partial}{\partial t}+v_0k^2\right]\overline{U}_\alpha(\vec{k},t)=\Pi_\alpha(\vec{k})+\sum_{\vec{j}}M_{\alpha\beta\gamma}(\vec{k})[\overline{U}_\beta(\vec{j},t)\overline{U}_\gamma(\vec{k}-\vec{j},t)+Q_{\beta\gamma}(\vec{k},t)] , \tag{2.20}$$

with a corresponding form for the covariance equation.

III. REFORMULATION OF THE STATISICAL EQUATIONS

A. Averaging by repeated smoothing

Let us begin by considering a generalization of the averaging process of (2.12), carried out over some period $2\tau_0$, where $\tau_0<<T$. We do this by introducing a weighting function $a_0(t)$, such that

$$\langle U_\alpha(\vec{x},t)\rangle_0=\int_{-\infty}^{\infty}U_\alpha(\vec{x},t+s)a_0(s)ds , \tag{3.1}$$

where

$$\int_{-\infty}^{\infty}a_0(t)dt=1 ,$$

and τ_0 is the time scale which characterizes $a_0(t)$. If we choose the weighting function to be

$$a_0(t)=(\tau_0)^{-1}\mathrm{sinc}(\pi t/\tau_0) ,$$

where $\mathrm{sinc}(\alpha)=\alpha^{-1}\sin(\alpha)$, then the operation of (3.1) will average out those frequencies which are greater than, say, ω_0, where $\omega_0=\pi/\tau_0$. If τ_0 is small enough, we note that the quantity

$$U_\alpha(x,t)-\langle U_\alpha(x,t)\rangle_0=u_{\alpha,0}(\vec{x},t)$$

is (a) small compared to U_α or $\overline{U}_\alpha$, and (b) its decay rate is governed by the molecular viscosity. Under these circumstances (which may be taken as the criterion for choosing τ_0), we may solve the equation of motion for the mean effect of the high-frequency fluctuation $u_{\alpha,0}$. This may then be eliminated from the equation of motion, with a consequent reduction of the overall problem.

One could hope to carry on like this, averaging over longer times (and eliminating progressively lower frequencies), until one gets back to the conventional mean value, when the nth averaging time is just T. We begin by introducing the set $\{\tau_0,\tau_1,\ldots,\tau_n,\ldots,\}$, such that $\tau_0<\tau_1<\cdots<\tau_n\cdots$; and the weighting function $a_n(t)$, which must satisfy

$$\int_{-\infty}^{\infty}a_n(t)dt=1 ,$$

and which we choose to be

$$a_n(t)=(\tau_n)^{-1}\mathrm{sinc}(\pi t/\tau_n) . \tag{3.2}$$

Then we define the general operation

$$\langle U_\alpha(x,t)\rangle_n=\int_{-\infty}^{\infty}U_\alpha(\vec{x},t+s)a_n(s)ds , \tag{3.3}$$

and the associated definition

$$u_{\alpha,n}(\vec{x},t)=U_\alpha(\vec{x},t)-\langle U_\alpha(\vec{x},t)\rangle_n . \tag{3.4}$$

With the choice of (3.2) as weighting function, (3.3) satisfies

$$\langle U(t)\rangle_n\to U(t) \text{ as } \tau_n\to 0$$

and

$$\langle U(t)\rangle_n\to\overline{U}(t) \text{ as } \tau_n\to T .$$

Thus, in the limit of large n, (3.3) and (3.4) just reduce to the conventional mean and fluctuation, as defined by (2.12) and (2.13).

Now in any cycle (i.e., for any value of n), let us write the instantaneous velocity field as

$$U_\alpha(\vec{x},t)=U_\alpha^-(\vec{x},t)+U_\alpha^+(\vec{x},t) , \tag{3.5}$$

where

$$\langle U_\alpha(\vec{x},t)\rangle_n=U_\alpha^-(\vec{x},t) \tag{3.6}$$

and

$$\langle U_\alpha^+(\vec{x},t)\rangle_n=0 . \tag{3.7}$$

This procedure is analogous to the derivation of the Reynolds equations which follows from (2.13). That is, if we substitute (3.5) into the equation of motion, and perform the operation $\langle\ \rangle_n$, then we obtain equations for U^- and $\langle U^+U^+\rangle_n$. In each cycle, U^- is the analog of the mean and U^+ is the analog of the fluctuation from the mean. As n increases $U^-\to\overline{U}$, but U^+ does not become the fluctuating velocity. This is because we eliminate frequencies in the band $\omega_0<\omega<\infty$ for $n=0$, frequencies $\omega_1<\omega<\omega_0$ for $n=1$, and so on. Thus for a given value of n, U^+ only contains frequencies in the band $\omega_n<\omega<\omega_{n-1}$. This is by contrast with the true fluctuating velocity as defined by Eq. (3.4). For a given value of n, $u_{\alpha,n}(\vec{x},t)$ contains frequencies in the range $\omega_n<\omega<\infty$.

Similarly, the analog of Reynolds stress in any cycle n, only contains frequencies $\omega_n<\omega<\omega_{n-1}$ and we have

$$\langle U_\alpha^+(\vec{x},t)U_\beta^+(\vec{x},t)\rangle_n=\int_{\omega_n}^{\omega_{n-1}}E_{\alpha\beta}(\vec{x},\omega)e^{i\omega t}d\omega\ , \tag{3.8}$$

where $E_{\alpha\beta}$ is the energy spectrum tensor. With these points in mind we are in a position to develop an iteration procedure for the equation of motion.

B. Iterative development of the mean-field equation

Following the scheme outlined above, we substitute (3.5) for the instantaneous velocity in the equation of motion (2.5). To begin with we average over τ_0. The result for the mean field is

$$\left[\frac{\partial}{\partial t}-\nu_0\nabla^2\right](U_\alpha^-\vec{x},t)-M_{\alpha\beta\gamma}\langle U_\beta^+(\vec{x},t)U_\gamma^+(\vec{x},t)\rangle_0$$
$$=-\frac{\partial p_{\rm ext}}{\partial x_\alpha}+M_{\alpha\beta\gamma}U_\beta^-(\vec{x},t)U_\gamma^-(\vec{x},t)\ . \tag{3.9}$$

Subtracting this from (2.5), multiplying through by $U_{\alpha'}^+(\vec{x},t)$, and averaging over τ_0 we obtain

$$\left[\frac{\partial}{\partial t}-\nu_0\nabla^2\right]\langle U_\alpha^+(\vec{x},t)U_{\alpha'}^+(\vec{x},t)\rangle_0$$
$$=2M_{\alpha\beta\gamma}\langle U_\beta^+(\vec{x},t)U_{\alpha'}^+(\vec{x},t)\rangle_0U_\gamma^-(\vec{x},t)\ , \tag{3.10}$$

where we have borne in mind that $\langle U_\gamma^-\rangle_0=U_\gamma^-$.

The next step is to solve Eq. (3.10). At this stage we shall only do this in a formal sense by introducing the Green's function $S^{(0)}$ such that

$$\left[\frac{\partial}{\partial t}-\nu_0\nabla^2\right]S_{\alpha\beta}^{(0)}(\vec{x},\vec{x}\,';t,t')$$
$$=\delta_{\alpha\beta}\delta(\vec{x}-\vec{x}\,')\delta(t-t')\ . \tag{3.11}$$

Now, if we use (3.11) to obtain $\langle U^+U^+\rangle_0$ in terms of U^- and substitute back into (3.9), the effect of $\langle U^+U^+\rangle_0$ is seen to be that of an additional viscosity acting on the mean field. However, instead of the scalar constant ν_0 due to molecular effects, we are now faced with a transport coefficient which is a tensor function of the variables $\vec{x}$ and t. In principle, it is convenient to anticipate this by generalizing the zero order to take a consistent (if somewhat degenerate) form. Thus if we rewrite (3.11) as

$$L^{(0)}(\vec{x},t)S_{\alpha\beta}^{(0)}(\vec{x},\vec{x}\,';t,t')$$
$$=\delta_{\alpha\beta}\delta(\vec{x}-\vec{x}\,')\delta(t-t') \tag{3.12}$$

we may make the rather simple generalization

$$\int d^3\vec{x}\,'\int dt'L_{\alpha\sigma}^{(0)}(\vec{x},\vec{x}\,';t,t')S_{\sigma\beta}^{(0)}(\vec{x}\,',\vec{x}\,'';t',t'')$$
$$=\delta_{\alpha\beta}\delta(\vec{x}-\vec{x}\,'')\delta(t-t')\ , \tag{3.13}$$

where $L_{\alpha\sigma}^{(0)}(\vec{x},\vec{x}\,';t,t')$ contains the appropriate combination of delta functions in order to allow the right-hand side of (3.13) to reduce to that of (3.12).

With some rearrangement of indices we may write the solution of (3.10) as

$$\langle U_\beta^+(\vec{x},t)U_\gamma^+(\vec{x},t)\rangle_0=2\int d^3x'\int dt'S_{\beta\alpha'}^{(0)}(\vec{x},\vec{x}\,';t,t')M_{\alpha'\beta'\gamma'}\langle U_{\beta'}^+(\vec{x}\,',t')U_{\gamma'}^+(\vec{x},t)\rangle_0U_\gamma^-(\vec{x}\,',t') \tag{3.14}$$

and upon substitution of this into (3.9), we obtain for the mean field

$$\int d^3x'\int dt'L_{\alpha\sigma}^{(1)}(\vec{x},\vec{x}\,';t,t')U_\sigma^-(\vec{x}\,',t')=-\frac{\partial p_{\rm ext}}{\partial x_\alpha}+M_{\alpha\beta\gamma}U_\beta^-(\vec{x},t)U_\gamma^-(\vec{x},t)\ , \tag{3.15}$$

where

$$L_{\alpha\sigma}^{(1)}(\vec{x},\vec{x}\,';t,t')=L_{\alpha\sigma}^{(0)}(\vec{x},\vec{x}\,';t,t')-\delta P_{\alpha\sigma}^{(0)}(\vec{x},\vec{x}\,';t,t') \tag{3.16}$$

and

$$\delta P^{(0)}_{\alpha\sigma}(\vec{x},\vec{x}\,';t,t')=2\int d^3x'\int dt' M_{\alpha\beta\gamma}S^{(0)}_{\beta\alpha'}(\vec{x},\vec{x}\,';t,t')M_{\alpha'\beta'\sigma}\langle U^+_{\beta'}(\vec{x}\,',t')U^+_{\gamma}(\vec{x},t)\rangle_0 \,. \tag{3.17}$$

We should note that $\delta P^{(0)}$ acting on U^- in (3.15) represents the increase in mean kinematic stress due to the elimination of these fluctuations with frequency greater than ω_0. The product $M_{\alpha\beta\gamma}M_{\alpha'\beta'\sigma}$ gives rise to second derivatives such as

$$\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_{\beta'}}$$

acting to the right on U^- and hence to a mean rate of strain. Thus $\delta P^{(0)}$ can be characterized by a transport coefficient $\delta\nu$ (in general, an anisotropic second-order tensor), which is the increase in effective viscosity due to $\langle U^+U^+\rangle_0$. We shall make use of this idea in Sec. IV when we analyze the iteration in more detail.

Now let us repeat the process for an averaging time $\tau_1<\tau_0$. We shall only write the mean-field equation for this stage. In Eq. (3.15) we make the replacement $U^-\to U$. That is, "old mean field" becomes "current total field" as the U^+ for $\omega>\omega_0$ has been eliminated. We then redivide into U^- and U^+ (current values) and average over τ_1. The result is

$$\int d^3x'\int dt' L^{(1)}_{\alpha\sigma}(\vec{x},\vec{x}\,';t,t')U^-_\sigma(\vec{x}\,',t')-M_{\alpha\beta\gamma}\langle U^+_\beta(\vec{x},t)U^+_\gamma(\vec{x},t)\rangle_1=-\frac{\partial p_{\rm ext}}{\partial x_\alpha}+M_{\alpha\beta\gamma}U^-_\beta(\vec{x},t)U^-_\gamma(\vec{x},t)\,, \tag{3.18}$$

along with appropriate generalizations of Eqs. (3.13), (3.14), (3.16), and (3.17).

If we carry this process on, then the forms for an averaging time τ_n follow inductively. Thus,

$$\int d^3x'\int dt' L^{(n)}_{\alpha\sigma}(\vec{x},\vec{x}\,';t,t')U^-_\sigma(\vec{x}\,',t)-M_{\alpha\beta\gamma}\langle U^+_\beta(\vec{x},t)U^+_\gamma(\vec{x},t)\rangle_n=-\frac{\partial p_{\rm ext}}{\partial x_\alpha}+M_{\alpha\beta\gamma}U^-_\beta(\vec{x},t)U^-_\gamma(\vec{x},t)\,, \tag{3.19}$$

where

$$L^{(n)}_{\alpha\sigma}(\vec{x},\vec{x}\,';t,t')=L^{(n-1)}_{\alpha\sigma}(\vec{x},\vec{x}\,';t,t')-\delta P^{(n-1)}_{\alpha\sigma}(\vec{x},\vec{x}\,';t,t')\,, \tag{3.20}$$

$$\delta P^{(n)}_{\alpha\sigma}(\vec{x},\vec{x}\,';t,t')=2\int d^3x'\int dt' M_{\alpha\beta\gamma}S^{(n)}_{\beta\alpha'}(\vec{x},\vec{x}\,';t,t')M_{\alpha'\beta'\sigma}\langle U^+_{\beta'}(\vec{x}\,',t')U^+_{\gamma}(\vec{x},t)\rangle_n\,, \tag{3.21}$$

and

$$\int d^3x'\int dt' L^{(n)}_{\alpha\sigma}(\vec{x},\vec{x}\,';t,t')S^{(n)}_{\sigma\beta}(\vec{x}\,',\vec{x}\,'';t',t'')=\delta_{\alpha\beta}\delta(\vec{x}-\vec{x}\,'')\delta(t-t'')\,. \tag{3.22}$$

We can establish the connection between (3.19) and the long-time-averaged Reynolds equation [as given by (2.16)], by taking $\tau_n\to T$. Evidently $U^-\to\overline{U}$ by definition, and the term $\langle U^+U^+\rangle_n$ may be taken to be zero in this limit. [From (3.8), $\langle U^+U^+\rangle_n$ is the integral over that part of the spectrum for which $\omega_n\le\omega\le\omega_{n-1}$. Remembering that $\omega_n\to 0$ as $u\to\infty$, and that $E(\omega)\to 0$ rather rapidly as $\omega\to 0$, the step $\langle U^+U^+\rangle_n\to 0$ as $\tau_n\to T$ is plausible.] Then (3.19) becomes

$$\lim_{n\to\infty}\int d^3x'\int dt' L^{(n)}_{\alpha\sigma}(\vec{x},\vec{x}\,';t,t')\overline{U}_\sigma(\vec{x}\,',t')=-\frac{\partial p_{\rm ext}}{\partial x_\alpha}+M_{\alpha\beta\gamma}\overline{U}_\beta(\vec{x},t)\overline{U}_\gamma(\vec{x},t)\,. \tag{3.23}$$

Comparison with (2.16) shows that the Reynolds stress $Q_{\beta\gamma}$ has been eliminated. Its effect is now represented by the appearance of $L^{(n)}$ acting on the mean velocity. At this stage it is still an open question whether $L^{(n)}$ will settle down something like

$$\left[\frac{\partial}{\partial t}-\nu^{(n)}(\vec{x})\nabla^2\right]\to\left[\frac{\partial}{\partial t}-\nu(\vec{x})\nabla^2\right].$$

However, in Sec. IV we shall show that, provided certain simplifying assumptions can be made, this is indeed the case.

C. Application of the renormalization group

We close the work of this section by considering how we may carry out a renormalization-group (RG) analysis of the above iteration procedure. The RG analysis is most conveniently discussed in terms of the Fourier components of the field. We shall consider its application to the equation of motion in the form given by (2.9). The procedure as applied to dynamical problems involves two stages[2-4]:

(1) The Fourier decomposition of the velocity field is taken to be cut off for $k > \Lambda$. Divide the velocity field into modes $U_\alpha^<(\vec{k},t)$ and $U_\alpha^>(\vec{k},t)$, where $U_\alpha^>$ are the modes such that $a\Lambda < |\vec{k}| < \Lambda$. Eliminate the high-$k$ modes by solving the equation for $U_\alpha^>$ and substituting the solution into the equation for $U_\alpha^<$. [Note, because of the "sum over modes" in (2.9), the solution for $U_\alpha^>$ will contain $U_\alpha^<$.] Average over $\Pi(\vec{k})$. [For the purposes of this brief discussion, $\Pi(\vec{k})$ should be taken as a random stirring force with known statistics, rather than the steady pressure drop of shear flow.]

(2) Rescale $\vec{k}$, t, $U_\alpha^<$, and $\Pi(\vec{k})$ so that the new equation looks like the original Navier-Stokes equation. This last step involves the introduction of renormalized-transport coefficents.

The RG analysis in just this form has been applied by Forster *et al.*[3,4] to the study of velocity correlations at small wave numbers. Three different models have been considered, according to the way the forcing function is chosen. In all cases, the cutoff $k=\Lambda$ is taken to be small enough to exclude cascade effects. Rose[5] has applied the method in a slightly different way to the problem of modeling the effect of those eddies which are too small to be taken into account in a numerical calculation based on a finite-difference grid. The passive convection of a scalar ϕ by the velocity field U_α is considered and both ϕ and U_α are divided up into subgrid and supergrid fields. In this work the upper cutoff Λ is taken to be the dissipation wave number, while $a\Lambda$ is bounded by the inverse of the smallest mesh length used in the calculation.

In applying the RG analysis to the present work, we should compare the analysis leading to Eq. (3.18) to stage (1) above. Clearly, the two procedures are similar in some respects. The division of the field into $U_\alpha^<$ and $U_\alpha^>$ has its analog in (3.5) with the division into U_α^- and U_α^+. An the elimination of $U_\alpha^>$ (in terms of $U_\alpha^<$) parallels that of U_α^+ (in terms of $\bar{U}_\alpha$).

On the other hand, there are nontrivial distinctions to be made. In the present work the averaging comes first and is the defining operation [i.e., see Eqs. (3.5)–(3.7)]. A second, perhaps more important difference, is that the iteration is generated by the averaging process which we have chosen [i.e., see Eq. (3.3)]. Thus convergence to the mean-field equation is guaranteed without the need for any rescaling of the basic variables. Of course, one must note that renormalized-transport coefficients in the form of series which only converge "in principle" may be difficult to handle in practice, but there are several ways of tackling this problem. Hence it is probably valid to draw a distinction between the present work and RG.

However, having made this point, we shall nevertheless force our analysis into the RG mode. This will be seen to provide us with a rather simple "first look" at the iteration procedure. This poses the problem of finding an appropriate change of variables in order to carry out stage (2) of the RG analysis. Evidently such a rescaling should be based on the set $\{\tau_n\}$, of the averaging procedure.

These averaging times are not arbitrary. For instance, in order to solve (3.10) for U_α^+, we require τ_0 to be such that the Reynolds number is of order unity. In turn, this means that the "viscous" and "inertial" contributions to the eddy turnover time should be of the same size. The viscous response time is $(\nu_0 k^2)^{-1}$ and if we take the value of this time at $k=k_d$, where k_d is the boundary between inertial and viscous regions of the spectrum, then we have

$$\tau_0=(\nu_0 k_d^2)^{-1}=\nu_0^{1/2}/\epsilon^{1/2} . \tag{3.24}$$

Here we have used the relation $k_d=(\epsilon/\nu^3)^{1/2}$, ϵ being the rate of dissipation of turbulent kinetic energy, which comes from dimensional analysis.[14]

It is tempting to make the generalization of (3.24).

$$\tau_n=\nu_n^{1/2}/\epsilon^{1/2} \tag{3.25}$$

(and this will later be seen to be valid). Also, if we take the spectrum to be given by a power law then it follows that the effective viscosity will be a power law.

In all then, we shall assume τ_n to be given by

$$\tau_n=h^{-n}\tau_0 , \tag{3.26}$$

where $0 \le h \le 1$. The implications of this will become clearer in Sec. IV when we carry out the analysis in detail.

IV. ANALYSIS OF THE ITERATION PROCEDURE FOR THE MEAN FIELD

A. The recursion relation

From now on, we shall simplify the algebra by working with the Fourier components of the field in k space. We shall also make the assumptions that the fluctuating field may be taken as both homogeneous and isotropic. These assumptions would not be valid near solid boundaries but we have ruled such regions out of consideration at the beginning of this work. We shall enlarge on the practical significance of this type of approximation at a later stage.

With these assumptions, the pair correlation and response tensors take particularly simple forms. They may be written[14]

$$Q_{\alpha\beta}(\vec{k},\vec{k}';t,t')=D_{\alpha\beta}(\vec{k})\delta(\vec{k}-\vec{k}')Q(k;t,t') \tag{4.1}$$

and

$$S^{(n)}_{\alpha\beta}(\vec{k},\vec{k}';t,t')=D_{\alpha\beta}(\vec{k})\delta(\vec{k}-\vec{k}')S_n(k;t,t') , \tag{4.2}$$

where $S^{(n)}_{\alpha\beta}(\vec{k},\vec{k}';t,t')$ is the Fourier transform of $S^{(n)}_{\alpha\beta}(\vec{x},\vec{x}';t,t')$, as defined by (3.22), and the pair correlation is defined by

$$\left[\frac{L}{2\pi}\right]^3\langle u_\alpha(\vec{k},t)u_\beta(-\vec{k}',t')\rangle=Q_{\alpha\beta}(\vec{k},\vec{k}';t,t') , \tag{4.3}$$

where $u_\alpha(\vec{k},t)$ is the Fourier transform of the fluctuation from the mean as defined in (2.13). Here $\langle\ \rangle$ means either realization or long-time average. For completeness, we should also have the following generalizations of (4.1) and (4.3):

$$(L/2\pi)^3\langle u_{\alpha,n}(\vec{k},t)u_{\beta,n}(-k',t')\rangle_n =D_{\alpha\beta}(\vec{k})\delta(\vec{k}-\vec{k}')Q_n(k;t,t') , \tag{4.4}$$

where $u_{\alpha,n}(\vec{k},t)$ is the Fourier transform of the derivation from the (nth) mean, as defined by Eq. (3.4), and

$$\left[\frac{L}{2\pi}\right]^3\langle U^+_\alpha(\vec{k},t)U^+_\beta(-\vec{k}',t')\rangle_n =D_{\alpha\beta}(\vec{k})\delta(\vec{k}-\vec{k}')Q^+_n(k;t,t') . \tag{4.5}$$

We now wish to obtain Eqs. (3.19)–(3.22) in terms of the Fourier components. The above assumptions, and the properties of the $D_{\alpha\beta}$ operators, allow a considerable reduction of the problem. For instance, the first term on the right-hand side of (3.19) must take the form

$$\int dt' L_n(k;t,t')D_{\alpha\sigma}(\vec{k})U^-_\sigma(\vec{k},t') =\int dt' L_n(k;t,t')U^-_\alpha(\vec{k},t') , \tag{4.6}$$

where L_n is defined by analogy with S_n in (4.2). A further simplification of the analysis is possible if we follow the example of Rose[5] who assumes that the $U^>_\alpha(\vec{k})$ modes evolve more rapidly than the lower-frequency $U^<_\alpha(\vec{k})$ modes, to such an extent that the time dependence of the $U^>_\alpha(\vec{k})$ may be neglected over time scales associated with the $U^<_\alpha(\vec{k})$. It would seem to be reasonably plausible to make the same assumption about the $U^+_\alpha(\vec{k})$ and $U^-_\alpha(\vec{k})$ of the present work. And, although it is not a necessary approximation, it does allow the underlying structure of the iteration procedure to be seen rather more clearly.

To see what this means in practice, let us return to zero order. After Fourier transformation, Eqs. (3.9) and (3.10) become

$$\left[\frac{\partial}{\partial t}+\nu_0k^2\right]U^-_\alpha(\vec{k},t)-\sum_{\vec{j}}M_{\alpha\beta\gamma}(\vec{k})\langle U^+_\beta(\vec{j},t)U^+_\gamma(\vec{k}-\vec{j},t)\rangle_0=\Pi_\alpha(\vec{k}) +\sum_{\vec{j}}M_{\alpha\beta\gamma}(\vec{k})U^-_\beta(\vec{j},t)U^-_\gamma(\vec{k}-\vec{j},t) , \tag{4.7}$$

and

$$\left[\frac{\partial}{\partial t}+\nu_0k^2\right]\langle U^+_\alpha(\vec{k},t)U^+_{\alpha'}(\vec{k},t)\rangle_0=2\sum_{\vec{j}}M_{\alpha\beta\gamma}(\vec{k})\langle U^+_\beta(\vec{j},t)U^+_{\alpha'}(\vec{k}',t)\rangle_0U^-_\gamma(k-j,t) . \tag{4.8}$$

Then the assumption that U^+ evolves much more rapidly than U^- implies that U^+ relaxes to the steady-state solution of (4.8) while U^- is still evolving. Hence we may write (4.8) as

$$\langle U_\alpha^+(\vec{k},t)U_{\alpha'}^+(\vec{k}',t)\rangle_0 = 2\sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})(\nu_0 k^2)^{-1}\langle U_\beta^+(\vec{j},t)U_{\alpha'}^+(\vec{k}'t)\rangle_0 U_\gamma^-(\vec{k}-\vec{j},t) \ . \tag{4.9}$$

This is the $\vec{k}$-space equivalent of (3.14), with the time dependence simplified. Renaming dummy variables we may rewrite (4.9) as

$$\langle U_\beta^+(\vec{j},t)U_\gamma^+(\vec{k}-\vec{j},t)\rangle_0 = 2\sum_{\vec{j}'} M_{\alpha'\beta'\gamma'}(\vec{j})D_{\beta\gamma'}(\vec{j})(\nu_0 j^2)^{-1}\langle U_{\beta'}^+(\vec{j}',t)U_\gamma^+(\vec{k}-\vec{j})\rangle_0 U_{\gamma'}^-(\vec{j}-\vec{j}',t) \tag{4.10}$$

and substitute into (4.7) to obtain for the mean field:

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]U_\alpha^-(\vec{k},t) - 2\sum_{\vec{j},\vec{j}'} M_{\alpha\beta\gamma}(\vec{k})M_{\alpha'\beta'\gamma'}(\vec{j})D_{\beta\alpha'}(\vec{j})(\nu_0 j^2)^{-1}\langle U_{\beta'}^+(\vec{j}',t)U_\gamma^+(k-j,t)\rangle_0 U_{\gamma'}^-(\vec{j}-\vec{j}',t)$$
$$= \Pi_\alpha(\vec{k}) + \sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})U_\beta^-(\vec{j},t)U_\gamma^-(\vec{k}-\vec{j},t) \ . \tag{4.11}$$

This may then be written in terms of an incremental change to the viscosity, thus

$$\left[\frac{\partial}{\partial t}+\nu_1 k^2\right]U_\alpha^-(\vec{k},t) = \Pi_\alpha(\vec{k}) + \sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})U_\beta^-(\vec{j},t)U_\gamma^-(k-j,t) \ , \tag{4.12}$$

where

$$\nu_1 = \nu_0 + \delta\nu_0 \tag{4.13}$$

and [using Eq. (4.5) and taking the limit $L \to \infty$]

$$\delta\nu_0 = k^{-2}\int d^3j\, L_{kj}(\nu_0 j^2)^{-1} Q_0^+(|\vec{k}-\vec{j}|,t) \ , \tag{4.14}$$

where

$$L_{kj} = -2M_{\alpha\beta\gamma}(\vec{k})M_{\alpha'\beta'\gamma'}(\vec{j})D_{\beta\alpha'}(\vec{j})D_{\beta'\gamma}(\vec{k}-\vec{j})D_{\gamma'\alpha}(\vec{k}) \ . \tag{4.15}$$

As in the previous section, we repeat the process for an averaging time $\tau_1 > \tau_0$, to obtain the Fourier-transformed (and simplified) form of Eq. (3.18), thus

$$\left[\frac{\partial}{\partial t}+\nu_1 k^2\right]U_\alpha^-(\vec{k},t) - \sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})\langle U_\beta^+(\vec{j},t)U_\gamma^+(\vec{k}-\vec{j},t)\rangle_1 = \Pi_\alpha(\vec{k}) + \sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})U_\beta^-(\vec{j},t)U_\gamma^-(\vec{k}-\vec{j},t) \ , \tag{4.16}$$

and so on. Again, the equations for an averaging time of τ_n follow inductively. Thus the analogous forms of (3.19) *et seq* are

$$\left[\frac{\partial}{\partial t}+\nu_n k^2\right]U_\alpha^-(\vec{k},t) - \sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})\langle U_\beta^+(\vec{j},t)U_\gamma^+(\vec{k}-\vec{j},t)\rangle_n = \Pi_\alpha(\vec{k}) + \sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})U_\beta^-(\vec{j},t)U_\gamma^-(\vec{k}-\vec{j},t) \ , \tag{4.17}$$

$$\nu_{n+1} = \nu_n + \delta\nu_n \ , \tag{4.18}$$

and

$$\delta\nu_n = \int d^3j\, L_{kj}\frac{Q_n^+(|\vec{k}-\vec{j}|,t)}{\nu_n(j,t)k^2 j^2} \ . \tag{4.19}$$

For completeness, we note that L_{kj} may be written as

$$L_{kj} = \frac{[\mu(k^2+j^2)-kj(1+2\mu^2)]kj(1-\mu^2)}{k^2+j^2-2kj\mu} \ , \tag{4.20}$$

where μ is the cosine of the angle between the wave vectors $\vec{k}$ and $\vec{j}$.

B. Rescaling of time, frequency, and wave number

We now consider a set of transformations based on $\{\tau_n\}$. In particular, we wish to relate Q_n^+ to the long-time-average spectrum Q. The smallest averaging time τ_0 is already fixed in Eq. (3.24). We now need to fix the the largest averaging time τ_N (say). This is not too critical providing it includes all cascade affects without including the bursting intermittency. As suggested in Sec. I, the time scale of the largest eddies, in the form

$$\tau_N = a/U \; , \tag{4.21}$$

would seem to meet both requirements.

Let us now define the long-time-averaged spectrum Q as

$$Q(t) \equiv Q_N = \frac{1}{2\tau_N} \int_{-\tau_N}^{\tau_N} f(t+s)ds \; , \tag{4.22}$$

where we have dropped the spatial dependence for convenience, and f stands for the square of the fluctuating velocity. Introducing the change of variables

$$s = \tau_n s' \tag{4.23}$$

we may put

$$Q_n = Q(\tau_n t) \; , \tag{4.24}$$

and using (3.26),

$$Q_n = Q(h^{-n}\tau_0 t) \; . \tag{4.25}$$

Evidently, one may write the same relationship for Q_n^+ with the stipulation that this holds only for an integral over the Fourier components in the frequency range $\omega_n \le \omega \le \omega_{n-1}$. As we are working with a wave-number decomposition, we invoke the well-known Taylor hypothesis, thus

$$kU = \omega \; . \tag{4.26}$$

Then, corresponding to the transformation

$$t \to h^{-n}\tau_0 t'$$

we have

$$k \to (\pi\tau_0^{-1}h^n U^{-1})k' \; , \tag{4.27}$$

and so we may express Q_n^+ in terms of Q as

$$Q_n^+(k,t) \to Q\left[\frac{\pi h^n}{\tau_0 U}k', h^{-n}\tau_0 t'\right] \; , \tag{4.28}$$

where $1 \le k' \le h^{-1}$.

C. Calculation of the effective viscosity

In order to make a definite calculation we shall take the turbulence to be statistically steady (in the usual sense). This allows the time dependence to be dropped from Q^+. We shall further assume that the steady spectrum is given by the well-known Kolmogoroff distribution,[14]

$$Q(k) = \alpha\epsilon^{2/3}k^{-11/3} \; , \tag{4.29}$$

where α is a constant whose experimental value is somewhere in the range $1.3-1.8$. Then, Eq. (4.19) may be written as

$$\delta\nu_n(k_n k') = \alpha\epsilon^{2/3}k_n^{-8/3}\int d^3j' \frac{L_{k'j'}\,|\vec{k}'-\vec{j}'|^{-11/3}}{\nu_n(k_n j')k'^2 j'^2} \; , \tag{4.30}$$

where $1 \le k', j', |\vec{k}'-\vec{j}'| \le h^{-1}$, and

$$k_n = \frac{2\pi h^n}{\tau_0 U} \; . \tag{4.31}$$

It follows directly that we may write ν_n in the form (renaming $k'=k$, etc.):

$$\nu_n(k_n k) = \alpha^{1/2}\epsilon^{1/3}k_n^{-4/3}\nu_n^*(k) \; , \tag{4.32}$$

with Eq. (4.30) now becoming

$$\delta\nu_n^*(k) = \int d^3j \frac{L_{kj}\,|\vec{k}-\vec{j}|^{-11/3}}{\nu_n^*(j)k^2 j^2} \; , \tag{4.33}$$

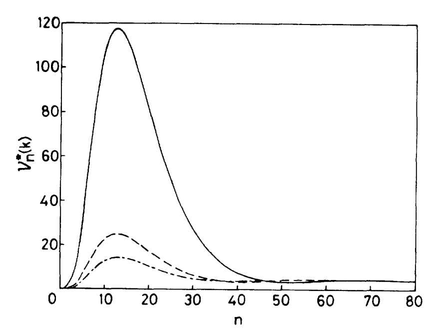

FIG. 1. Variation of the effective viscosity $\nu_n^*(k)$ with iteration-cycle number n, for $k = 1.011k_n$; ——: $\nu_0^* = 0.01$; —— ——: $\nu_0^* = 0.05$; —— – ——: $\nu_0^* = 0.10$.

where $1 \le k,j$, $|\vec{k}-\vec{j}| \le h^{-1}$. For consistency, it is readily shown that the recursion relation of Eq. (4.18) should be written as

$$h^{-4/3}\nu^*_{n+1}(k)=\nu^*_n(hk)+\delta\nu^*_n(hk)\ . \tag{4.34}$$

Equations (4.33) and (4.34) have been computed numerically. The recursion relation is found to reach a fixed point, such that

$$\lim_{n\to\infty} \nu^*_n(k)\to\nu^*(k)\ . \tag{4.35}$$

The effective (or eddy) viscosity $\nu^*(k)$ does not depend on the molecular viscosity ν_0^*, but does depend on the value chosen for the parameter h. Results are shown in Fig. 1 for the particular case of $h=0.9$. The effective viscosity $\nu_n^*(k)$ is plotted against n for three different values of the molecular viscosity ν_0^*, and for $k=1.011k_n$.

V. EQUATIONS FOR THE FLUCTUATING FIELD

A. The covariance equation

In the Reynolds analysis, Eq. (2.17) for the covariance of the velocity field leads to the statistical-closure problem. In the present analysis, the analogous equation (4.7) [or, more generally, Eq. (3.14)] does not. Both these equations deal with the contribution to the covariance from a band of frequencies which depends on the iteration cycle. In order to obtain a quantity which tends to the covariance as $n\to\infty$ (i.e., we want Q_n rather than Q_n^+) we need to invoke the deviation from the nth mean as defined by (3.4). In k space this becomes

$$u_{\alpha,n}(\vec{k},t)=U_\alpha(\vec{k},t)-\langle U_\alpha(\vec{k},t)\rangle_n\ . \tag{5.1}$$

We obtain an equation of motion for $u_{\alpha,n}$ as follows. Substituting (5.1) into the instantaneous equation of motion (2.9), we find

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu_0k^2\right][u_{\alpha,n}(\vec{k},t)+\langle U_\alpha(\vec{k},t)\rangle_n]&=\Pi_\alpha(\vec{k})+\sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})u_{\beta,n}(\vec{j},t)u_{\gamma,n}(\vec{k}-\vec{j},t)\\&\quad+\sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})[2\langle U_\beta(\vec{j},t)\rangle_n u_{\gamma,n}(\vec{k}-\vec{j},t)\\&\quad+\langle U_\beta(\vec{j},t)\rangle\langle U_\gamma(\vec{k}-\vec{j},t)\rangle_n]\ .\end{aligned} \tag{5.2}$$

Remembering that $\langle U_\alpha\rangle_n=U_\alpha^-$ on each iteration cycle, we may use Eq. (4.17) for U_α^- to reduce (5.2) to the form

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu_0k^2\right]u_{\alpha,n}(\vec{k},t)&=2\sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})[\langle U_\beta(\vec{j},t)\rangle_n u_{\gamma,n}(\vec{k}-\vec{j},t)+u_{\beta,n}(\vec{j},t)u_{\gamma,n}(\vec{k}-\vec{j},t)]\\&\quad+(\nu_n-\nu_0)k^2\langle U_\alpha(\vec{k},t)\rangle_n-\sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})\langle U_\beta^+(\vec{j},t)U_\gamma^+(\vec{k}-\vec{j},t)\rangle_n\ .\end{aligned} \tag{5.3}$$

It should be noted that the $(\nu_n-\nu_0)k^2\langle U\rangle_n$ and $M\langle U^+U^+\rangle$ terms do not cancel. Quite the reverse: On each cycle of iteration, some more of the $M\langle U^+U^+\rangle$ term is transferred into $(\nu_n-\nu_0)k^2\langle U\rangle_n$. Together these terms make up the $M\langle uu\rangle$ term which occurs in the traditional derivation leading to Eq. (2.17). This term vanishes when one multiplies through by u and averages. The same is true for (5.3) when we multiply through by $u_{\alpha,n}$ and average over τ_n. It is trivial to show that the resulting equation reduces to (Fourier-transformed) Eq. (2.17) in the limit of large n.

In order to study the closure problem in its simplest form, we again consider regions remote from the solid boundaries of the system. This implies that the fluctuations may be treated as isotropic and that we may neglect gradients of the mean velocity. Thus (5.3) reduces to

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_{\alpha,n}(\vec{k},t)=\sum_{\vec{j}} M_{\alpha\beta\gamma}(k)[u_{\beta,n}(\vec{j},t)u_{\gamma,n}(\vec{k}-\vec{j},t)-\langle U_\beta^+(\vec{j},t)U_\gamma^+(\vec{k}-\vec{j},t)\rangle_n]+f_{\alpha,n}(\vec{k},t)\ , \quad (5.4)$$

where we have added the stirring force $f_{\alpha,n}(\vec{k},t)$ to represent the production of turbulent energy. As n becomes large we have

$$u_{\alpha,n}(\vec{k},t)\rightarrow u_\alpha(\vec{k},t)\ . \quad (5.5)$$

We can also infer that, as the contribution from $\langle U^+U^+\rangle_n$ is transferred into $f_{\alpha,n}$ (in reality, this is achieved through the gradient of the mean velocity, and we have already seen that this process is well behaved), we may expect

$$f_{\alpha,n}(\vec{k},t)\rightarrow f_\alpha(\vec{k},t)\ , \quad (5.6)$$

and

$$\sum_{\vec{j}} M_{\alpha\beta\gamma}(\vec{k})\langle U_\beta^+(\vec{j},t)U_\gamma^+(\vec{k}-\vec{j},t)\rangle_n\rightarrow 0\ , \quad (5.7)$$

as $n\rightarrow\infty$. The last step was discussed in connection with the mean-field equation and follows plausibly if $E(\vec{k},t)\rightarrow 0$ as $(\vec{k})\rightarrow 0$.

Evidently, Eq. (5.4) just reduces to the Navier-Stokes equation for the fluctuating field, in the limit $n\rightarrow\infty$. Thus the iteration process that establishes the mean field does not in itself assist us with the covariance field or the general-closure problem. One possible approach is to tackle Eq. (5.4) using perturbation theory. We shall briefly examine this idea below.

B. The closure problem and the infrared divergence

The application of renormalized-perturbation theory to the Navier-Stokes equation has had its successes. But there is also a serious difficulty to be faced. We begin by discussing this.

Let us consider the effect of letting the molecular viscosity ν_0 tend to zero, while keeping the dissipation rate constant. In this process, the effects of viscous dissipation are pushed to even higher wave numbers. In the limit $\nu_0\rightarrow 0$, the viscous dissipation is represented by a delta function at $k=\infty$. In a purely isotropic field, the stirring forces are arbitrary—only the rate of doing work is fixed—so we take the balancing input term to be a delta function at the origin of $\vec{k}$ space. Under these circumstances the inertial-range form of the spectrum will aply for all values of k. And, although there is currently some debate about the precise form of the inertial-range solution, we shall take it to be the Kolmogoroff distribution, as given by Eq. (4.29).

Now this may seem rather an extreme situation but, nevertheless, it poses a well-defined problem for any turbulence theory. The Kolmogoroff power law is derived using dimensional methods and one might expect a general theory to provide a value for the unknown constant of proportionality α which may then be compared with the experimental result. While the theoretical problem is unrealistic in that it requires the system to contain an infinite amount of energy, it is not unphysical in that the dissipation rate is finite and there is an infinite amount of $\vec{k}$ space to absorb the energy. Thus, a theoretical value of α should also be finite.

Unfortunately this simple test has proved a stumbling block for the straightforward application of renormalized-perturbation methods. In most modern theories, closure is in terms of the energy spectrum and a response function. When the Kolmogoroff spectrum is substituted as a solution, equations for the energy spectrum are found to be well behaved, but integrals in the equation for the response function diverge at $k=0$. This is sometimes referred to as an "infrared" divergence. Some attempts have been made to introduce arbitrary cutoffs in wave number but evidently this arbitrariness is reflected through in the dependence of calculated values of α on the value chosen for the cutoff.[14,15] Various, more elaborate, closure techniques have been tried,[16–20] but it is probably fair to say that all possess one or more unsatisfactory features. A full discussion of this problem will be found in the book by Leslie.[14]

Let us now consider a rather simple-minded application of the direct-interaction approximation (DIA)[14,21] to our present system, as defined by Eq. (5.4). We proceed in the manner of the DIA and introduce the infinitesimal response function $\hat{g}_{\alpha\sigma,n}$ such that

$$\delta u_{\alpha,n}(k,t)=\int_{-\infty}^{t}\hat{g}_{\alpha\sigma,n}(\vec{k};t,t')\delta f_{\sigma,n}(\vec{k},t')\ , \quad (5.8)$$

where $k_n\le k\le\infty$. As $n\rightarrow\infty$, $\hat{g}_{\alpha\sigma,n}$ becomes $\hat{g}_{\alpha\sigma'}$, the DIA response function, and is itself a random variable. The same solution applies for infinitesimal changes in the U^+ field, thus

$$\delta U^+_\alpha(k,t)=\int_{-\infty}^{t}\hat{g}_{\alpha\sigma,n}(\vec{k};t,t')\delta f_{\sigma,n}(\vec{k},t') , \qquad (5.9)$$

where $k_n \le k \le k_{n-1}$.

By linearizing Eq. (5.4) in terms of δu, δU^+, and δf, one may obtain an equation for $\hat{g}$, that is,

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]\hat{g}_{\alpha\sigma,n}(\vec{k};t,t')=\sum_{\vec{j}}2M_{\alpha\beta\gamma}(\vec{k})[\hat{g}_{\beta\sigma,n}(\vec{j};t,t')u_{\gamma,n}(\vec{k}-\vec{j},t)-\langle\hat{g}_{\beta\sigma,n}(j;t,t')U^+_\gamma(\vec{k}-\vec{j},t)\rangle_n]$$
$$+D_{\alpha\sigma}(\vec{k})\delta(t-t') . \qquad (5.10)$$

Averaging, and putting

$$\langle\hat{g}_{\alpha\sigma,n}\rangle_n=g_{\alpha\sigma,n} , \qquad (5.11)$$

we then have

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]g_{\alpha\sigma,n}(\vec{k};t,t')=\sum_{\vec{j}}2M_{\alpha\beta\gamma}(k)[\langle\hat{g}_{\beta\sigma,n}(\vec{j};t,t')u_{\gamma,n}(\vec{k}-\vec{j},t)\rangle_n-\langle\hat{g}_{\beta\sigma,n}(j;t,t')U^+_\gamma(\vec{k}-\vec{j},t)\rangle_n]$$
$$+D_{\alpha\sigma}(k)\delta(t-t') . \qquad (5.12)$$

DIA then involves expanding u, U^+, and $\hat{g}$ in terms of a book-keeping parameter λ. Zero-order fields are treated as Gaussian, the lowest-order nonvanishing terms are retained and λ put equal to unity. The result for (5.12) may be simplified by again invoking isotropy, with

$$g_{\alpha\sigma,n}(\vec{k};t,t')=D_{\alpha\sigma}(\vec{k})g_n(k;t,t') , \qquad (5.13)$$

and corresponding forms for $\langle uu\rangle$ and $\langle U^+U^+\rangle$. Hence, one obtains

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]g_n(\vec{k};t,t')=-\int d^3\vec{j}\,L_{kj}\int_{-\infty}^{t'}ds\,g_n(j;t,s)$$
$$\times g_n(k;s,t')[Q_n(|\vec{k}-\vec{j}|;t,s)-Q^+_n(|\vec{k}-\vec{j}|;t,s)] . \qquad (5.14)$$

We may make a comparison with the results in Sec. IV for the mean field by introducing an effective viscosity through the eddy-decay rate $\omega(k)$. Assuming exponential time dependences,

$$g(k,t-t')=\begin{cases}\exp[-\omega(k)(t-t')] & t>t'\\ 0 & t<t'\end{cases} \qquad (5.15)$$

and

$$Q(k,t-t')=Q(k)\exp[-\omega(k)(t-t')] , \qquad (5.16)$$

we may reduce (5.14) to the form

$$\omega_n(k)=\int d^3\vec{j}\,\frac{L_{kj}}{\omega_n(j)+\omega_n(|k-j|)}$$
$$\times[Q_n(|\vec{k}-\vec{j}|)-Q^+_n(|\vec{k}-\vec{j}|)] . \qquad (5.17)$$

Finally, with the transformations $k\to k_n k'$, and the assumption of the Kolmogoroff distribution for the energy spectrum, we may write (5.17) as

$$\omega_n(k_n k')=\alpha\epsilon^{2/3}k_n^{4/3}$$
$$\times\int d^3\vec{j}\,'\frac{L_{k'j}|k'-j'|^{-11/3}}{\omega_n(k_n j')+\omega_n(k_n|k'-j'|)} , \qquad (5.18)$$

where $h^{-1}\le|k'-j'|\le\infty$. Comparison with (4.30) for the mean-field case (note: eddy-decay rate$=k^2\times$an effective viscosity) indicates that, detailed differences aside, the two forms of the effective viscosity have the same basic structure.

At this point all we have done is apply DIA to the nth-order equation of motion; knowing that, as $n\to\infty$, this equation becomes the Navier-Stokes equation. It might seem tempting to extend this by building up an iterative solution from Eq. (5.12). That is, we take $n = 0$, solve (5.12) by perturbation methods, recalculate the viscosity from (5.17) and then proceed to solve for $n = 1$. This way we would obtain a recursion relation for $\omega_n(k)$. And a comparison with the work of Sec. IV suggests that such a process would converge, leading to a finite $\omega(k) \equiv \lim_{n\to\infty} \omega_n(k)$, even for the infinite Reynolds-number limit. One could therefore expect to avoid the infrared divergence. Unfortunately, there are serious difficulties in the way. One can most easily see this by considering why the method works so well for the mean field $\bar{U}$.

In general terms, $\bar{U}$ is the zero-frequency part of the instantaneous field U. Thus any technique for progressively eliminating the higher frequencies will automatically tend to $\bar{U}$ as a limit. Moreover, the *mean* effect of the fluctuations is always positive (i.e., they always give rise to a net dissipation of energy from the mean field). This does not, of itself, guarantee that the iteration will converge. But if does allow one to make an unambiguous interpretation of, for example, Eq. (4.11) in terms of an effective viscosity. In contrast, the effect of other fluctuations on a particular mode $\vec{k}$ is rather complicated. At its simplest—the cascade picture—there is a flow of energy into mode $\vec{k}$ from lower wave numbers. Correspondingly, energy flows out of mode $\vec{k}$ to still higher wave numbers. Clearly the behavior of mode $\vec{k}$—whether it grows, decays, or remains the same—depends on the relative magnitudes of these two energy flows. If we take the example of a decay mode, then it follows that the eddy-decay rate (and, hence, the effective viscosity) is determined by the *difference* between the two energy flows. Thus the effective viscosity for the fluctuating field mode $\vec{k}$ must also depend on modes below $\vec{k}$. We cannot therefore just scale away wave numbers above $\vec{k}$ as we did in the mean-field case.

It is no more than a definition of the inertial range to state that there exist modes which are entirely determined by these nonlinear energy transfers. For such modes, the role of the stirring forces is played by other modes at lower wave numbers. Thus if we wish to obtain the velocity-field covariance from (5.4), it is not sufficient just to renormalize the viscosity, we must also renormalize the stirring forces. Clearly this raises the question of how we divide up the nonlinear inertial transfer between these two effects. A fuller discussion of this point in the context of the Navier-Stokes equations will be found in Refs. 18 and 19.

We shall not pursue these points in the present work. Evidently the use of (5.4) does not solve the problems inherent in the Navier-Stokes equation. One still faces serious difficulties. However, it seems reasonable to claim that the present procedures offer a useful and new approach to closure. This will be the subject of future work.

VI. DISCUSSION

In Sec. III we have introduced a new form of mean-field equation by repeated averaging of the Navier-Stokes equation over progressively longer averaging times. In the limit of long averaging times, this mean-field equation [i.e., Eq. (3.19)] becomes equivalent to the Reynolds equation. However, the term involving the mean square of the fluctuating field (the Reynolds stress) is replaced by a constitutive equation which is linear in the mean field. This constitutive equation is specified by the recursion relations (3.20)–(3.22). These equations are quite general but the set of averaging times $\{\tau_n\}$ is determined by the need to keep the Reynolds number small in each iteration cycle n. This is a necessary condition for the solution of Eqs. (3.21) and (3.22) which is crucial for the iteration to work.

If we wish to make progress with a definite problem, we have to specialize Eqs. (3.19)–(3.22) to some tractable form. In order to make a first analysis of these equations we have assumed the following:

(1) The fluctuating field is homogeneous and isotropic.

(2) The dependence on wave number is related to the frequency dependence through the mean velocity (the Taylor hypothesis).

(3) The turbulent energy spectrum is a power law.

(4) In each cycle of iteration, the high-frequency components may be taken as quasistationary over the time scales associated with the low-frequency components.

(5) The averaging times $\{\tau_n\}$ can be related to each other through a power law.

In Sec. IV we carry out this analysis and show by numerical calculation that the recursion relation for the effective viscosity reaches a fixed point.

This analysis does not impinge on the closure problem associated with the moments of the fluctuating field. Essentially one is making progress by assuming a known form for the covariance (or spectrum) rather than attempting to solve an equation for it. This technique offers a method of attacking engineering problems in a fairly systematic fashion. And, although some of the simplifying assumptions of Sec. IV may seem less than realistic for real shear flows, we have by no means exhausted the practical (and possibly more realistic) assumptions that could be made in order to simplify Eqs. (3.19)–(3.22).

A secondary motivation for the present work was the need for a method of tackling time-dependent phenomena that are slowly varying in comparison to the turbulent cascade. We shall conclude therefore with a brief discussion of how the analysis of Sec. IV might be expanded to the problem of the intermittent generation of turbulence. In particular, we shall consider Eq. (4.17) and suppose that $n=N$, where N is large enough for us to have $\nu_N(k)\equiv\nu(k)$. We shall further suppose that U^- then contains only the long-time averaged mean and the low-frequency fluctuations of the bursting process. (In fact, for the choice $h=0.8$, the calculation in Sec. IV satisfies these conditions. In general, one would intuitively suppose, from the wide separation of the cascade and bursting time scales, that this would be the case. Nevertheless, in any particular application, this point would require further examination.)

Equation (4.17) may then be written as

$$\left[\frac{\partial}{\partial t}+\nu(k)k^2\right]U_\alpha^-(\vec{k},t)=\Pi_\alpha(\vec{k})+\sum_{\vec{j}}M_{\alpha\beta\gamma}(\vec{k})U_\beta^-(\vec{j},t)U_\gamma^-(\vec{k}-\vec{j},t)\ . \tag{6.1}$$

A further simplification results if we restrict our attention to two-dimensional mean flow (e.g., flow in a straight pipe). Under these circumstances, the term which is bilinear in the mean field is zero. Thus if we further averaged each term in (6.1) over infinite time, the equation would become

$$\left[\frac{\partial}{\partial t}+\nu(k)k^2\right]\bar{U}_\alpha(\vec{k})=\Pi_\alpha(\vec{k})\ . \tag{6.2}$$

This is just the familiar Reynolds equation with the mean-square fluctuation (the Reynolds stress) replaced by an effective viscosity $\nu(k)$. With our assumption of a steady-state pressure gradient $\Pi_\alpha(k)$, the mean velocity would be steady and determined by $\nu(k)$.

Now reverting to the short-averaging-time case τ_N, as in Eq. (6.1), we can interpret the term in U^-U^-. This contains the frequency band $0\le\omega\le\omega_N$, and goes to zero as $\omega\to 0$ (due to the vanishing of $\sum MUU$). In other words U^-U^- is just (less an additive constant due to the averaging of frequencies greater than ω_N) what experimentalists call the short-sample time autocorrelation (that is, short enough not to smooth out the coherent structure associated with the bursting process). We may summarize this by writing Eq. (6.1) in the form

$$\left[\frac{\partial}{\partial t}+\nu(k)k^2\right]U_\alpha^-(\vec{k},t)=\Pi_\alpha(\vec{k})+\tilde{\Pi}_\alpha(k,t)\ , \tag{6.3}$$

where $\tilde{\Pi}$ is just the term $\sum MU^-U^-$ and is introduced to emphasize that it may be modeled as a pressure-gradient input due to the (almost) regular low-speed fluctuations.

We know from experiment what forms $\tilde{\Pi}$ must take. In general, $\tilde{\Pi}$ will be an oscillatory function with a small random variation in period and amplitude. The mean value of the period will be T_B, the mean time between turbulent bursts. If off-diagonal components of $U_\beta^-U_\gamma^-$ are involved, the oscillation wil be stepwise between zero and some finite value, i.e., a square wave. On the other hand, if diagonal components of $U_\beta^-U_\gamma^-$ are involved, the result will be nearly sinusoidal.

Although we shall not pursue this here, it is clear that one could go on and derive equations for both the on-diagonal and off-diagonal forms of $U_\beta^-U_\gamma^-$, for frequencies less than ω_N. Of course, we should then encounter the problem of closing the moment hierarchy. And, as we have seen in Sec. V, some nontrivial problems will be involved. Nevertheless, one consequence of scaling away the high frequencies is that we are left with a much reduced problem overall. Experimental results tell us that U^- may be thought of as the sum of a mean and a near-deterministic fluctuation. At the very least, this quasideterminism should have interesting implications for the closure problem.

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