PHYSICAL REVIEW A VOLUME 26, NUMBER 2 AUGUST 1982

Invariants for forced time-dependent oscillators and generalizations

John R. Ray
Department of Physics and Astronomy, Clemson University, Clemson, South Carolina 29631

James L. Reid
Department of Physics and Astronomy, University of Georgia, Athens, Georgia 30602

(Received 29 March 1982)

Several methods for deriving invariants for time-dependent systems have been developed. Four of these methods are as follows: (1) dynamical algebra, (2) Noether's theorem, (3) transformation group, and (4) Ermakov's method. These four methods are related by their use in deriving invariants for the forced, time-dependent oscillator. For this system the methods give the same results. We also discuss forced, nonlinear oscillators possessing analogous invariants. The nonlinear superposition laws for the forced oscillator are derived and discussed.

I. INTRODUCTION

There has been a recent revival of interest in invariants (constants of the motion) for time-dependent systems. Ermakov systems[1] are one class of such systems which have been extensively studied. There are two main uses of such invariants. In classical mechanics the invariant leads to a nonlinear superposition law that allows the solution of the problem of interest to be expressed in terms of particular solutions to other auxiliary equations. A review paper[2] contains a discussion of this theory along with earlier references. In quantum mechanics the invariant allows for the exact solution to the time-dependent Schrödinger equation for certain time-dependent potentials.[3] Besides these exact solutions for certain potentials, the theory is also useful in numerical solutions of the time-independent Schrödinger eigenvalue problem for arbitrary potentials.[4,5]

There have been several methods developed for deriving invariants for time-dependent systems: there is Lutzky's form of Noether's theorem,[6] Ray and Reid's use of the Ermakov method,[1] Korsch's use of the dynamical algebra for the system,[7,8] Sarlet's differential form integrability conditions,[9,10] the Lie theory of extended groups,[11,12] and the group transformation method.[13] If this is not an exhaustive list of methods, it is sufficient for our purposes.

In this paper we discuss the forced time-dependent harmonic oscillator using some of these different approaches. We also show how to enlarge the class of systems and still obtain invariants.

Our starting point is the system

$$\ddot{y}+h(t)\dot{y}+\bar{\omega}^2(t)y=F(t)\,, \tag{1.1}$$

where $h(t)$, $\bar{\omega}^2(t)$, and $F(t)$ are arbitrary functions. By use of the well-known transformation

$$x=y\exp\left[\tfrac{1}{2}\int^{(t)} h\,dt\right]$$

we can eliminate $\dot{y}$ from (1.1) to obtain

$$\ddot{x}+\omega^2(t)x=f(t)\,, \tag{1.2}$$

where the transformed frequency and driving term are given by

$$\omega^2(t)=\bar{\omega}^2(t)-\tfrac{1}{2}\dot{h}-\tfrac{1}{4}h^2\,,$$

$$f(t)=F(t)\exp\left[\tfrac{1}{2}\int^{(t)} h\,dt\right],$$

respectively.

Recently, Takayama[14] derived an Ermakov-type invariant for this system using the dynamical-algebra approach of Korsch. This same problem has been treated by Khandekar and Lawande[15] and Ray,[13] primarily in a quantum context. First we review the results of Takayama.

II. DYNAMICAL ALGEBRA

The method of the dynamical algebra is clearly explained and applied in Refs. 7 and 8. Takayama applies this approach to (1.2) and obtains the invariant

$$I=\tfrac{1}{2}(C\dot{x}-\dot{C}x)^2+\frac{k}{2}(x/C)^2 + \int^{(t)}\psi(\lambda)f(\lambda)d\lambda-\psi(t)\dot{x}+\dot{\psi}x-C^2xf \ , \tag{2.1a}$$

where $C(t)$ and $\psi(t)$ are auxiliary functions satisfying the equations

$$\ddot{C}+\omega^2(t)C=k/C^3 \ , \tag{2.1b}$$

$$\ddot{\psi}+\omega^2(t)\psi=C^2\dot{f}+3C\dot{C}f \ , \tag{2.1c}$$

where k is an arbitrary constant. I in (2.1a) is an invariant for the system (1.2) for C any solution to (2.1b) and ψ any solution to (2.1c). Note that we have made certain changes in notation from Takayama[14] for later simplicity. Next we shall present these results using Lutzky's version of Noether's theorem.

III. NOETHER'S THEOREM

Here we may use the results presented in the paper by Ray and Reid[16] on Noether's theorem and time-dependent invariants. Equation (2.9) of Ref. 15 is the requirement for (1.2) to allow a Noether symmetry, with ρ of Ref. 15 replaced by x, $G(t)$ replaced by $f(t)$, and $F(x)$ by x. Equation (2.9) of Ref. 15 yields

$$(-\omega\dot{\omega}\xi-\omega^2\dot{\xi}-\tfrac{1}{4}\dddot{\xi})x^2-[\ddot{\psi}+\omega^2(t)\psi]x + (\xi\dot{f}+\dot{\xi}f)x+\tfrac{1}{2}fx\dot{\xi}+f\psi-\dot{\chi}=0 \ , \tag{3.1}$$

where $\xi(t)$, $\psi(t)$, and $\chi(t)$ are at this stage arbitrary functions. Equation (3.1) must be identically satisfied in order for (1.2) to have a Noether symmetry. This fact requires that the separate powers of x in (3.1) must vanish. The x^2 terms yield

$$\dddot{\xi}+4\omega\dot{\omega}\xi+4\omega^2\dot{\xi}=0 \tag{3.2}$$

or

$$\ddot{C}+\omega^2(t)C=k/C^3 \ , \tag{3.3}$$

with $\xi=C^2(t)$ and k an arbitrary constant of integration. The linear terms in x in (3.1) give an equation for ψ,

$$\ddot{\psi}+\omega^2(t)\psi=C^2\dot{f}+3C\dot{C}f \ . \tag{3.4}$$

The remaining terms of (3.1) produce

$$\dot{\chi}=\psi(t)f(t)$$

or

$$\chi=\int^{(t)}\psi(\lambda)f(\lambda)d\lambda \ . \tag{3.5}$$

Finally, the Noether invariant calculated using Ref. 15, has the form

$$I=\tfrac{1}{2}(C\dot{x}-x\dot{C})^2+\frac{k}{2}(x/C)^2 + \int^{(t)}\psi(\lambda)f(\lambda)d\lambda-\psi\dot{x}+\dot{\psi}x-C^2xf \ . \tag{3.6}$$

The results (3.3), (3.4), and (3.6) are exactly the same as the results obtained by Takayama using Korsch's method. The symmetry operator for the Noether symmetry has the form

$$X=C^2\frac{\partial}{\partial t}+(C\dot{C}x+\psi)\frac{\partial}{\partial x} \ . \tag{3.7}$$

Thus, the first integral (3.6) is associated with the five-parameter Lie group defined by the group operators (3.7) (X depends on five parameters k, two integration constants for C, and two integration constants for ψ).

Thus, we see that Noether's theorem reproduces the same results as the dynamical-algebra approach for (1.2). These same results can also be obtained from Leach's treatment of symmetries using Noether's theorem.[17] These results of Leach are discussed in detail by Reid and Ray.[12]

IV. TRANSFORMATION-GROUP METHOD

This technique was applied for finding invariants by Ray[13] in his study of the quantum problem for (1.2). The method uses the transformation-group techniques introduced by Burgan *et al.*[18,19]

We transform the equation (1.2) via the transformation

$$x'=x/C(t)+A(t) \ , \tag{4.1a}$$

$$t'=D(t) \ , \tag{4.1b}$$

where $C(t)$, $A(t)$, and $D(t)$ are at this stage arbitrary functions. Under this transformation (1.2) takes the form

$$C\dot{D}^2\frac{d^2x'}{dt'^2}+(2\dot{C}\dot{D}+C\ddot{D})\frac{dx'}{dt'}+[\ddot{C}+\omega^2(t)C]x' + [-\ddot{C}A-2\dot{C}\dot{A}-\omega^2(t)CA-C\ddot{A}-f]=0 \ . \tag{4.2}$$

Following Burgan *et al.* we demand (1.2) be form invariant under the transformation (4.1). Thus the coefficient of dx'/dt' in (4.2) must vanish which yields

$$\dot{D}=\frac{dt'}{dt}=\frac{1}{C^2}\ . \tag{4.3}$$

Equation (4.2) becomes

$$\frac{d^2x'}{dt'^2}+C^3(\ddot{C}+\omega^2(t)C)x' +C^3(-\ddot{C}A-2\dot{C}\dot{A}-\omega^2(t)CA-C\ddot{A}-f)\ . \tag{4.4}$$

Up to this point $C(t)$ and $A(t)$ have been arbitrary, now we choose them to simplify (4.4). We make the choices

$$\ddot{C}+\omega^2(t)C=k/C^3\ , \tag{4.5}$$

$$\ddot{A}+kA/C^4+2\dot{C}\dot{A}/C+f/C=0\ , \tag{4.6}$$

which reduces (4.4) to the simple harmonic-oscillator equation

$$\frac{d^2x'}{dt'^2}+kx'=0\ , \tag{4.7}$$

which is the autonomous equation associated with (1.2). The energy integral for (4.7) has the form

$$I=\frac{1}{2}\left[\frac{dx'}{dt'}\right]^2+\frac{k}{2}x'^2\ . \tag{4.8}$$

By carrying out the inverse transformations, we can write the invariant I in the form

$$I=\tfrac{1}{2}(C\dot{x}-\dot{C}x+C^2\dot{A})^2+\frac{k}{2}(x/C+A)^2\ . \tag{4.9}$$

Thus, I is an invariant for (1.2) if C is any solution to (4.5) and A is any solution to (4.6). We shall show in Sec. V that these results are equivalent to the results obtained in Secs. II and III. Leach[20] has also employed transformations such as (4.1) to find invariants by this method of reduction to autonomous systems.

V. DYNAMICAL ALGEBRA, NOETHER, AND GROUP-METHOD EQUIVALENCE

The group methods of Sec. IV led to the invariant (4.9) for (1.2), where C and A satisfy (4.5) and (4.6), respectively. If we define ψ by

$$\psi=-C^3\dot{A}\ , \tag{5.1}$$

then calculating $\ddot{\psi}$ and forming $\ddot{\psi}+\omega^2(t)\psi$ yields, using (4.5) and (4.6),

$$\ddot{\psi}+\omega^2(t)\psi=C^2\dot{f}+3C\dot{C}f\ , \tag{5.2}$$

or the equation for ψ found in (2.1c) and (3.4). Thus, the equation (5.1) relates the dynamical-algebra–Noether theorem variable $\psi(t)$ to the group variable $A(t)$. The invariant I in (4.9) also is converted into the form (2.1a) or (3.6) under the transformation from A to ψ (5.1). Writing out (4.9) we have

$$I=\tfrac{1}{2}(C\dot{x}-x\dot{C})^2+\frac{k}{2}(x/C)^2+C^3\dot{A}\dot{x}-C^2\dot{C}\dot{A}x +kAx/C+kA^2/2+C^4\dot{A}^2/2\ . \tag{5.3}$$

With (4.6) we can derive

$$C^3\dot{A}f=-\frac{d}{dt}\left[\frac{1}{2}C^4\dot{A}^2+\frac{k}{2}A^2\right]\ . \tag{5.4}$$

Using this latter result along with (5.1) in the invariant (5.3) we obtain

$$I=\tfrac{1}{2}(C\dot{x}-x\dot{C})^2+\frac{k}{2}(x/C)^2-\psi\dot{x} +(-C^2\dot{C}\dot{A}+kA/C+fC^2)x-fC^2x -\int^{(t)}C^3\dot{A}f\,d\lambda\ , \tag{5.5}$$

where we have also added and subtracted the term fC^2x. From (5.1) and (4.6) follows

$$\dot{\psi}=-C^2\dot{C}\dot{A}+kA/C+fC^2\ , \tag{5.6}$$

which when used in (5.5) gives

$$I=\tfrac{1}{2}(C\dot{x}-x\dot{C})^2+\frac{k}{2}(x/C)^2-\psi\dot{x}+\dot{\psi}x-fC^2x +\int^{(t)}\psi(\lambda)f(\lambda)d\lambda\ , \tag{5.7}$$

which is exactly the invariant (2.1a) or (3.6). Thus, starting from the equations of motion for C and A we find the equation of motion for ψ, defined in (5.1), is exactly the same as (2.1c) and (3.4). Also the invariant found by the group method is the same as (2.1a) or (3.6). These results prove that the group method used in Sec. IV contains the results derived by the dynamical-algebra approach in Sec. II and from Noether's theorem in Sec. III. Therefore, these three methods lead to equivalent results when applied to the forced harmonic-oscillator equation (1.2) In Sec. VI we turn to our final method of deriving these results, the Ermakov method.

VI. ERMAKOV'S METHOD

The Ermakov method is more heuristic than the previous three methods of arriving at invariants, but it can sometimes lead to more general systems possessing invariants. This is true mainly because this method is so simple that one can often guess more general systems to which the method can be applied.

Our starting point for the Ermakov method is the set of equations

$$\ddot{x}+\omega^2(t)x=f(t)\ , \tag{6.1}$$

$$\ddot{C}+\omega^2(t)C=k/C^3\ , \tag{6.2}$$

$$\ddot{\psi}+\omega^2(t)\psi=C^2\dot{f}+3C\dot{C}f\ . \tag{6.3}$$

Here (6.1) is the primary equation and (6.2) and (6.3) are auxiliary equations. First we eliminate the frequency $\omega^2(t)$ between (6.1) and (6.2) and multiply the resulting equation by $C\dot{x}-x\dot{C}$ which gives

$$\frac{1}{2}\frac{d}{dt}(C\dot{x}-x\dot{C})^2+\frac{k}{2}\frac{d}{dt}\left[\frac{x}{C}\right]^2=(C\dot{x}-x\dot{C})Cf\ . \tag{6.4}$$

Next we eliminate the frequency $\omega^2(t)$ between (6.2) and (6.3) and rearrange the resulting equation to obtain

$$(C\dot{x}-x\dot{C})Cf=\frac{d}{dt}(C^2fx)+\frac{d}{dt}(\psi\dot{x}-x\dot{\psi})-\dot{\psi}f\ . \tag{6.5}$$

Employing this latter result in (6.4) results in the invariant

$$I=\tfrac{1}{2}(C\dot{x}-x\dot{C})^2+\frac{k}{2}(x/C)^2-C^2fx+x\dot{\psi}-\psi\dot{x}+\int^{(t)}\dot{\psi}(\lambda)f(\lambda)d\lambda\ , \tag{6.6}$$

which is exactly the same as found by the other three methods. Note that in the original Ermakov method[1] we had only one auxiliary equation and, therefore, only one elimination of frequency was necessary. Here, with two arbitrary functions $\omega^2(t)$, $f(t)$ in the original equation (1.2) we need two auxiliary equations C and A in order to arrive at the invariant. This example illustrates the weakness of the Ermakov method as compared to the three earlier methods since here the auxiliary equations must be known *a priori*, whereas in the other methods they are derived in a systematic manner.

On the other hand, it is easy to see how to generalize to more general nonlinear equations and still obtain invariants using the Ermakov method. For example, the equation

$$\ddot{x}+\omega^2(t)x=f(t)+\frac{1}{x^2C}g(C/x)\ , \tag{6.7}$$

where g is an arbitrary function, possesses the invariant

$$I=\tfrac{1}{2}(C\dot{x}-x\dot{C})^2+\frac{k}{2}(x/C)^2-C^2fx+x\dot{\psi}-\psi\dot{x}+\int^{(t)}\dot{\psi}(\lambda)f(\lambda)d\lambda+\int^{(C/x)}g(\lambda)d\lambda\ . \tag{6.8}$$

As an explicit case we take $g=k'C/x$, and obtain the invariant

$$I=\tfrac{1}{2}(C\dot{x}-x\dot{C})^2+\frac{k}{2}(x/C)^2+\frac{k'}{2}(C/x)^2-C^2fx+x\dot{\psi}-\psi\dot{x}+\int^{(t)}\dot{\psi}(\lambda)f(\lambda)d\lambda\ , \tag{6.9}$$

for the driven Pinney equation

$$\ddot{x}+\omega^2(t)x=f(t)+k'/x^3\ , \tag{6.10}$$

where k' is an arbitrary constant.

Other generalizations involving, for example, velocity-dependent forces in the x equation can also be obtained. However, we shall not discuss these examples further here.[21] It is interesting that the group method will work for the equation of motion

$$\ddot{x}+\omega^2(t)x=f(t)+\frac{1}{C^3}m(x/C+A)\ , \tag{6.11}$$

where m is an arbitrary function. Under the transformations (4.1) this equation reduces to the autonomous equation

$$\frac{d^2x'}{dt'^2}+kx'=m(x')\ . \tag{6.12}$$

From this equation we can derive the energy integral

$$I=\frac{1}{2}\left[\frac{dx'}{dt'}\right]^2+\tfrac{1}{2}kx'^2-\int^{(x')}m(\lambda)d\lambda\ . \tag{6.13}$$

Applying the inverse transformations to (6.13) we obtain a time-dependent invariant for (6.11). The system (6.11) can also be treated using the dynamical-algebra or the Noether theorem approach.[22] This fact means that the several methods that we have discussed, and whose equivalence we proved in Sec. V, are all valid for a larger class of systems than just the forced oscillator. In Sec. VII we discuss the idea of nonlinear superposition as applied to the equation (1.2).

VII. NONLINEAR SUPERPOSITION

Following previous usage,[22,23] a solution to the x equation, written in terms of particular solutions to the auxiliary equations A and C, is called a nonlinear superposition law. Here we investigate this idea for (1.2). We use the notation introduced in Sec. IV.

The autonomous equation (4.7) has the solution

$$x' = a\sin(\sqrt{k}\,t' + \delta)\ , \tag{7.1}$$

where a and δ are integration constants. Inverting the transformation (4.1a) we obtain the solution to the x equation in the form

$$x = C_p[a\sin(\sqrt{k}\,t' + \delta) - A_p]\ , \tag{7.2}$$

where C_p and A_p are any particular solutions to (4.5) and (4.6). In (7.2) t' is calculated from

$$t' = \int \frac{dt}{C_p^2}\ . \tag{7.3}$$

The advantage of the nonlinear superposition law (7.2) is that, for all initial conditions that can be secured by a and δ in (7.1), it gives the solution to the equation (1.2) in terms of particular solutions C_p and A_p. Thus, if we tabulate C_p, A_p, and t', then (7.2) gives the general solution to (1.2) without further numerical integrations.

In the classical theory such nonlinear superposition laws are interesting but perhaps not of much practical use. However, the mathematical structure that leads to nonlinear superposition in the classical theory carries over and is useful in quantum theory. This structure allows the exact solution to certain time-dependent Schrödinger equations[3,13] as well as the efficient numerical solution to the time-independent Schrödinger eigenvalue equation.[4,5]

VIII. CONCLUSIONS

In this paper we have related four different methods for deriving invariants for time-dependent systems by concentrating on the forced time-dependent harmonic oscillator. The three basic methods: (i) Dynamical algebra, (ii) Noether's theorem, and (iii) group method, lead to equivalent results for this problem. Given the auxiliary equations, Ermakov's method of eliminating the frequency gives the same results as these three methods. As we were able to show in Sec. VI, these methods can be generalized in various ways to give invariants for different forced nonlinear systems.

The quantum theory of the forced, time-dependent harmonic oscillator was given in Ref. 13 using the group method. The exact solution to the Schrödinger equation for this problem is analogous to the nonlinear superposition laws discussed in Sec. VII. This point is also made in Ref. 3. The extension of these and related techniques to (1) more general time-dependent equations, (2) higher-order differential equations,[24] and (3) systems of differential equations[25,26] is an important task for future work.

[1] J. R. Ray and J. L. Reid, Phys. Lett. 71A, 317 (1979).
[2] J. R. Ray, Prog. Theor. Phys. 65, 877 (1981).
[3] J. G. Hartley and J. R. Ray, Phys. Rev. A 24, 2873 (1981).
[4] H. J. Korsch and H. Laurent, J. Phys. B 44, 4213 (1981).
[5] H. J. Korsch, H. Laurent, and R. Möhlenkamp, J. Phys. B 15, 1 (1982).
[6] M. Lutzky, Phys. Lett. 68A, 3 (1978).
[7] H. J. Korsch, Phys. Lett. 74A, 317 (1980).
[8] R. S. Kaushal and H. J. Korsch, J. Math. Phys. 22, 1904 (1981).
[9] W. Sarlet, Phys. Lett. 82A, 161 (1981).
[10] W. Sarlet and J. R. Ray, J. Math. Phys. 22, 2504 (1981).
[11] P. G. L. Leach, J. Math. Phys. 22, 465 (1981).
[12] J. L. Reid and J. R. Ray, Phys. Rev. A (in press).
[13] J. R. Ray, Phys. Rev. A 26, 729 (1982).
[14] K. Takayama, Phys. Lett. 88A, 57 (1982).
[15] D. C. Khandekar and S. V. Lawande, J. Math. Phys. 20, 1870 (1979).
[16] J. R. Ray and J. L. Reid, J. Math. Phys. 20, 2054 (1979).
[17] P. G. L. Leach, Phys. Lett. 84A, 161 (1981).
[18] J. R. Burgan, J. Gutierrez, A. Munier, E. Fijalkow, and M. R. Feix, in *Strongly Coupled Plasmas,* edited by G. Kalman and P. Carini (Plenum, New York, 1978).
[19] J. R. Burgan, M. R. Feix, E. Fijalkow, and A. Munier, Phys. Lett. 74A, 11 (1979).
[20] P. G. L. Leach, J. Math. Phys. 22, 468 (1981).
[21] J. R. Ray and J. L. Reid, J. Math. Phys. 22, 91 (1981).

[22]J. L. Reid and J. R. Ray, J. Math. Phys. 21, 1583 (1981).
[23]J. R. Ray, Phys. Lett. 78A, 4 (1980).
[24]J. L. Reid and J. R. Ray, J. Math. Phys. 23, 503 (1982).
[25]J. R. Ray and J. L. Reid, Phys. Lett. 74A, 23 (1979).
[26]J. R. Ray and J. G. Hartley, Phys. Lett. 88A, 125 (1982).