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# Quantum theory of stimulated Čerenkov radiation

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Basic features of stimulated Čerenkov radiation are investigated by means of two different quantum-mechanical approaches: (a) a solution of the Klein-Gordon equation in the presence of a plane wave in a refractive medium, and (b) a classical current interacting with a quantized radiation field, with recoil corrections introduced by hand. Both approaches lead to identical results. The electron energy spread is obtained in perfect agreement with earlier classical calculations which have been indirectly verified by a recent experiment. The relation between gain and spread turns out to be more favorable than for the conventional free-electron laser.

## I. INTRODUCTION

With the experimental realization of a free-electron laser,[1] interest in other free-electron devices has also increased. In order that an electron can exchange energy with a laser field, the presence of a third agent is necessary, which is the wiggler field in case of the free-electron laser. The very simplest case is when this third agent is merely a homogeneous medium represented by a refractive index $n(\omega)$. If an externally applied laser beam is nearly tuned to the spontaneous Čerenkov radiation, we can expect the same phenomena as in the case of the free-electron laser: the electron energy suffers a comparatively large spread which is almost symmetric with respect to the initial electron energy. The amount of detuning of the laser beam versus spontaneous radiation, determines whether positive or negative gain occurs, which leads to bunching of the electron beam at the optical wavelength. If started from spontaneous emission, the device could also be run as an oscillator. Theoretical and experimental aspects of Čerenkov amplifiers and lasers are extensively dealt with in the review papers (Ref. 2), which also refer to the earlier literature.

In this paper we will derive the basic features of stimulated Čerenkov radiation via two different quantum-mechanical approaches which have been applied earlier to the free-electron laser. Although all essential results turn out to be classical, the derivation by relativistic quantum mechanics is by no means more complicated than a classical approach and enhances understanding of the phenomena by presenting them from a very different viewpoint. Some results, e.g., the relation between gain and spread, even seem to be more easily accessible via quantum mechanics. We use a highly idealized model, a monochromatic electron beam interacting with a monochromatic plane-wave field for a finite time $T$, which is applicable in the small-signal cold-beam noncollective regime. Under these conditions a consequence of stimulated Čerenkov radiation, namely, the induced spread in the electron energy distribution, has recently been observed experimentally.[3] Our simple model describes this experiment satisfactorily, showing that detailed features such as the exact geometry of the laser beam and the electron distribution, only play a minor role. An earlier quantum approach to the Čerenkov laser[4] deals with a different regime (very long interaction time, warm beam). Hence it does not exhibit the electron energy spread, which is characteristic of the cold-beam regime.

In the second section we employ the Klein-Gordon equation in the presence of a monochromatic plane-wave field propagating at some angle with respect to the electron through a refractive medium to describe spread and gain of a Čerenkov amplifier. In the third section we apply a different approach, a classical electron current interacting with a quantized radiation field, with the quantum kinematics reintroduced afterwards by hand. The second approach, albeit less rigorous in principle, also includes spontaneous emission. Both have been successfully applied to the conventional free-electron laser, cf. Ref. 5 and Refs. 6 and 7, respectively.

 

## II. SEMICLASSICAL FORMULATION

In this approach we consider a relativistic, quantum-mechanical electron interacting with a classical electromagnetic field in the presence of a medium described by an index of refraction $n \geq 1$. The interaction is illustrated in Fig. 1. An electron with momentum $p$ and energy $\epsilon$ propagates along the $z$ axis, while an electromagnetic wave propagates in the $y$-$z$ plane at an angle $\theta$ to the electron beam. The electromagnetic wave is assumed to be plane polarized in the $y$-$z$ plane. The interaction is turned on at $t=0$ and endures for a time $T=L/v$, where $v$ is the electron velocity and $L$ is the interaction length.

The external electromagnetic field in the presence of the medium is given by the vector potential,

$$\vec{A}=A(\sin\theta\hat{z}-\cos\theta\hat{y})\cos(\vec{k}\cdot\vec{r}-\omega t)\ , \tag{2.1}$$

where $A$ is the amplitude of the wave and $\theta$ is the angle between the direction of propagation of the electron and wave (see Fig. 1). The only effect of the medium in the problem is to modify the free-space dispersion relation

$$k=|\vec{k}|=n\frac{\omega}{c}\ . \tag{2.2}$$

Because spin effects can be shown to be unimportant,[5] the motion of the electron is determined by the solution of the Klein-Gordon equation with the external field (2.1). After some simplifications this equation can be written as

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial y^2}+2i\frac{eA}{\hbar}\sin\theta\cos(\vec{k}\cdot\vec{r}-\omega t)\frac{\partial}{\partial z}+\frac{m^2c^2}{\hbar^2}\right]\psi=0\ , \tag{2.3}$$

where $e$ and $m$ are the charge and mass of the electron. In particular, we have made use of the geometry illustrated in Fig. 1 and the important inequality

$$\epsilon=mc^2\gamma>>mc^2>>\hbar\omega\ . \tag{2.4}$$

Also, the small coupling between the electron and the radiation field due to the $|\vec{A}|^2$ term has been dropped.

The expansion of the electron wave function in terms of plane waves

$$\psi=\sum_{\vec{p}}C(\vec{p},t)\exp\{i[\vec{p}\cdot\vec{r}-\epsilon(\vec{p})t]/\hbar\}\ , \tag{2.5}$$

reduces Eq. (2.3) to the following system of equations for the coefficients $C(\vec{p},t)$:

$$\left[\frac{d^2}{dt^2}-2i\frac{\epsilon(\vec{p})}{\hbar}\frac{d}{dt}\right]C(\vec{p},t)-\frac{ecpA}{\hbar^2}\sin\theta e^{i\epsilon(\vec{p})t/\hbar}(C(\vec{p}-\vec{k},t)\exp\{-i[\epsilon(\vec{p}-\vec{k})+\omega]t/\hbar\}$$
$$+C(\vec{p}+\vec{k},t)\exp\{-i[\epsilon(\vec{p}+\vec{k})-\omega]t/\hbar\})=0\ , \tag{2.6}$$

where $p=|\vec{p}|$ and $\epsilon(\vec{p})=(p^2c^2+m^2c^4)^{1/2}$. By comparing the second and third terms in Eq. (2.6), an estimate can be made for the characteristic time scale $\tau$ for the variation of $C(\vec{p},t)$:

$$\tau\simeq\hbar\epsilon(\vec{p})/(eApc\sin\theta)\ .$$

From this result, the relative size of the first two terms in Eq. (2.6) can be determined:

$$\frac{d^2}{dt^2}C(\vec{p},t)\Big/\left[\epsilon(\vec{p})\frac{d}{dt}C(\vec{p},t)\right]\sim[\epsilon(\vec{p})\tau]^{-1}=\frac{eAp\sin\theta}{\epsilon^2(\vec{p})}\ ,$$

which is equal to $10^{-9}$ for the conditions of the experiment of Ref. 3. We can therefore neglect the second derivatives of the coefficients $C(\vec{p},\epsilon)$ with respect to time. With this approximation and the following definitions:

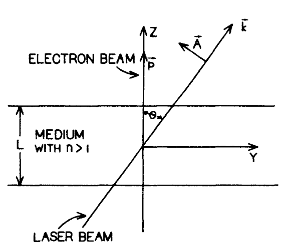


FIG. 1. Geometry for stimulated Čerenkov interaction.

$$\vec{p}_l \equiv \vec{p} + l\hbar\vec{k}, \quad \epsilon_l \equiv \epsilon(\vec{p}_l) = (p_l^2 c^2 + m^2 c^4)^{1/2},$$
$$C(\vec{p}_l, t) = a_l(t)\exp[i(\epsilon_l - l\hbar\omega - \epsilon)t/\hbar],$$
$$l = 0, \pm 1, \pm 2, \ldots \quad (2.7)$$

the system of Eqs. (2.6) can then be rewritten as

$$i\hbar\dot{a}_l - (\epsilon_l - l\hbar\omega - \epsilon)a_l = -\frac{ep_l cA\sin\theta}{2\epsilon_l}(a_{l+1} + a_{l-1}), \quad (2.8)$$

with the boundary condition

$$a_l(t=0) = \delta_{l,0}. \quad (2.9)$$

The above set of equations depends on $l$ in a complicated manner through the energy $\epsilon_l$. In order to further simplify the set (2.8), we use the inequality (2.4) to expand $\epsilon_l$ about its initial value $\epsilon$ as

$$\epsilon_l = \epsilon\left[1 + \frac{2l\hbar kc\beta\cos\theta}{\epsilon} + \left[\frac{l\hbar kc}{\epsilon}\right]^2\right]^{1/2}$$
$$\simeq \epsilon + l\hbar kc\beta\cos\theta + \frac{1}{2}\frac{(l\hbar kc)^2}{\epsilon}(1 - \beta^2\cos^2\theta), \quad (2.10)$$

where $\beta = pc/\epsilon$. With the substitution of this expansion, the set of Eqs. (2.8) takes the form

$$i\dot{a}_l - \left[l\omega(n\beta\cos\theta - 1) + \frac{l^2}{2}\frac{\hbar(kc)^2}{\epsilon}(1 - \beta^2\cos^2\theta)\right]a_l = -\frac{eA\beta\sin\theta}{2\hbar}(a_{l+1} + a_{l-1}). \quad (2.11)$$

The expression has been derived from Eq. (2.8) by neglecting terms of order $l^3$ and higher and by replacing $p_l$ and $\epsilon_l$ on the right-hand side of Eq. (2.8) with their initial values. Explicit calculations of the corrections due to using more exact expressions for these terms and retaining higher powers of $l$ show that they are small with respect to the retained terms.

At this point it is worth identifying the principal parameters that characterize the system. These parameters are the interaction energy

$$\epsilon_i = \hbar\omega_i = \frac{e\beta A\sin\theta}{2},$$

the anharmonicity energy

$$\epsilon_a = \hbar\omega_a = \frac{\hbar^2 k^2 c^2}{2\epsilon}(1 - \beta^2\cos^2\theta),$$

the frequency detuning from resonance

$$\Delta = \omega(n\beta\cos\theta - 1),$$

and the interaction time $T$.

In order to solve for the wave function of the electron, i.e., $a_n$'s, we will consider the quasienergy (stationary) solution of Eq. (2.11): $a_n(t) = b_n\exp(-i\gamma t)$, where $\hbar\gamma$ is the quasienergy and the $b_n$ are constant coefficients that satisfy

$$\gamma b_l - (\Delta l + \omega_a l^2)b_l = \omega_i(b_{l+1} + b_{l-1}). \quad (2.12)$$

If Eq. (2.12) is multiplied by $\exp(ilx)$, $0 < x < 2\pi$, and the resulting expression summed over $l$, Eq. (2.12) changes from a recurrence relation to the following second-order differential equation in $x$:

$$\left[\omega_a\frac{d^2}{dx^2} + i\Delta\frac{d}{dx} - 2\omega_i\cos x + \gamma\right]\psi = 0, \quad (2.13)$$

where

$$\psi = \sum_l b_l\exp(ilx). \quad (2.14)$$

Equation (2.13) is simply Mathieu's equation and can therefore be solved in terms of Mathieu functions. Unfortunately, a formal solution is not particularly useful. Instead of pursuing this exact solution we will follow the procedure used to investigate the free-electron laser[5] and develop an approximate solution of Eq. (2.13) that is more useful than the exact solution. In order to apply this approximate method it is necessary to determine the relative sizes of the various terms of Eq. (2.12) or (2.13).

In order to roughly estimate the maximum size of $l$ in Eq. (2.11) or (2.12), we investigate the steady-state equation derived from Eq. (2.11) by setting $\dot{a}_l = 0$. If, in turn, $n\beta\cos\theta = 1$ and we assume $a_l \sim a_{l-1}$ and $a_{l+1} \simeq 0$ (all levels with $l \le l_{\max}$ filled), then $r \equiv l_{\max} \simeq (|\omega_i|/\omega_a)^{1/2}$.

On the other hand, Eq. (2.12) shows that the time to excite the levels $l = \pm 1$ is approximately $|\omega_i|^{-1}$, so that during an interaction time $T$, $l = T\omega_i$ levels will be excited. If $T\omega_i << r$, then the influence of the anharmonic term is small because $l^2\omega_a \simeq (l/r)^2\omega_i << \omega_i$. This means that the anharmonic term in Eq. (2.12) and the term $d^2\psi/dx^2$ in

Eq. (2.13) can be treated as a perturbation as long as $(\omega_i\omega_a)^{1/2}T << 1$. When the second derivative is treated as a perturbation, the eigensolutions and eigenfrequencies of Eq. (2.13) to first order are given by

$$\gamma_n = n\Delta + n^2\omega_a \ , \tag{2.15}$$

$$\psi_n = \frac{1}{\sqrt{2\pi}}\left[1+\frac{2\omega_i\omega_a}{\Delta^2}[(n+\tfrac{1}{2})e^{ix}-(n-\tfrac{1}{2})e^{-ix}]\right] \times\exp\left[i\left[nx-\frac{2\omega_i}{\Delta}\sin x\right]\right] . \tag{2.16}$$

Now that the quasienergy solutions are known, the time-dependent solution can be determined. In the same manner by which Eq. (2.12) was transformed into Eq. (2.13), Eq. (2.11) can be transformed into the differential equation

$$-\frac{i\partial\psi(x,t)}{\partial t} = \left[\omega_a\frac{\partial^2}{\partial x^2}+i\Delta\frac{\partial}{\partial x} - 2\omega_i\cos x\right]\psi(x,t) \ , \tag{2.17}$$

where

$$\psi(x,t)=\sum_n a_n(t)e^{inx}, \quad 0<x<2\pi \ . \tag{2.18}$$

The condition on $\psi(x,t)$ at $t=0$ is determined from Eq. (2.9) to be

$$\psi(x,0)=1 \ . \tag{2.19}$$

It is also possible to expand the function $\psi(x,t)$ in terms of the quasienergy function (2.15) and (2.16) as

$$\psi(x,t)=\sum_m c_m\psi_m(x)e^{i\gamma_m t} \ , \tag{2.20}$$

so that

$$a_n(t)=\sum_m c_m e^{i\gamma_m t}\int_0^{2\pi} e^{-inx}\psi_m(x)\,dx \ , \tag{2.21}$$

with

$$c_m=\int_0^{2\pi}\psi_m^*(x)\,dx \ . \tag{2.22}$$

Equations (2.15), (2.16), (2.21), and (2.22) determine the wave function of the electron subject to the condition $T(|\omega_i|\omega_a)^{1/2} << 1$.

We are now in a position where the small-signal gain and the spread of the electron momentum can be calculated. The energy gained or lost by the electron is given by

$$\Delta\epsilon \simeq \hbar\omega\sum_n n\,|a_n|^2 = \frac{\omega}{2\pi}\sum_n\left\{\left[n-\frac{4\omega_a\omega_i^2}{\Delta^3}\right]d_n^2-\frac{\omega_i}{\Delta}d_n d_{n+1}\left[\left[1+\frac{\omega_a}{\Delta}(2n+1)\right]e^{i(\gamma_n-\gamma_{n+1})t} + \left[1+\frac{\omega_a}{\Delta}(2n-1)\right]e^{-i(\gamma_n-\gamma_{n+1})t}\right]\right\} , \tag{2.23}$$

where

$$d_n=\sqrt{2\pi}\left\{J_n\left[\frac{2\omega_i}{\Delta}\right]+\frac{2\omega_i\omega_a}{\Delta^2}\left[(n+\tfrac{1}{2})J_{n+1}\left[\frac{2\omega_i}{\Delta}\right]+(n-\tfrac{1}{2})J_{n-1}\left[\frac{2\omega_i}{\Delta}\right]\right]\right\} . \tag{2.24}$$

The gain experienced by the radiation field is related to the energy lost by the electron through

$$G=-\frac{8\pi}{k^2A^2}\rho\Delta\epsilon \ , \tag{2.25}$$

where $\rho$ is the electron density. Although tedious, the sums in Eq. (2.24) can be performed with the aid of the recurrence relations for Bessel functions.

The resulting expression for the small-signal gain is given by

$$G=\frac{\pi\rho\omega e^2\beta^2(n^2-1)}{m\gamma n^2}T^3\sin^2\theta\frac{d}{du}\left[\frac{\sin u}{u}\right]^2 , \tag{2.26}$$

where

$$u=\frac{\omega T}{2}(1-n\beta\cos\theta) \ . \tag{2.27}$$

In a similar manner the energy spread for the small-signal region can be found to be

$$\Delta E = \hbar\omega \left[ \sum_n n^2 |a_n|^2 \right]^{1/2} = \frac{e\beta A\omega T}{\sqrt{2}} \frac{\sin u}{u} . \tag{2.28}$$

Here $u$ is again given by Eq. (2.27). Equations (2.26) and (2.28) are valid only as long as

$$\mu = T(|\omega_i|\omega_a)^{1/2} << 1 . \tag{2.29}$$

If numbers that are typical of the experiment[3] are substituted in Eq. (2.29), then $\mu \simeq 1$. This would seem to indicate that the small-signal-gain formula is not applicable for the present experiment. The same problem occurs with the free-electron laser where, according to a relation[5] similar to Eq. (2.29), the corresponding small-signal-gain expression should not be applicable to the conditions of the original experiment.[1] Yet, agreement between the original experiment and the predictions of the small-signal gain are rather good.[5] It is therefore likely that Eq. (2.26) will give reasonable predictions for the gain under present experimental conditions.

Although the expression for the energy spread (2.28) agrees well with a previously published result (see Ref. 3 and the discussion of this paper) the formula for the small-signal gain (2.26) differs from the result published by Gover *et al.*[12] Our expression is larger by the not insignificant factor $(n^2-1)\beta^2\gamma^2$. In view of this important difference, the expressions for the small-signal gain and the energy spread are rederived in Sec. III using a different method.

## III. MULTIPHOTON FORMALISM

In this section we shall rederive the results of Sec. II by a multiphoton formalism which had been earlier developed in the context of the free-electron laser.[6,7] In this approach the problem of a quantized electron interacting with a quantized radiation field in the presence of a medium is made tractable by replacing, to lowest order, the current operator of the electron by its classical current, while retaining the quantized radiation field. This lowest-order approximation is then exactly soluble. In the process of replacing the current operator by the classical current, however, the quantum recoil has been lost. Hence the lowest order does not yield gain. We then investigate energy-momentum conservation for the case in which the electron emits or absorbs a definite number of photons into or out of the radiation field. This can be used to introduce the quantum recoil by hand into the classical current results, which then allows for calculation the gain. For a detailed description of the method we refer to Ref. 7. Recently, it has proven in the case of the free-electron laser,[8] by means of a rigorous expansion of the fully quantized transition amplitudes to first order in the recoil, that this simple approach leads to correct results.

All information is contained in the quantity

$$z = \frac{2\pi}{\hbar\omega V n^2} \left| \int_{-T/2}^{T/2} dt \int d\vec{x}\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \vec{\epsilon}\cdot\vec{j}(x) \right|^2 , \tag{3.1}$$

where

$$\vec{j}(x) = ec\vec{\beta}\delta(\vec{x}-\vec{x}_0-c\vec{\beta}t), \quad \vec{\beta}=(0,0,\beta) \tag{3.2}$$

is the classical current of the electron, and $\vec{\epsilon}$ is the polarization vector of the radiation field with $\vec{k}\cdot\vec{\epsilon}=0$. The total power radiated spontaneouly is obtained from Eq. (3.1) by integrating over the phase space of the emitted photons and summing over polarizations,

$$W = \sum_{\text{pol}} \int \frac{d\vec{k}V}{(2\pi)^3} \hbar\omega z , \tag{3.3}$$

while the spread of the final electron energies is

$$\Delta E = \hbar\omega[z^2 + (2N+1)z]^{1/2} . \tag{3.4}$$

Here $N$ denotes the number of initial photons of the radiation field. Since $z << 1$, we have for $N >> 1$, i.e., for stimulated Čerenkov radiation,

$$\Delta E \cong \hbar\omega\sqrt{2Nz} . \tag{3.5}$$

The probability that $n$ photons are emitted is, for $N >> 1$, given by[6,7] $J_n^2(2\sqrt{Nz})$. The Bessel functions are practically zero when the index exceeds the argument. Hence, invoking energy conservation, we can conclude that the maximum energy exchange between the electron and the laser is

$$\Delta E_{\max} = 2\hbar\omega\sqrt{Nz} = \sqrt{2}\Delta E . \tag{3.6}$$

For specified $\vec{k} = n\omega/c(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, a system of two mutually orthogonal polarization vectors is given by[9]

$$\vec{\epsilon}^{(1)} = (\sin\phi, -\cos\phi, 0) ,$$

$$\vec{\epsilon}^{(2)} = (-\cos\theta\cos\phi, -\cos\theta\sin\phi, \sin\theta) .$$

Hence in view of Eqs. (3.1)–(3.3) the spontaneous radiation is linearly polarized with the direction specificed by $\vec{\epsilon}^{(2)}$. With $\vec{\epsilon}=\vec{\epsilon}^{(2)}$ we have

$$z=\frac{8\pi}{\hbar\omega V}\left[\frac{e\beta cT\sin\theta}{2n}\frac{\sin u}{u}\right]^2, \qquad (3.7)$$

with $u$ defined in Eq. (2.27). Equation (3.5) reproduces the spread of Eq. (2.28), because $N/V=\omega A^2n^2/8\pi\hbar c^2$ with $A$ the amplitude of the vector potential (2.1).

In the limit of $T\to\infty$, in view of

$$\frac{\sin^2 ax}{x^2}\overset{a\to\infty}{\longrightarrow}\pi a\delta(x),$$

we obtain for the spontaneous power radiated per unit time:

$$\lim_{T\to\infty}\frac{W}{T}=e^2\beta^2c^2n\int\omega\,d\omega\,d(\cos\theta)\sin^2\theta \times\delta(1-n\beta\cos\theta), \qquad (3.8)$$

which exhibits the well-known power spectrum of Čerenkov radiation.

To calculate the gain we investigate the quantum kinematics for the case that the electron emits $s$ photons ($s>0$) or absorbs $-s$ photons ($s<0$):

$$p'=p-s\hbar k, \qquad (3.9)$$

when $p\,(p')$ is the energy momentum four vector of the initial (final) electron, $p^2=(E/c)^2-\vec{p}^{\,2}=m^2c^2=p'^2$, and $k=(\omega/c,\vec{k})$ with $\vec{k}^2=n^2\omega^2/c^2$, is the wave vector of the laser field. Squaring Eq. (3.9) we get

$$1-n\beta\cos\theta-\frac{2(1-n^2)\hbar\omega}{2mc^2\gamma}=0, \qquad (3.10)$$

which exhibits the recoil as a quantum correction to the classical Čerenkov radiation condition. Replacing $1-n\beta\cos\theta$ in Eq. (3.7) by the left-hand side Eq. (3.10) and expanding the modified $z$ to first order with respect to the recoil according to

$$z\left[1-n\beta\cos\theta-s\frac{(1-n^2)\hbar\omega}{2mc^2\gamma}\right]=z(1-n\beta\cos\theta)+s\Delta z \qquad (3.11)$$

defines

$$\Delta z=\frac{\pi e^2\sin^2\theta(n^2-1)\beta^2\omega T^3}{2Vm\gamma n^2}\frac{\partial}{\partial u}\left[\frac{\sin u}{u}\right]^2. \qquad (3.12)$$

The first-order expansion will restrict us to the small-signal regime. The condition for its validity turns out to be identical to Eq. (2.29). The gain is than given by[6,7]

$$G=\frac{V\rho}{N}\Delta z\frac{\partial}{\partial z}\left[\frac{\Delta E}{\hbar\omega}\right]^2=2V\rho\Delta z \qquad (3.13)$$

in agreement with Eq. (2.26). The bunching amplitude is exhibited in the square of the wave function of the electron when leaving the wiggler,[7]

$$|\phi_{\text{out}}|^2=1+\frac{2N\Delta z}{\sqrt{Nz}}\sin(\omega t-\vec{k}\vec{x}+\phi). \qquad (3.14)$$

Equation (3.13) shows that we actually derived the gain via a gain-spread relation. Using

$$\Delta z\frac{\partial}{\partial z}=-\frac{(n^2-1)\hbar\omega\beta^2\gamma^2}{2mc^2}\frac{\partial}{\partial\gamma},$$

we can rewrite this relation in the common form Madey's theorem[10]

$$\langle\gamma_f-\gamma_i\rangle=\tfrac{1}{2}(n^2-1)\beta^2\gamma^2\frac{\partial}{\partial\gamma_i}\langle(\gamma_i-\gamma_f)^2\rangle, \qquad (3.15)$$

where $\gamma_i$ and $\gamma_f$ are related to the electron's initial and final energy, respectively. The corresponding relation for the standard helical wiggler free-electron laser[10] is identical with Eq. (3.15) except that the factor $(n^2-1)\beta^2\gamma^2$ is missing. For $n\beta=1$, i.e., $\cos\theta=1$, this factor is unity. For $\cos\theta$ slightly smaller than unity, however, it increases significantly. For the values of the Stanford experiment[3] it is approximately 15, indicating a much more favorable gain-spread ratio than in the case of the conventional free-electron laser.

In order that our gain formula applies, we must have

$$\delta u=\delta\left[\frac{\omega T}{2}(1-n\beta\cos\theta\right]<\pi, \qquad (3.16)$$

where $\delta$ means variations with respect to either $\beta$, $\cos\theta$, or $\omega$ due to uncertainties in the initial electron and laser beam. Varying with respect to $\gamma$, $\theta$, and $\omega$ we obtain, respectively,

$$\delta\gamma<\frac{\gamma^3\lambda}{nL}, \qquad (3.17a)$$

$$\delta\theta<\frac{\lambda}{L}\cot\theta, \qquad (3.17b)$$

$$\frac{\delta\omega}{\omega}<\frac{\lambda}{L}\left[1-n\beta\cos\theta-\omega\frac{dn}{d\omega}\beta\cos\theta\right]^{-1}. \qquad (3.17c)$$

Among these, Eq. (3.17a) seems most easy and Eq. (3.17b) most difficult to meet. For example, the parameters of the Stanford experiment[3] do not satisfy Eq. (2.17b). Equation (2.17c) requires a highly frequency-independent refractive index. Equations (2.17a)–(2.17c) are mainly important if the gain is of interest, whereas the spread essentially survives even if these conditions are not fully satisfied by the experimental parameters. We emphasize that this discussion refers to a Čerenkov amplifier and not a Čerenkov laser. In the latter case much more restrictive conditions would apply.

## IV. DISCUSSION

In the recent Stanford experiment,[3] attention was focused on the spread as the most pronounced feature. Although the spread could not directly be measured, data are consistent with the formula

$$\Delta E_{\max}(\text{keV}) = 38.8[P(\text{MW})]^{1/2} \qquad (4.1)$$

giving the maximum energy change of the electron due to its interaction with the laser in terms of the peak power $P$ of the laser in megawatts. It is remarkable that the energy change depends solely on $P$. Equation (4.1) was derived[11] by solving the Lorentz force equation for an electron in the presence of a plane wave with a Gaussian intensity variation and corroborated by computer calculations with a more accurate laser profile. We obtain from Eq. (2.29)

$$\Delta E = \frac{eA}{mc^2}\omega T \frac{v}{nc} \sin\theta mc^2 , \qquad (4.2)$$

assuming exact resonance so that $(\sin u/u)^2 = 1$. Relating the amplitude $A$ of the vector potential of the laser field with wavelength $\lambda$ to its peak intensity $I$ in $\text{W/cm}^2$ by

$$\frac{eA}{mc^2} = 8.52\times 10^{-6}\ \lambda(\text{cm})[I(\text{W/cm}^2)]^{1/2} \qquad (4.3)$$

and noting that the length $L = vT$ of the interaction region is related to the diameter $W$ of the laser beam by $L\sin\theta = W$, we obtain for $v \simeq c$, $n \sim 1$

$$\Delta E(\text{keV}) = 27.4\ [P(\text{MW})]^{1/2} . \qquad (4.4)$$

This specifies the average spread; the approximate maximum energy change (in a quantum treatment there is no definite upper limit for the energy change) is in view of Eq. (3.6) $\Delta E_{\max} \sim \sqrt{2}\Delta E = 38.7P^{1/2}$. Since we used a highly idealized description of the laser field, the almost complete agreement with Eq. (4.1) is surprising. It indicates that the spread, being the most pronouned feature of stimulated Čerenkov radiation, is not at all sensitive to the detailed geometry of the experiment.

Our expression for the gain differs from an expression obtained earlier[12] by means of the plasma physics dispersion relation approach essentially by the factor $(n^2-1)\gamma^2\beta^2$. This is the same factor which occurs on the right-hand side of the gain-spread relation (3.15). As has been pointed out above, this is not an insignificant discrepancy. At present we do not have an explanation for this difference.

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