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Schrödinger solitons and kinks behave like Newtonian particles

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Sine-Gordon kinks under the influence of a constant force do not travel along the classical trajectory. It is shown here that solitons and kinks which obey one of the nonlinear Schrödinger equations and which are also subject to a constant force, on the contrary, do travel along the classical trajectory.

I. INTRODUCTION

In a recent letter, Fernandez *et al.*[1] have shown that the velocity of sine-Gordon kinks in a constant external field increases for small times with the third power in time and for large times approaches a constant. Since this is in contrast to the classical law of a linearly increasing velocity, they refer to the sine-Gordon kinks as non-Newtonian particles. On the contrary, we prove here that there exist soliton and kink solutions of nonlinear Schrödinger equations (NLSE's) such as the nonlinear Schrödinger equation,[2] the logarithmic NLSE,[3] the derivative NLSE,[4] or others,[5] which are also subject to a constant field, but have the same shape as without external field and that they move with the classical velocity.

II. SOLITONS

First we consider solitons in an arbitrary external potential $\phi(x)$ under the time-dependent Schrödinger equation

$$i\dot{\psi}(x,t)=H\psi(x,t) \tag{1}$$

and the nonlinear Hamiltonian

$$H=\tfrac{1}{2}p^2+W+\phi\ . \tag{2}$$

Here, $p=-i\partial/\partial x$ is the momentum operator and, for the time being, the nonlinearity W is merely a function of the modulus of the wave function,

$$\psi=\varphi e^{iS} \tag{3}$$

and S is its phase. Using Heisenberg's equation of motion for expectation values, we find

$$\begin{aligned}&\frac{d}{dt}\langle x\rangle=\langle p\rangle\\&\frac{d}{dt}\langle p\rangle=-\langle\phi'\rangle-\langle W'\rangle\ .\end{aligned} \tag{4}$$

The expectation value $\langle W'\rangle$ is converted into a φ integration which, in turn, vanishes by means of the vanishing conditions $\varphi(-\infty)=\varphi(\infty)=0$, provided that W is a faster function of φ than φ^{-2}. Equations (4) then form the quantum equations for motion in an external potential,

$$\frac{d^2}{dt^2}\langle x\rangle=-\langle\phi'\rangle \tag{5}$$

which coincide with the classical equation if the potential is at most quadratic. We shall show below the existence of soliton solutions in a linear potential. Whether there exist solitons in other potentials is an open question.

The proof for the derivative NLSE,

$$W=\frac{\alpha}{2}(p\varphi^2+\varphi^2p)\ , \tag{6}$$

is analogous. Here we get

$$\begin{aligned}&\frac{d}{dt}\langle x\rangle=\langle p\rangle+\alpha\langle\varphi^2\rangle\ ,\\&\frac{d}{dt}\langle p\rangle=-\langle\phi'\rangle+\frac{\alpha}{2}\langle\varphi^2S''\rangle\ ,\end{aligned} \tag{7}$$

which can be combined to also yield

$$\frac{d^2}{dt^2}\langle x\rangle=-\langle\phi'\rangle\ . \tag{8}$$

Note that in this case by Eq. (7) the canonical momentum differs from the kinetic momentum.

III. KINKS AND SOLUTIONS

Step-function-like kinks obey only one vanishing condition, $\varphi(-\infty)=0$, $\varphi(\infty)\neq 0$ and, since expectation values do not exist, they are no quantummechanical particles in the strict sense. In many instances, however, they behave like particles and also obey NLSE's of the type (1,2). In the following, we have to restrict ourselves to a constant external field,

$$\phi = -gx \ , \tag{9}$$

and prove that kinks follow the classical trajectory and solve for the shapes of the kinks and the solitons.

We follow Ref. 5 and split up the Schrödinger equation into real and imaginary parts. This results in the Hamilton-Jacobi equation

$$\dot{S} + W - gx + \tfrac{1}{2}(S')^2 = \frac{\varphi''}{2\varphi} \ , \tag{10}$$

where the rhs is the Madelung-Bohm quantum potential, and in the equation of continuity,

$$\dot{\varphi} + \varphi' S' + \tfrac{1}{2}\varphi S'' = 0 \ . \tag{11}$$

For a nonspreading soliton or kink the modulus is merely a function of $x - X(t)$, where $X(t)$ is a yet undetermined function of time. Hence $\dot{\varphi} = -\dot{X}\varphi'$ and the solution of Eq. (11) demands $S' = \dot{X}$ to yield the phase

$$S = \dot{X}x + (\epsilon - \tfrac{1}{2}\dot{X}^2)t \ , \tag{12}$$

where ϵ is a constant which is related to the energy. Equation (10) then becomes

$$(\ddot{X} - g)x + \epsilon - \dot{X}\ddot{X}t + W = \frac{\varphi''}{2\varphi} \ , \tag{13}$$

which admits

$$\ddot{X} = g \tag{14}$$

being the constant external acceleration. Equation (13) then turns into the same differential equation as if the external force were absent to give the inverse of the modulus[5]

$$\pm 2(x - X) = \int_{\varphi_0}^{\varphi} \left[\int_0^{\varphi} \varphi(\epsilon + W(\varphi))d\varphi \right]^{-1/2} d\varphi \ , \tag{15}$$

where $\varphi_0 = \varphi(X)$. Equations (5), (8), and (14) are the classical or quantum equations of motion.

IV. SUMMARY

As a result, for a given NLSE there exist solitons and kinks in a constant external field. They move with constant acceleration and have the same shapes as their counterparts without external field. We have also shown that solitons in a quadratic potential, if they exist, also move on the classical trajectory and those in other potentials, if they exist, move on the corresponding quantum trajectory. This behavior is as expected because nonspreading in a NLSE is the result of the interplay between spreading and nonlinearity. The former effect but is the same with or without external field. It would therefore be interesting to study numerically the solutions of NLSE's sliding down an inverted parabolic potential in order to possibly find nonspreading solitons or kinks.

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