PHYSICAL REVIEW A VOLUME 25, NUMBER 1 JANUARY 1982

Analysis of Bose-Einstein condensation for an ideal relativistic Bose gas in d dimensions

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(Received 26 January 1981)

We investigate the conditions for Bose-Einstein condensation of an ideal relativistic Bose gas in various spatial dimensions. From the direct numerical evaluations of the equations for the density and the specific-heat anomaly, we are able to show the qualitative difference between the massive and massless relativistic systems. As a consequence we are able to discuss the dependence of the critical exponents upon the spatial dimensionality for this system.

I. INTRODUCTION

The investigation of the dependence of the thermodynamics on the spatial dimensions for the ideal relativistic Bose gas leads to results relating to phase transitions and critical phenomena in much more complicated systems. In this work we want to look more extensively into the analytical structure of the phenomenon of Bose-Einstein condensation (BEC) in various spatial dimensions[1,2] in order to further develop an approach to the related physical systems.

Although for a long time the general concept of condensation had been related to the liquid-gas first-order phase transition below the critical point, it was first explicitly stated by Einstein[3] in 1925 that such a situation could exist for an ideal quantum many-particle system, which now is called the ideal Bose gas. In particular, he showed that from the basic structure of the total noninteracting many-particle state it would be possible under the right thermodynamical conditions to have a finite fraction of the system drop into the ground state. Somewhat later, this type of condensation was considered by London,[4] who regarded it as a momentum-space condensation, to relate to the then known properties of superfluidity, which even today has its place in the study of the collective phenomena using macroscopic models for superfluid helium and superconductivity.

The effects of dimensionality have also previously been investigated[5-7] for nonrelativistic thermodynamical systems possessing an energy spectrum of the form

$$\epsilon(p)=\sum_{i=1}^{d} c_i p_i^{\sigma}, \tag{1.1}$$

where the c_i are the particular coefficients of the respective components of the momenta p_i, d is the number of spatial dimensions, and σ is a free parameter relating to the power of the spectrum. Several interesting special cases arise in the detailed investigation of this proposed energy spectrum. It is clear that the special case with all the values of c_i equal to $1/2m$, where m is the particles' mass, and $\sigma=2$ corresponds to an isotropic system of free particles. Another physically interesting case is that of a gas of massless particles where we have $\sigma=1$ together with all the coefficients equal to c, the velocity of light, which we hereafter take as $c=1=\hbar$. It has already been pointed out[5-7] that the quantity $q\equiv d/\sigma$ is the determining factor for the existence of BEC with the energy spectrum of the type (1.1). Thus for $q\leq 1$, there exists no BEC; while $q>1$ guarantees that it takes place.

In this work we want to discuss in detail the ideal relativistic Bose gas in relation to the above observations. We replace the energy spectrum (1.1) with the usual relativistic energy

$$\epsilon(p)=(m^2+|p|^2)^{1/2}, \tag{1.2}$$

which yields the special cases of (1.1), with $\sigma=2$ and $\sigma=1$, respectively, the nonrelativistic and ultrarelativistic limits.

In order to write our partition function as an integral, we must use the appropriate phase space measure for this relativistic system. We have shown in another work[8] how in general the grand partition for the relativistic quantum gases may be gotten from the relativistic constraint formulation. Furthermore, we demonstrated there that the form of this phase space measure reduces to the known Touschek[9] measure $d\nu_3(p)$ for an ideal gas in the usual three-dimensional volume V_3. The direct generalization, which we shall use, into d spatial dimensions then becomes

$$d\nu_d(p)=\frac{1}{(2\pi)^d}\frac{V_{d\mu}p^{\mu}}{p^0}\,d^dp, \tag{1.3}$$

where $V_{d\mu}$ and p^μ are, respectively, the volume and momentum vectors belonging to a $(d+1)$-dimensional generalized Minkowski space.

Now we give a short summary of our work. In the next section of this paper we elaborate on the above mentioned criterion for BEC for the relativistic ideal Bose gas with an energy spectrum (1.2) in relation to the particle density n. In the following section the specific heat anomaly[10] is discussed in relationship to the dimensionality. Then we are able to make some statements about the relationship of the critical exponents to the dimensionality. In conclusion we make some remarks about extensions of this work to some related areas.

II. BOSE-EINSTEIN CONDENSATION IN d-DIMENSIONS

In our previous work[1,2] we investigated the ideal Bose gas in d-dimensions using the grand canonical ensemble. The thermodynamical potential Ω is gotten from the grand partition function Ξ in the form $-\beta^{-1}\ln\Xi$, so that we find[8]

$$\Xi(V_d,\beta,A)=\sum_N A^N \int_{V_d} d\nu_{d,N}(p)\, e^{-\beta_\mu p^\mu}, \qquad (2.1)$$

where β_μ and p^μ are the inverse temperature and momentum vectors of a $(d+1)$-dimensional generalized Minkowski space and A is the relativistic fugacity[11,12] with a direct relation to the relativistic chemical potential μ in the form of $\exp(\beta\mu)$. The nonrelativistic fugacity z is given by

$$z=A\exp(-\beta m). \qquad (2.2)$$

The evaluation of Ω in the rest frame of the system yields

$$\beta\Omega=\frac{V_d}{(2\pi)^d}\int d^d p\,\ln(1-Ae^{-\beta\epsilon(p)}) + \ln(1-Ae^{-m\beta}), \qquad (2.3)$$

where we have used (1.3) for $d\nu_d(p)$. The second term of the right-hand side of (2.3) arises from the contributions from the ground state, which becomes singular when A reaches $\exp(m\beta)$ or as $z \to 1$, in the same way as it is for the nonrelativistic Bose gas.[13] We use the spherical symmetry of the system in momentum space for (2.3) to obtain

$$\beta\Omega=\frac{V_d}{(2\pi)^{d-1}}\frac{\pi^{(d-2)/2}}{\Gamma(d/2)}\int_0^\infty dp\,p^{d-1}\ln(1-Ae^{-\beta\epsilon(p)}) + \ln(1-Ae^{-m\beta}). \qquad (2.4)$$

The evaluation of this integral is carried out[14] using the modified Bessel functions[15] $K_\nu(x)$. The particle number density n can be simply calculated by taking a derivative with respect to A on (2.4).

We have previously introduced a d-dimensional generalization[1,2] of a form which has been valuable in $d=3$ for the thermodynamics of the ideal quantum gases.[11] This dimensionally dependent quantity we defined as

$$L_d(m,\beta)\equiv\frac{\pi^{-d'}}{\beta^d}\,2^{-d'+3/2}(m\beta)^{d'}e^{m\beta}K_{d'}(m\beta). \qquad (2.5)$$

This form yields the simple relation for the ideal relativistic Boltzmann gas, $n=L_d z$. However, for the ideal relativistic Bose gas we find that

$$n=L_d\sum_{k=1}^\infty \frac{z^k e^{(k-1)m\beta}}{k^{d'-1}}\frac{K_{d'}(km\beta)}{K_{d'}(m\beta)}+n_0, \qquad (2.6)$$

where n_0 is the density of particles in the ground state and $d'=(d+1)/2$. n_0 is given by

$$n_0=\lim_{V_d\to\infty}\frac{A}{V_d}\frac{e^{-m\beta}}{1-Ae^{-m\beta}}=\lim_{V_d\to\infty}\frac{z}{V_d(1-z)}. \qquad (2.7)$$

We see that n_0 has the same form as in the nonrelativistic Bose gas. Likewise, it can be shown that at sufficiently low temperatures for $V_d\to\infty$ and $z\to 1$, we have $n_0\neq 0$. The critical density n_c is determined by[1]

$$n_c=L_d\sum_{k=1}^\infty \frac{e^{(k-1)m\beta}}{k^{d'-1}}\frac{K_{d'}(km\beta)}{K_{d'}(m\beta)}. \qquad (2.8)$$

The significance of this relationship comes from the fact that (2.8) defines at a given temperature $T=\beta^{-1}$ the critical density n_c. Furthermore, it is possible to turn around this statement so that at a given density we may solve for a critical temperature T_c. Thus (2.8) defines a critical line in the density-temperature plane.

Now we discuss the numerical computation of these quantities. In our previous work[1,2] we have already looked into the behavior of the quantity $L_d^{-1}n_c$ in relation to various dimensions, which we include for the sake of comparison in Fig. 1(a). In Fig. 1(b) between $d=3$ and $d=7$ we show how $L_d^{-1}n_c$ looks as a function of $m\beta$ for integer dimensions between 3 and 7. Particular numerical values for these two plots are given in Table I. We notice in general the lowering of the critical density at higher dimensions and temperatures in comparison with the density of the Boltzmann gas at the temperature. Thus, the nonrelativistic and ultrarelativistic limit form, respectively, upper and lower bounds for the curves in Fig. 1(a) as well as ending and beginning point for Fig. 1(b). In Sec. III we shall contrast this behavior with the critical phenomena related to the specific heat C_V.

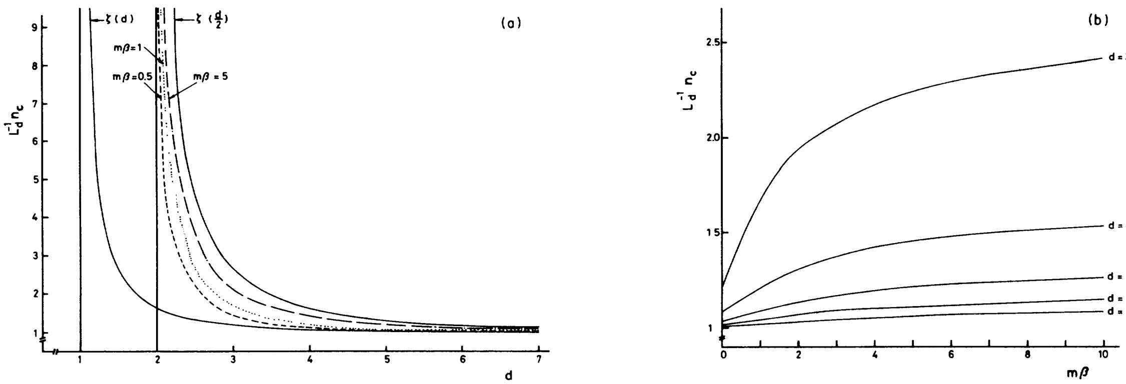


FIG. 1. The ratio of the critical density in terms of $L_d^{-1} n_c$ of a Bose gas to the corresponding Boltzmann gas (a) in relation to the spatial dimensions d for various stated values of the parameter $m\beta$, and (b) as a function of $m\beta$ for the listed d.

III. DIMENSIONAL BEHAVIOR OF THE SPECIFIC HEAT

The properties of the specific heat C_V at the critical point can also be studied for BEC. Landsberg and Dunning-Davies[10] have examined the relativistic Bose gas for $d=3$. They found that in massive systems there is no discontinuity of the specific heat while for the massless gas there exists a jump in C_V at the critical temperature T_c. S. de Groot *et al.*[5] have shown that for nonrelativistic gases with energy spectra of the form (1.1), a jump in C_V exists for $d/\sigma > 2$, whereas there is no such jump for $1 < d/\sigma \leq 2$. We investigate in this section the structure of C_V for the relativistic Bose gas with the energy spectrum (1.2). Our results are consistent with the above statements.

Now we discuss the behavior of the specific heat for arbitrary dimensions d. The specific heat C_V is related to the energy E, which may be obtained in the usual way from Ω with a change in variables[16] from the temperature and particle number to the inverse temperature β and the fugacity $z = \exp(\alpha)$ so that

$$C_V = -\beta^2 \left[\left(\frac{\partial E}{\partial \beta}\right)_{\alpha, V} - \left(\frac{\partial E}{\partial \alpha}\right)_{\beta, V} \left(\frac{\partial N}{\partial \beta}\right)_{\alpha, V} \Big/ \left(\frac{\partial N}{\partial \alpha}\right)_{\alpha, V} \right]. \tag{3.1}$$

With the substitution x for the energy spectrum $\epsilon(p)$ in (1.2), so that x^2 replaces $\epsilon^2 - m^2$, and the definition

$$\gamma_d \equiv \frac{\pi^{-d/2}}{2^{d-1}\Gamma(d/2)} V_d , \tag{3.2}$$

we may write the integrals for the particle number N and energy E by using (1.3) together with the positivity of the energy and the mass-shell condition as well as the spherical symmetry in momentum space. The form of these quantities is a generalization to d dimensions of the usual relativistic invariant phase space expressions.[16] Thus, the particle number and energy are given by

$$N = \gamma_d \int_0^\infty dx (x+m)(x^2 + 2mx)^{(d-2)/2} (e^{-\alpha} e^{\beta x} - 1)^{-1} , \tag{3.3}$$

$$E = \gamma_d \left(\int_0^\infty dx (x^2 + mx)(x^2 + 2mx)^{(d-2)/2} (e^{-\alpha} e^{\beta x} - 1)^{-1} + m \int_0^\infty (x+m)(x^2 + 2mx)^{(d-2)/2} (e^{-\alpha} e^{\beta x} - 1)^{-1} \right). \tag{3.4}$$

Here we have dropped the term that belongs to the states with $\vec{p} = 0$ because they contribute to the energy only through the rest mass, which only de-

TABLE I. The values of $L_d^{-1} n_c$ in Fig. 1 are tabulated together with the nonrelativistic [$\zeta(d/2)$] and the ultrarelativistic [$\zeta(d)$] cases for the stated d and $m\beta$.

d	$\zeta(d/2)$	$m\beta = 10$	$m\beta = 5$	$m\beta = 1$	$m\beta = 0.5$	$m\beta = 0.1$	$\zeta(d)$
2	∞	∞	∞	∞	∞	∞	1.645
3	2.612	2.406	2.084	1.665	1.460	1.251	1.202
4	1.645	1.522	1.451	1.241	1.150	1.095	1.082
5	1.341	1.245	1.214	1.089	1.062	1.043	1.037
6	1.202	1.135	1.112	1.041	1.028	1.021	1.017
7	1.127	1.100	1.061	1.020	1.013	1.012	1.008

pends on the particle number. However, since the specific heat is evaluated at constant particle number N, this term always vanishes upon differentiation. The second term in (3.4) is the contribution of the rest mass from the nonzero momentum states. Thus, by the same argument we may drop this term before continuing with our calculations.

For the sake of simplicity of the evaluation of C_V we introduce the following expressions:

$$I_d(n,\alpha) \equiv \int_0^\infty dx\,(x^n + m\beta x^{n-1}) \times (x^2+2m\beta x)^{(d-2)/2}\frac{e^x}{(e^{(x-\alpha)}-1)^2} \qquad (3.5)$$

and

$$J_d(n,\alpha) \equiv \int_0^\infty dx\,(x^n + m\beta x^{n-1}) \times (x^2+2m\beta x)^{(d-2)/2}(e^{x-\alpha}-1)^{-1}\,. \qquad (3.6)$$

After performing the indicated differentiations in (3.1) using (3.3) and (3.4), we may express C_V with the expressions of (3.5) and (3.6) as follows:

$$C_V = \begin{cases} Ne^{-\alpha}\dfrac{I_d(3,\alpha)}{J_d(1,\alpha)} - \dfrac{[I_d(2,\alpha)]^2}{I_d(1,\alpha)J_d(1,\alpha)}\,, & T>T_c \qquad (3.7a)\\[2ex] N\dfrac{I_d(3,0)}{J_d(1,0)}\,, & T<T_c \qquad (3.7b)\end{cases}\,.$$

We found these two differing expressions for C_V, which belong to the indicated ranges of temperature, from the fact that in (3.1) the second term vanishes for $T<T_c$, since α is constant there.

Now we investigate the behavior of the jump in C_V, which we previously mentioned[1,2] but did not evaluate. We define

$$\Delta_d \equiv |C_V(T_c+) - C_V(T_c-)|\,, \qquad (3.8)$$

which after using (3.7a) and (3.7b) yields

$$\Delta_d = Ne^{-\alpha}\frac{[I_d(2,\alpha)]^2}{I_d(1,\alpha)\,J_d(1,\alpha)}\,. \qquad (3.9)$$

We are particularly interested in the limit $\alpha\to 0$. However, the integrands of $I_d(n,0)$ and $J_d(n,0)$ are singular at the lower limit of integration. We analyze $I_d(n,\alpha)$ in the Appendix; $J_d(n,\alpha)$ can be similarly analyzed. We find that $I_d(n,\alpha)$ converges in the limit $\alpha\to 0$ if $d>2\ (3-n)$, whereas it diverges if $d\leq 2\ (3-n)$. The same procedure for $J_d(n,\alpha)$ yields in the limit $\alpha\to 0$ convergence if $d>2\ (2-n)$, and divergence if $d\leq 2\ (2-n)$. Now we consider the jump in C_V given by (3.9). The combination of convergence conditions yields the result as $\alpha\to 0$,

$$\Delta_d = \begin{cases} 0, & 2<d\leq 4\\ N\dfrac{[I_d(2,0)]^2}{I_d(1,0)J_d(1,0)}\,, & d>4\end{cases}\,. \qquad (3.10)$$

The case of $d=2$ need not be considered since massive particles do not undergo BEC in two spatial dimensions. Therefore, one cannot define a critical temperature T_c.

Now we discuss briefly some special cases of this form of Δ_d in (3.9) involving the nonrelativistic and ultrarelativistic cases. We find that the integrals simplify so that we have the following:

(i) the ultrarelativistic limit ($m\beta\to 0$)

$$\Delta_d = Nd^2\zeta(d)/\zeta(d-1)\,,\quad d>2 \qquad (3.11)$$

(ii) the nonrelativistic limit ($m\beta\to\infty$)

$$\Delta_d = N\,(d/2)^2\zeta(d/2)/\zeta[(d/2)-1]\,,\quad d>4\,. \qquad (3.12)$$

Thus we see a further confirmation that the relativistic gas of massive particles behaves similarly to the nonrelativistic gas for all values $m\beta\neq 0$. Furthermore, we again obtained the result that the ultrarelativistic gas in d dimensions should behave like the nonrelativistic gas in 2 d dimensions. The results for these limiting cases have been noted previously by Landsberg,[7,17] where he finds from the general form of the energy spectrum (1.1) that $\Delta_d>0$ when $d/\sigma>2$.

Now we turn to a discussion of the numerical evaluation of this jump in the specific heat Δ_d/N. We have tabulated the quantity Δ_d/N in the spatial dimensions from $d=2$ to $d=8$ for selected values of the parameter $m\beta$ including the limiting cases in Table II. The absence of entries for $d=2$ except for $m\beta=0$ comes from the fact established in Sec. II that there exists no BEC aside from the massless Bose gas. Furthermore, we see that only the massless Bose gas has positive values for Δ_d/N in $d=3$ and 4 in spite of the fact that BEC takes place in all cases, which we saw analytically in (3.10). This dependence upon dimensionality is illustrated in Fig. 2(a), in which we notice that the massless gas and the nonrelativistic limit form the upper and lower bounds for the curves in the sense opposite to Fig. 1(a). The dependency of Δ_d/N on the parameter $m\beta$ is shown for dimensions 5 to 10 in Fig. 2(b) to be monotonically decreasing. The limiting values are listed in the first and last columns of Table II. In Sec. IV we shall show how these analytical and numerical facts can be incorporated into concrete statements[2] about the critical exponents.

IV. CRITICAL PHENOMENA IN d DIMENSIONS

In this section we want to look into the nature of BEC near to its onset through the investigation of

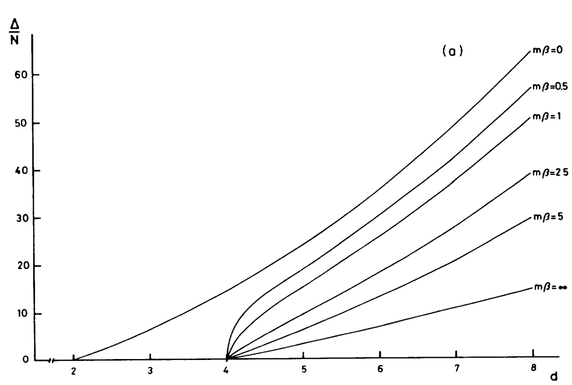
TABLE II. The jump in the specific heat per particle Δ/N in Fig. 2 is given for stated values of d and $m\beta$.

d	$m\beta=\infty$	$m\beta=10$	$m\beta=7.5$	$m\beta=5$	$m\beta=2.5$	$m\beta=1$	$m\beta=0.5$	$m\beta=0.1$	$m\beta=0$
2								0	0
3	0	0	0	0	0	0	0	0	6.58
4	0	0	0	0	0	0	0	0	14.41
5	3.21	4.73	5.26	6.32	9.29	14.97	18.76	22.81	23.95
6	6.58	9.73	10.77	12.77	17.68	25.28	29.63	34.06	35.32
7	10.29	15.48	17.14	20.21	27.25	36.94	42.07	47.15	48.57
8	14.41	22.17	24.58	28.93	38.37	50.34	56.34	62.13	63.73
9	18.96	29.88	33.20	39.05	51.18	65.59	72.51	79.05	80.83
10	23.95	38.71	43.10	50.69	65.78	82.76	90.63	97.94	99.90

the analytical structure around the critical point. This analysis is usually characterized by using the critical exponents,[18] where the spatial dimensions are generally treated as a continuous parameter d. Here we investigate these quantities in relationship to the scaling laws for BEC.

The work of the preceeding sections was discussed for integer-valued dimensions although the curves characterizing the critical density [Fig. 1(a)] and the jump in the specific heat [Fig. 2(a)] show a continuous dimensionality dependence. It is precisely this continuous structure that is relevant for the critical exponents. Especially important to us here are the properties of the specific heat above four spatial dimensions, which we shall elaborate on soon. In order to see the behavior of Δ_d/N more clearly, we have extended Table II between four and five spatial dimensions in smaller intervals at various values of $m\beta$ in Table III. We notice the almost linear growth of the nonrelativistic limit ($m\beta=\infty$) near $d=4$ as opposed to the very steep increase of the small values of $m\beta$ as can be seen in Fig. 2(a). However, the massless Bose gas ($m\beta=0$) has no onset of Δ_d/N at $d=4$ but instead at $d=2$. This behavior around $d=4$ is analogous to the situation for n_c as illustrated in Fig. 1(a) at $d=2$, which we have previously discussed.[1]

Now we reexamine some aspects of the critical phenomena related to the relativistic Bose gas.[2] Although the analytical structure of the nonrelativistic Bose gas has already been investigated[19-21] for arbitrary spatial dimensions d with the isotropic form of the energy spectrum (1.1), there has been little concern for the critical structure of the relativistic system with $\epsilon(p)$ in (1.2), which in the massless case was known to have an anoma-



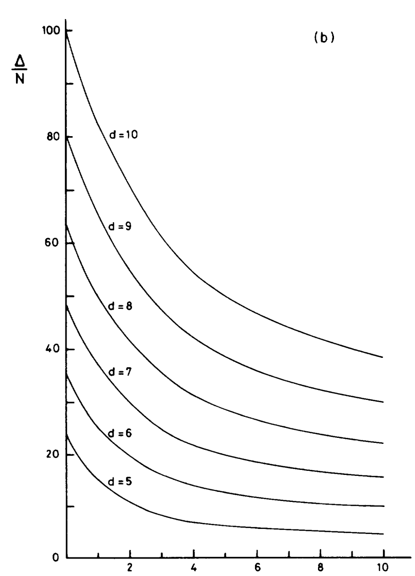


FIG. 2. The jump in specific heat per particle Δ/N at T_c (a) in relation to the spatial dimensions d for given values of $m\beta$, and (b) as a function of $m\beta$ for the stated d.

TABLE III. An extension of Table II to noninteger dimension given Δ/N for d between $d=4$ and $d=5$.

d	$m\beta=\infty$	$m\beta=10$	$m\beta=7.5$	$m\beta=5$	$m\beta=2.5$	$m\beta=1$	$m\beta=0.5$	$m\beta=0.1$	$m\beta=0$
4.0	0	0	0	0	0	0	0	0	14.41
4.2	0.65	0.98	1.11	1.38	2.33	5.47	9.30	14.87	16.17
4.4	1.29	1.93	2.17	2.66	4.26	8.48	12.31	16.85	18.01
4.6	1.92	2.86	3.20	3.89	6.00	10.83	14.59	18.79	19.92
4.8	2.57	3.79	4.22	5.10	7.66	12.94	16.70	20.77	21.90
5.0	3.21	4.73	5.26	6.32	9.29	14.97	18.76	22.81	23.95

ly[10] for $d=3$ $(d>2)$. However, in the massive system we have seen that this jump first appears above $d=4$.

It is known that in the usual nonrelativistic Bose gas most of the critical exponents[18] with the exception of α associated with the specific heat show the expected behavior of a classical system for d above four dimensions. The usual definition of α for temperatures near to the critical temperature T_c expressed in terms of the reduced temperature $t=(T-T_c)T_c$ as

$$C_V \sim (t)^{-\alpha} . \tag{4.1}$$

However, it has been known[19] for a long time that the description in dimensions higher than four, demands a special exponent α_s defined by taking the mth partial derivative, which then yields a singularity of the form

$$\frac{\partial^m C_V}{\partial T^m} \sim (t)^{-(\alpha_s+m)} . \tag{4.2}$$

This definition is necessary because of the fact that for $d>4$, C_V is finite at T_c as we calculated in Sec. III. Nevertheless, there is a singularity at dimensions smaller than four. One finds the corresponding critical exponent α_h whose definition comes from the usual free energy, $F(T)$, by subtracting off its nonsingular part $F_{ns}(T)$, so that

$$\frac{\partial^2}{\partial T^2}[F(T)-F_{ns}(T)] \sim (t)^{-\alpha_h} . \tag{4.3}$$

Furthermore, it was shown[21] that the difficulties in relation to α for $d>4$ can be resolved by using the quantity $\alpha_h-\alpha_s$ for some scaling relations,[18] which relate the various critical exponents to each other. In particular, the famous Rushbrooke scaling relation[18] has β as the exponent of the order parameter and γ as that for the isothermal compressibility, which together with the usual definition from (4.3) for α_h yields

$$\alpha_h+2\beta+\gamma=2 . \tag{4.4}$$

The validity of this equation can be seen by a substitution of the values for the critical exponents listed in Table IV. When we introduce the exponents[15] δ from the equation of state, η from the correlation function, and ν from the correlation length, then we find the scaling laws[22] of Widom

$$\gamma=\beta(\delta-1) , \tag{4.5}$$

and of Fisher

$$\gamma=(2-\eta)\nu . \tag{4.6}$$

These three scaling relations can be seen to be completely consistent with the classical values for $d>4$. In fact, the nonrelativistic case with $\sigma=2$ can be seen to give identically the classical values for the exponents for $d>4$, while the ultrarelativistic case with $\sigma=1$ differs from the classical system only for η and ν, when both have the numerical value 1. However, the significant difference from the classical system comes with the Josephson scaling law

$$d\nu=2-\alpha_s , \tag{4.7}$$

which reflects the dependence on the dimensionality in α_s. From Table IV it is also clear that α_h instead of α_s in (4.7) would result in an inequality for $d>2\sigma$. For other dimensionally dependent scaling relations it was shown by C. Hall[18] that the apparent scaling violations can be restored for $d>2\sigma$ by introducing a factor $\alpha_h-\alpha_s$. Thus, the remaining major scaling laws become

$$d\nu-\gamma=2\beta+(\alpha_h-\alpha_s) , \tag{4.8}$$

TABLE IV. A list of some particular critical exponents with stated conditions on the dimensionality in relation to the parameter σ of the energy spectrum given in (1.1).

	α_h	α_s	β	γ	δ	η	ν
$d<\sigma$							
$\sigma<d<2\sigma$	$\frac{d-2\sigma}{d-\sigma}$	$\frac{d-2\sigma}{d-\sigma}$	$\frac{1}{2}$	$\frac{\sigma}{d-\sigma}$	$\frac{d+\sigma}{d-\sigma}$	$2-\sigma$	$\frac{1}{d-0}$
$d>2\sigma$	0	$\frac{2\sigma-d}{\sigma}$	$\frac{1}{2}$	1	3	$2-\sigma$	$\frac{1}{\sigma}$

$$\nu(d-2-\eta)=2\beta+(\alpha_h-\alpha_s)\,, \qquad (4.9)$$

$$2-\eta=d\gamma/(2\beta+\gamma+\alpha_h-\alpha_s)\,, \qquad (4.10)$$

$$d-2+\eta=d(2\beta+\alpha_h-\alpha_s)/(2-\alpha_s)\,. \qquad (4.11)$$

Furthermore, we note that the Eqs. (4.7)–(4.11) all have the dimensionality d entering directly into the formulas, which relates with property of hyperscaling.

We have listed in Table IV the stated critical exponents in terms of their dependence upon d in relation to the power σ of the energy spectrum in (1.1). From the point of view of critical phenomena, this discussion of the dimensionality can be related to the $n=\infty$ line of the spherical model in the $d-n$ plane,[23] where n is the dimension of the order parameter. The nonrelativistic case with $\sigma=2$ is the crossing point of the $n=\infty$ line, with the line for the classical systems at $d=4$. The ultrarelativistic limit represents the lower end of the line $n=\infty$ at $d=2$.

As a conclusion for this section we remark that our calculations show an anomaly in the specific heat, which has the same dimensional behavior as the violations in the hyperscaling relations of (4.7)–(4.11). However, a direct analytical relationship to Δ_d is not apparent since Δ_d given in (3.10) remains constant, which implies that even values C_V very near to T_c always remains bounded. Therefore, we get directly only the classical value $\alpha=0$. Nevertheless, the definition of α_s in (4.2) remains the same so that the given forms of the hyperscaling relations keep their validity. Thus, we have only established the consistency of the relativistic Bose gas with what was already known for its nonrelativistic counterpart.

V. CONCLUSIONS

In this last section we want to bring this work to an end with a discussion of some applications in various branches of physics. Many of these applications arise naturally in relation to synergetic systems,[2, 24] where the structure of the cooperative phenomena are dominated by the competing states of order and chaos (condensate and gas). The simple structure of the ideal relativistic Bose gas included in the energy spectrum (1.2) can be extended to a generalized energy spectrum[25] for certain types of relativistic interactions, which bring about changes in the dimensional character for BEC. In a similar way it can also be seen that the walls (boundaries) can also change drastically some of the conclusions[26] concerning the dimensional structure of BEC. The nature of magnetic confinement[27] of bosons with a magnetic moment (for example, monoatomic hydrogen or deuterium gas) also has a similar effect. Finally, we mention briefly that various processes in nuclear and particle physics in relation to the relativistic many-particle systems show a sort of condensation process.

Although synergetics[24] draws its subject matter from many diverse fields throughout the sciences, we remark upon two of the most important cases in relation to physics, the laser and the electron-hole pairs, as a demonstration of a type of BEC. The laser,[28] even though complicated by nonlinear couplings between its different internal fields, clearly demonstrates in a particular one-dimensional model[29] a condensation into a collective state corresponding to BEC. Although these models are not generally formulated in a manifestly covariant way, their fundamental structure due to the electromagnetic fields is relativistic. Similarly for an excited semiconductor, the particle-hole pair can also undergo a BEC in the form of a droplet[30] instead of forming only the usual gas of excitons. However, it must be further remarked that both of these examples are typical of this branch of science in that they represent systems far from equilibrium in contrast with our above calculations for an ideal relativistic Bose gas.

Another problem enters when we consider the walls of the system. A Bose gas with either attractive or repulsive boundary conditions brings about a significant change in the dimensional structure of the BEC. In fact, it has been shown[26] that with sufficiently attractive boundary conditions at the walls, the BEC in an ideal Bose gas can take place in two or even one dimension.

A closely related problem is the effect of a strong magnetic field, which confines the magnetized Bose particles. This external force changes the relation of the BEC to the dimensionality[27] in a manner similar to an ideal gas of higher dimensions. In particular, it is shown that with a "strong" confinement potential, the related phase transition does not occur. However, in the case of the weak field for a gas with $d=3$ and a field, which confines in a plane (acts only perpendicular to the direction of the magnetic field), it is found that the system possesses the same structure as the ideal system with $d=5$. Thus, there is shown to exist a jump in the specific heat,[27] which corresponds directly with our Δ in (3.12). An additional interesting fact that comes together with the presence of an external field H is the existence of a critical applied field $H_c(T)$, which yields a magnetic induction $B=H-H_c$ for $H>H_c$, and $B=0$ for $H\leq H_c$.

The importance of relativistic properties is seen in relation to various condensation processes at high energy. In the strong interactions between nucleons, the pions at very high densities have a state into which they condense.[31] This possibility has interesting applications in nuclear matter as well as in astrophysics.[32] A similar phenomenon happens at very high energies where the gluons undergo a similar type of condensation.[33] This effect has its importance in the study of the transition between hadronic and quark matter.

After this work was finished we found that similar calculations for Δ_d were in progress. The reported results[34] are similar to our Fig. 2(a).

ACKNOWLEDGMENTS

We would like to thank H. Satz for many discussions and suggestions in relation to this work. We appreciate numerous discussions and helpful remarks on this work by V. V. Dixit. We are also grateful to P. T. Landsberg for discussions with us and for communication of similar work prior to publication. Our work was supported in part by the Deutsche Forschungsgemeinschaft under Contract No. Mi 211/1.

APPENDIX

For the explicit evaluation of C_V we need the integrals $I_d(n,\alpha)$ and $J_d(n,\alpha)$ in (3.5) and (3.6). We want to explicitly analyze $I_d(n,\alpha)$ here. A similar analysis holds for $J_d(n,\alpha)$. We write $I_d(n,\alpha)$ as a sum of two integrals $I_d^1(n,\alpha)$ and $I_d^2(n,\alpha)$ defined by

$$I_d^1(n,\alpha)\equiv\int_0^\epsilon dx\,(x^n+m\beta x^{n-1})(x^2+2m\beta x)^{(d-2)/2}\frac{e^x}{(e^{x-\alpha}-1)^2} \tag{A1}$$

and

$$I_d^2(n,\alpha)\equiv I_d(n,\alpha)-I_d^1(n,\alpha)\,. \tag{A2}$$

For (A1) we choose the real number ϵ in such a way that we may achieve the necessary exactness. The integrand of $I_d^2(n,\alpha)$ contains no singularities so that it vanishes for $x\to\infty$ like $\exp(-x)$. If we substitute $y+|\alpha|$ for x, we may write

$$I_d^1(n,\alpha)=m\beta(2m\beta)^{(d-2)/2}\int_{|\alpha|}^{\epsilon+|\alpha|}dy\,y^{-2}(y+|\alpha|)^{n+(d/2)-2}\,. \tag{A3}$$

In the limit $\alpha\to 0$,

$$I_d^1(n,\alpha)=\begin{cases}\left(m\beta(2m\beta)^{(d-2)/2}\dfrac{1}{n+(d/2)-3}\,y^{n+(d/2)-3}\right)\Bigg|_{|\alpha|}^{|\alpha|+\epsilon} & \text{for } n+(d/2)\neq 3 \quad \text{(A4a)}\\[2ex] \left(m\beta(2m\beta)^{(d-2)/2}\ln y\right)\Bigg|_{|\alpha|}^{|\alpha|+\epsilon} & \text{for } n+(d/2)=3\,. \quad \text{(A4b)}\end{cases}$$

Thus we have the conditions for Δ_d in the limit $\alpha\to 0$ as we stated in (3.10).

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