New expansion technique for the decay of an unstable state

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(Received 4 August 1981)

We show that the stochastic dynamic approach to the decay of an unstable state can be developed in the form of an expansion which allows a systematic resummation of the usual perturbative series in the noise strength. The approach is applied to the laser transient radiation very close to threshold and is shown to give rather good results.

I. INTRODUCTION

In recent times there has been a growing interest in the transient-phenomena characteristic of the decay of an unstable state. From the experimental point of view, the study of these phenomena has been performed in the case of transient-laser radiation,[1,2] spinodal decomposition in fluid mixtures,[3] and hydrodynamic instabilities.[4] Theoretically, the approach based on the Fokker-Planck equation has been widely used both through the calculation of the eigenvalues,[5,6,7] and more recently, with the use of instantons.[8,9] Specific models for the transient-behavior have been formulated by Haake,[10] Arecchi and Politi,[11] Suzuki,[12] and ourselves.[13,14]

In the language of stochastic dynamics the evolution of a system from an unstable situation can be represented as the overdamped motion of a particle (or a field) under the influence of a deterministic and a stochastic force. Because of the random force the system moves from the initial unstable configuration to a final stable one. In the case of systems with a single degree of freedom, the situation is typically represented by the well-known model of a particle in a double-well potential.

The method used in Refs. 12, 13, and 14 to study the transient in this model is to work directly on the stochastic differential equation equivalent to the Fokker-Planck equation. In particular, we tried to approach the problem considering the trajectories of the stochastic process and approximating them directly. The reason for this choice is related to the fact that the trajectories have very simple qualitative features, as shown in Fig. 1. As soon as the system leaves the instability point because of the random fluctuations, it moves along a trajectory which is very close to a deterministic one. As a consequence, since the various trajectories leave the instability point at different instants, a range of times exists in which anomalously large fluctuations are observed. When the particle reaches the stable state in either well, the motion is again essentially determined by the fluctuating force. In Refs. 10—14 an attempt was

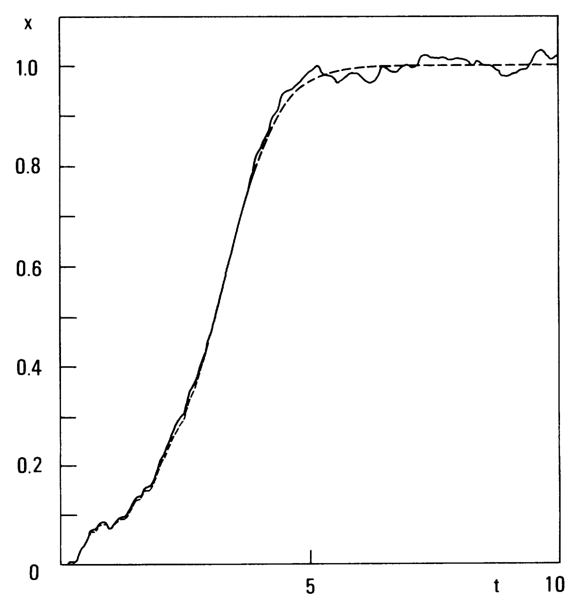

FIG. 1. A typical trajectory of the stochastic process $dx = x(1-x^2)dt + \sqrt{\epsilon}\,dW(t)$ $(\epsilon = 10^{-3})$. The full line represents the computer simulation and the dashed line is the QDT process mentioned in the text.

given to understand the behavior of the system when it starts from an unstable initial state and we refer to it as the quasideterministic theory (QDT). The reason for the use of new methods is due to the failure of a straightforward perturbative expansion in the noise strength.

In our version of the QDT we introduce a stochastic process which approximates very accurately the trajectories of the original process. The detailed scheme for the derivation of the QDT process will be given in Sec. II with reference to the laser-transient radiation. A drawback of the quasideterministic theory is the absence of fluctuations when it is extrapolated to long times. The fluctuations in the stationary state are easily calculated by means of the usual perturbation theory around a stable point, but in this way one needs a connection time to match the results at intermediate times. It would be convenient to formulate a theory which includes the short- and intermediate-times behavior together with the steady state. A first step in this direction has been made in Refs. 12, while in Refs. 14 and 15 we introduced an expansion of the process in terms of the QDT process, in the case of the double-well potential. It turns out that the QDT process gives the anomalous fluctuations which do not scale with the noise strength, while the correcting ones, which arise from the mentioned expansion, give the fluctuations that scale with it. We have also shown that it is necessary to use at least two correction processes to improve the QDT. The results for the first moment are quite satisfactory far from the instability threshold[16] and still meaningful very close to it.[14,15]

A system well suited for a conclusive check of the relevance of the proposed expansion is the switching of a laser, since accurate experiments have been performed both far[1] and close[2] to threshold. In the first case, it has been shown[17] that the QDT describes quite well the laser-intensity buildup and its enhanced fluctuations. In the second case, an accurate numerical solution exists[5] and has been experimentally tested.[2] The QDT only gives qualitative indications in this case, especially for the intensity fluctuations,[17] since it is valid only for very short times. Therefore, it is evident that the correction processes to the QDT play an important role when close to threshold in order to get a good quantitative agreement with experimental data.[2,5] It must be stressed that since we are interested in the intensity and its fluctuations we only consider the process for the field amplitude which is decoupled from the diffusive motion of the phase of the field. The specific problem for the QDT in this case is to take into account the repulsive barrier which prevents the amplitude from becoming negative. In Sec. II we present the well-known model for the unimode laser in terms of Itô's stochastic differential equations (SDE) and we describe the QDT applied to this case. Section III concerns the correction processes giving an analytical approximation for the light intensity of the laser and its fluctuations. In Sec. IV we analyze the results making a comparison with the numerical solution of Ref. 5.

II. THE QDT FOR THE LASER MODEL

The laser model[18] is given by a two-component stochastic process for the electric field $\alpha=\alpha_1+i\alpha_2$ which satisfies Itô's SDE:

$$d\alpha_i(\tau)=\alpha_i(1-|\alpha^2|)d\tau + \sqrt{\epsilon}\,dW_i(\tau) \quad (i=1,2) \tag{2.1}$$

where the variables are normalized as in Ref. 17 and the strength of the noise $\sqrt{\epsilon}$ is related to the pump parameter a, which gives the instability threshold at $a=0$, by $\epsilon=2/a^2$.

The differential of a two-component Wiener process $W_i(\tau)$ is $dW_i(\tau)=W_i(\tau+d\tau)-W_i(\tau)$, with

$$\langle dW_i(\tau)\rangle=0\ ,$$
$$\langle dW_i(\tau)dW_j(\tau')\rangle=\delta_{ij}\delta(\tau-\tau')d\tau\, d\tau' \quad (i,j=1,2)\ . \tag{2.2}$$

In polar coordinates one gets the two processes $r=(\alpha_1^2+\alpha_2^2)^{1/2}$ and $\varphi=\arctan(\alpha_2/\alpha_1)$:

$$dr(\tau)=\left[r(1-r^2)+\frac{\epsilon}{2r}\right]d\tau+\sqrt{\epsilon}\,dW_r(\tau)\ , \tag{2.3}$$

$$d\varphi(\tau)=\frac{\sqrt{\epsilon}}{r}dW_\varphi(\tau)\ , \tag{2.4}$$

with

$$\begin{aligned} dW_r(\tau)&=\cos\varphi\, dW_1(\tau)+\sin\varphi\, dW_2(\tau)\ ,\\ dW_\varphi(\tau)&=-\sin\varphi\, dW_1(\tau)+\cos\varphi\, dW_2(\tau)\ , \end{aligned} \tag{2.5}$$

and we used Itô's calculus rules.[19] The specific difficulty of the model is apparent from the fact that the steady state is degenerate with respect to the phase φ. In fact, φ still performs a diffusive motion for long times when the modulus fluctuates around its stationary equilibrium value.

In order to introduce the QDT we adopt the following procedure. Let us consider the usual perturbative expansion[20] for the stochastic process $r(\tau)$ in terms of $\sqrt{\epsilon}$:

$$r(h,\tau)=z(h,\tau)+\sqrt{\epsilon}z_1(h,\tau)+\epsilon z_2(h,\tau)+\cdots\ , \tag{2.6}$$

with h the initial condition $r(h,0)=h$. The equations for z, z_1, and z_2 are obtained by collecting terms of the order in powers of $\sqrt{\epsilon}$:

$$dz(\tau)=z(1-z^2)d\tau,\ \ z(h,0)=h \tag{2.7}$$

$$dz_1(\tau)=(1-3z^2)z_1 d\tau+dW_r(\tau),\qquad z_1(h,0)=0 \tag{2.8}$$

$$dz_2(\tau)=\left[(1-3z^2)z_2-3zz_1^2+\frac{1}{2z}\right]d\tau,\qquad z_2(h,0)=0\ . \tag{2.9}$$

$z(h,\tau)$ represents the deterministic motion, i.e., the solution of Eq. (2.3) with $\epsilon=0$,

$$z(h,\tau)=\frac{he^{\tau}}{[1+h^2(e^{2\tau}-1)]^{1/2}}\ . \tag{2.10}$$

It is also apparent that the process z_1 gives a large contribution for long times when the initial condition is in the region of the instability (h close to 0). In fact, in that case the term linear in z_1 becomes positive and tends to amplify the effect of the correction. On the contrary, when the initial condition is near a point of stability ($h\approx 1$) the perturbative theory approximates, to order ϵ, the original process. To overcome this difficulty associated with the unstable initial regime, let us consider what happens for small times τ. In this case we can linearize the original processes (2.1),

$$d\bar{\alpha}_i(\tau)=\bar{\alpha}_i d\tau+\sqrt{\epsilon}\,dW_i(\tau)\ \ (i=1,2) \tag{2.11}$$

which are equivalent to the following parametric representation:

$$\bar{\alpha}_i(\tau)=h_i(\tau)e^{\tau},\qquad dh_i(\tau)=\sqrt{\epsilon}\,e^{-\tau}dW_i(\tau)\ . \tag{2.12}$$

As a consequence we can write the linearized process $\bar{r}(\tau)$ corresponding to $r(\tau)$ as

$$\bar{r}(\tau)=h(\tau)e^{\tau}\ , \tag{2.13a}$$

$$dh(\tau)=\frac{\epsilon}{2h}e^{-2\tau}d\tau+\sqrt{\epsilon}\,e^{-\tau}dW_r(\tau)\ , \tag{2.13b}$$

with $h(\tau)=(h_1^2+h_2^2)^{1/2}$ and $dW_r=(h_1/h)dW_1+(h_2/h)dW_2$. The set of equations (2.13) represents a mapping between the two stochastic processes $\bar{r}(\tau)$ and $h(\tau)$ which has the same form as the correspondence between $\bar{r}$ and its initial condition in the deterministic motion. The essential point of the QDT is to identify the initial condition h in Eq. (2.10) as the same stochastic process as the one in Eq. (2.13b). $z(h,\tau)$ thus becomes a stochastic process represented parametrically by

$$z(h(\tau),\tau)=h(\tau)e^{\tau}[1+h^2(\tau)(e^{2\tau}-1)]^{-1/2}\ , \tag{2.14a}$$

$$dh(\tau)=\frac{\epsilon}{2h(\tau)}e^{-2\tau}d\tau+\sqrt{\epsilon}\ e^{-\tau}dW_r(\tau)\ . \tag{2.14b}$$

In a sense, the QDT amounts to a resummation of the perturbative expansion (2.6) in terms of the strength of the random noise. The reason why we expect the QDT process $z(h(t),\tau)$ to be a good candidate to represent the original process $r(\tau)$, is that it has all the qualitative characteristics we described earlier in the initial and transient regime. In fact, close to the instability point [i.e., for small times when $h^2(\tau)(e^{2\tau}-1)<<1$] the two processes z and r coincide. For longer times ($e^{\tau}>>1$) the single trajectory of $h(\tau)$ reaches a constant value and as a consequence the motion of z becomes deterministic. As we already noted, the trajectories of z remain deterministic as time goes on and therefore they are unable to describe fluctuations around the steady state. As we mentioned in the Introduction, the mapping (2.14) gives very good results when applied to a laser well above threshold.[17]

III. CORRECTION PROCESSES

A systematic way of determining the limits of validity of the QDT and to improve it is to introduce an expansion of the form of Eq. (2.6), but with the essential difference that h must now be interpreted as the stochastic process (2.14b) and not as the initial condition of the motion. Consequently, the correction processes obtainable from Eq. (2.3) become, to order ϵ,

$$dz_1(\tau)=(1-3z^2)z_1 d\tau+\left[1-e^{-\tau}\frac{\partial z}{\partial h}\right]dW_r(\tau),\quad z_1(0)=0 \tag{3.1}$$

$$dz_2(\tau)\left[(1-3z^2)z_2-3zz_1^2+\frac{1}{2}\left[\frac{1}{z}-\frac{e^{-2\tau}}{h}\frac{\partial z}{\partial h}\right]-\tfrac{1}{2}e^{-2\tau}\frac{\partial^2 z}{\partial h^2}\right]d\tau\,,\quad z_2(0)=0\,. \tag{3.2}$$

Note that z_1 represents oscillations around the quasideterministic motion, while z_2 gives a systematic correction with respect to z. Both processes would be amplified by the linear terms close to the instability, but the driving terms are vanishingly small in the same region and this avoids the divergences typical of the ordinary perturbation theory. Actually, z_1 gives negligible contributions— and *a fortiori* z_2—for

$$\langle |h|^2\rangle(e^{2\tau}-1)<<1\,, \tag{3.3}$$

where the average is taken with respect to the probability distribution function of the process $h(\tau)$, that is,

$$P(h,\tau)=\frac{1}{\pi\sigma^2(\tau)}e^{-|h|^2/\sigma^2(\tau)}\,, \tag{3.4}$$

$$\langle |h|^2\rangle=\sigma^2(\tau)=\epsilon(1-e^{-2\tau})\,. \tag{3.5}$$

We are thus able to give a self-consistent estimate of the times interval during which the correction processes are negligibly small:

$$0\le\tau<<\tfrac{1}{2}\ln\left[1+\frac{1+\sqrt{1+4\epsilon}}{2\epsilon}\right]. \tag{3.6}$$

The asymptotic solutions of Eqs. (2.14), (3.1), and (3.2) for $\tau\to\infty$ are

$$\begin{aligned} z&=1\,,\\ z_1&=\int_0^\tau dW_r(\tau')e^{-2(\tau-\tau')}\,,\\ z_2&=\int_0^\tau d\tau'(\tfrac{1}{2}-3z_1^2)e^{-2(\tau-\tau')}\,, \end{aligned} \tag{3.7}$$

and give the correct analytic contribution up to order ϵ to the averages

$$\begin{aligned} \langle r^2\rangle_{\rm st}&=\langle z^2\rangle_{\rm st}+2\sqrt{\epsilon}\langle zz_1\rangle_{\rm st}\\ &\quad+\epsilon\langle z_1^2\rangle_{\rm st}+2\epsilon\langle zz_2\rangle_{\rm st}=1\,,\\ \langle(\delta r^2)^2\rangle_{\rm st}&=\langle r^4\rangle_{\rm st}-\langle r^2\rangle_{\rm st}^2=\epsilon\,, \end{aligned} \tag{3.8}$$

where $\langle\cdots\rangle_{\rm st}$ means the average with $P(h,\tau)$ of Eqs. (3.4) and (3.5) as $\tau\to\infty$. The nonanalytic contributions are clearly impossible to get with an expansion of the type we use. They are presumably associated with the rare trajectories which go from the stable to the unstable equilibrium position. The value of the latter contribution should however be negligibly small at least for the moments of r.

The full solutions of Eqs. (3.1) and (3.2) are

$$z_1(\tau)=\int_0^\tau dW_r(\tau')\left[1-e^{-\tau'}\frac{\partial z}{\partial h}\right]\exp\left[\int_{\tau'}^\tau d\tau''[1-3z^2(\tau'')]\right], \tag{3.9}$$

$$z_2(\tau)=\int_0^\tau d\tau'\left[-3zz_1^2+\frac{1}{2}\left[\frac{1}{z}-\frac{e^{-2\tau'}}{h}\frac{\partial z}{\partial h}\right]-\tfrac{1}{2}e^{-2\tau'}\frac{\partial^2 z}{\partial h^2}\right]\exp\left[\int_{\tau'}^\tau d\tau''[1-3z^2(\tau'')]\right], \tag{3.10}$$

together with Eqs. (2.14), where z_1 is given by a stochastic integral, and z_2 by an integral over stochastic functions.

A direct way of computing the averages by means of Eqs. (2.14), (3.9), and (3.10) is to calculate the realizations of z_1 and z_2 on a computer. This has been done and gives very good results when compared with the numerical solution of the Fokker-Planck equation of Ref. 5.

An analytical approximation for the contributions of the processes z_1, z_2 to the first nonzero moments of r can be easily obtained at least in the asymptotic regime where they are relevant. In this regime we can consider the stochastic process $h(\tau)$ as slowly varying in time with respect to the Wiener process $W_r(\tau)$. We approximate Eqs. (3.9) and (3.10) considering h to the upper time τ. Since $[1-e^{-\tau'}(\partial z/\partial h)]\simeq 0$ in the time interval (3.6) and $e^{-\tau'}(\partial z/\partial h)\simeq 0$ outside it, we can approximate Eq. (3.9) by

$$z_1(h(\tau),\tau)\simeq\int_0^\tau dW_r(\tau')e^{-2(\tau-\tau')}\exp\left[3\int_{\tau'}^{\tau}d\tau''[1-z^2(h(\tau),\tau'')]\right] . \tag{3.11}$$

Moreover, since only the expectation with respect to $W_r(\tau)$ is used we can write

$$z_1^2(h(\tau),\tau)\simeq\int_0^\tau d\tau' e^{-4(\tau-\tau')}\exp\left[6\int_{\tau'}^{\tau}d\tau''[1-z^2(h(\tau),\tau'')]\right] , \tag{3.12}$$

considering

$$1-z^2(h(\tau),\tau'')\simeq\frac{1-h^2(\tau)}{h^2(\tau)}e^{-2\tau''}<<1$$

we obtain

$$z_1^2(h(\tau),\tau)\simeq 1-\tfrac{3}{4}z^2(h(\tau),\tau) , \tag{3.13}$$

where $z(h(\tau),\tau)$ is given by Eq. (2.14). On the same grounds, after some algebra, we obtain for the process z_2 the approximate expression

$$z_2(h(\tau),\tau)\simeq-\tfrac{3}{8}z(h(\tau),\tau)+\frac{1}{4z(h(\tau),\tau)} . \tag{3.14}$$

As far as the calculation of the moments is concerned, these two approximations are valid up to the leading order in a formal expansion in $1-z^2(h(\tau),\tau)$, which obviously vanishes for $\tau\to\infty$. To this order we obtain

$$\langle r^2(\tau)\rangle\simeq(1-\tfrac{3}{2}\epsilon)\langle z^2(\tau)\rangle+\tfrac{3}{2}\epsilon , \tag{3.15}$$

$$\langle[\delta r^2(\tau)]^2\rangle\simeq(1-6\epsilon)[\langle z^4(\tau)\rangle-\langle z^2(\tau)\rangle^2] -3\epsilon\langle z^2(\tau)\rangle^2+4\epsilon\langle z^2(\tau)\rangle . \tag{3.16}$$

A systematic approach that is capable to justify rigorously the previous approximation and to improve it can be introduced with the procedure of Bouc and Pardoux.[15,21]

IV. RESULTS AND CONCLUSIONS

From Eqs. (3.15) and (3.16) we see that up to order ϵ the laser intensity and its fluctuations are given in terms of the moments of the quasideterministic approximation which from (2.14), (3.4), and (3.5) are[17]

$$\langle z^2(\tau)\rangle=\frac{e^{2\tau}}{e^{2\tau}-1}\int_0^\infty dy\, e^{-y}y[\eta(\tau)+y]^{-1} =\frac{e^{2\tau}}{e^{2\tau}-1}U(1,0,\eta(\tau)) , \tag{4.1}$$

$$\langle z^4(\tau)\rangle=2\left[\frac{e^{2\tau}}{e^{2\tau}-1}\right]^2U(2,0,\eta(\tau)) , \tag{4.2}$$

where $U(\cdots)$ is the confluent hypergeometric function[22] and

$$\eta^{-1}(\tau)=\epsilon(1-e^{-2\tau})(e^{2\tau}-1) . \tag{4.3}$$

In Figs. 2 and 3 the results for the laser-intensity buildup $a\langle r^2\rangle$ and its fluctuations $a^2\langle(\delta r^2)^2\rangle$ (curves labeled TH) are plotted as functions of τ/a for $a=8$. The results of the QDT are also shown, together with the computation of Risken and Vollmer[5] (curves labeled RV). These quite nontrivial results make us confident of the method we developed above and of the validity of the approximation we used.

As a conclusion we want to stress that the point of view we assumed at the beginning of this work, i.e., to consider and approximate directly the trajectories of the process appears to be successful. It

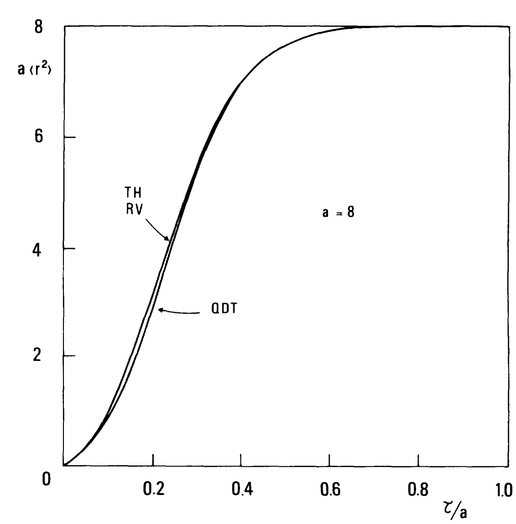

FIG. 2. Transient-laser intensity as a function of time for the value $a=8$ of the pump parameter. Comparison of the solution of Ref. 5 (RV), the QDT, and the result of this work (TH).

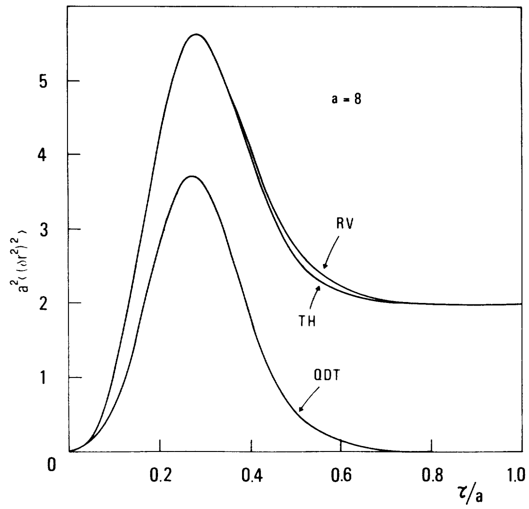

FIG. 3. The same as in Fig. 2 for the transient-laser-intensity fluctuations.

would be of interest to assume the same point of view also in the case of the decay from an unstable configuration for a system with infinite degrees of freedom.

As far as a comparison with the traditional eigenfunction expansion of the Fokker-Planck equation is concerned,[5] we note that this method requires cumbersome numerical calculations of many eigenvalues in the initial regime. Moreover, it allows to take into account the long-time behavior associated, in our scheme, with the trajectories very far from the deterministic ones. These trajectories give irrelevant contribution to the statistical moments but are essential to explain the long-time behavior of correlation functions. The expansion around the QDT process fails to describe such trajectories.

We finally note that recently Suzuki[23] proposed several approximate methods for the calculation of the moments with the inclusion of the corrective process z_1, but no check has been performed to test their validity. Moreover, the fact that the process z_2 is not considered should fail to give the correct behavior at intermediate times and in the steady state.

This work was supported by Gruppo Nazionale di Struttura della Materia del Consiglio Nazionale delle Ricerche, Rome, Italy.

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