PHYSICAL REVIEW A VOLUME 25, NUMBER 1 JANUARY 1982

Harmonic generation in the free-electron laser. II. cw calculation for the linearly polarized wiggler

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(Received 3 August 1981)

Harmonic generation in the free-electron laser offers a possible means to extend the wavelength range of the device towards high frequency. Numerical solutions to the basic equations describing this process are shown for cw operation using a linearly polarized wiggler. Higher harmonic emission becomes enhanced as the magnetic field is increased and as the energy spread in the electron beam is reduced.

One of the most attractive features of free-electron lasers (FEL) is that they can be made to lase over a wide spectral domain, which, in principle, could extend from the infrared to at least the ultraviolet. In these devices a highly relativistic electron beam (with energy $mc^2\gamma$) passes through a periodic static field of period λ_q, and amplification is achieved in the direction of the electron beam at a wavelength[1] $\lambda_s \simeq \lambda_q/2\gamma^2$. Most theoretical work on the FEL has been done using a helical wiggler (circularly polarized magnetic field). This magnet gives a narrow gain bandwidth near the lasing frequency. With an arbitrary nonperiodic magnet geometry the gain, in general, is not concentrated near any one frequency and is consequently small. However, in the case of a nonhelical periodic or quasiperiodic magnet, the gain can be appreciable at the fundamental frequency ω_s, while higher harmonics $n\omega_s$ of this frequency are generated in the forward direction by the nonlinear dynamics of the FEL.[2,3] Therefore, harmonic generation may provide a means to extend the tunable range of the FEL to shorter wavelengths in the uv. Mechanisms of harmonic generation in the FEL have been also described by other workers.[4–6]

In this paper we investigate higher harmonic emission in the cw regime of the FEL with a linearly polarized magnetic field, and present numerical results for the case that a strong laser field at the fundamental frequency produces harmonic up conversion in a single pass. For the linear magnet only odd harmonics $(3\omega_s, 5\omega_s, \ldots)$ can be generated.[3,6] Higher harmonic emission becomes enhanced as the magnetic field is increased and as the energy spread in the electron beam is reduced. However, the larger the magnetic field, the smaller the magnet wavelength must be in order to keep the wavelength λ_s of the fundamental radiated field in a given range. We calculate the amount of harmonic generation in a single pass, assuming that only the fundamental frequency ω_s is present at the entrance of the wiggler.

The basic equations describing optical harmonic generation in the FEL are obtained through the technique of multiple-scaling perturbation theory.[7] The perturbation procedure leads in the lowest order to the following coupled Maxwell and single-particle equations[3] for the cw operation of the FEL:

$$\sigma_n^*(\xi)\partial[\sigma_n(\xi)E_n(\xi)]/\partial\xi = \frac{e}{2mc\gamma_0}A_n(\xi)\int d\mu_0 \mathscr{F}(\mu_0)\frac{1}{2\pi}\int_0^{2\pi} d\theta_0 \exp[-in\hat{\theta}(\xi,\mu_0,\theta_0)] \;, \tag{1}$$

$$d\hat{\theta}(\xi,\mu_0,\theta_0)/d\xi = \hat{\mu}(\xi,\mu_0,\theta_0) \;, \tag{2}$$

$$d\hat{\mu}(\xi,\mu_0,\theta_0)/d\xi = -\frac{e^2 k_s}{2m^2c^3\gamma_0^4}\sum_{n\neq 0}\{A_n^*(\xi)E_n(\xi)\exp[in\hat{\theta}(\xi,\mu_0,\theta_0)] + \text{c.c.}\} \;. \tag{3}$$

In these equations, k_s is the wave vector of the fundamental mode, which is related to the magnet wave vector k_q according to the Doppler up-shift condition $k_s = 2\gamma_0^2 k_q/\bar{\Delta}$, where $\bar{\Delta}$ is the average of the mass shift over a magnet period and $mc^2\gamma_0$ is the center of the ultrarelativistic distribution of electron energies. Furthermore, ξ is a generalized position coordinate which includes the effect of relativistic mass shift and is defined by

$$\xi = \int_0^z \Delta(z')dz' \ . \tag{4}$$

Also, θ_0 and μ_0 are the initial phase angles and energy detunings at the entrance ($\xi=0$) to the magnet, $\mathscr{I}(\mu_0)$ is the current per unit interval in μ_0 entering the magnet, and $A_n(\xi)$ are the complex coefficients in an appropriate Fourier-series expansion of the periodic wiggler.[3] The amplitudes $E_n(\xi)$ are the on-axis optical electric field amplitudes for radiation at frequency nck_s. The angle $\hat{\theta}$ measures the position of electrons with respect to the bunching potential, and the energy-detuning variable $\hat{\mu}$ is defined by

$$\gamma = \gamma_0(1+\gamma_0^2\hat{\mu}/k_s) \ . \tag{5}$$

The functions $|\sigma_n(\xi)|^2$ are proportional to the laser mode area and account approximately for diffractive beam spreading and phase shift. We see from Eq. (1) that, if $A_n \neq 0$, radiation of frequency $n\omega_s$ will be generated even if not initially present. All that is required is that the electron bunching have higher harmonic content, which is the case if the FEL is saturated. We assume the radiation at frequency ω_s lases in the fundamental Gaussian mode of a resonator. In this case the complex function $\sigma_1(\xi)$ for the fundamental mode is determined by projecting Maxwell's equation onto the Gaussian mode,[7] which gives (assuming the beam waist is located in the center of the wiggler)

$$\sigma_1(\xi) = (\tfrac{1}{2}\pi\epsilon_0 c)^{1/2} w_1[1+i(2\xi-L')/\overline{\Delta}k_s w_1^2] \ , \tag{6}$$

where $L'=\overline{\Delta}L$ is the effective magnet length and w_1 is the beam waist.

If we now specialize to the case of a uniform linearly polarized wiggler[3] (indicated by subscript l), then A_n is constant and the linear gain for the harmonics can be easily calculated in the small-gain, small-diffraction, small-signal, cold-beam regime, where $E_n(\xi)$ is approximately constant and the harmonics are uncoupled and can be treated separately. By using Eqs. (1)–(3), the gain per pass in this regime is found to be[8]

$$G_n = \frac{ne^3 IL^3 \overline{\Delta}_l^2 k_l A_n^2}{\pi\epsilon_0 m^3 c^5 \gamma_0^3 w_n} \sin\eta(\eta^{-3}\sin\eta - \eta^{-2}\cos\eta) \ , \tag{7}$$

where $\eta = \tfrac{1}{2}n\mu_0 L'$, I is the current, and w_n is the beam waist of the nth harmonic (mode area $=\tfrac{1}{2}\pi w_n^2$). The mode area is proportional to λ_s/n for a given resonator and so decreases for the higher harmonics, which tends to increase the gain. However, to ensure optimal interaction between the field and the electrons, the field mode area should be larger than the electron-beam area. This is the case in the calculations presented in this paper. Equation (7) is the usual antisymmetric gain formula, but we notice that for higher harmonics the gain bandwidth narrows, so a higher electron-beam quality is needed to generate higher harmonics. Figure 1 shows the maximum gain (in the small-gain, small-diffraction, small-signal, cold-beam regime) for the fundamental and the first few harmonics, using numbers typical of the Stanford experiment, but for a linear magnet. We see that the gain of the harmonics can easily be too small for lasing (even smaller when diffraction and finite beam aperture are taken into account). However, the optical harmonics can be generated in appreciable amount (even when below threshold) by radiation from higher harmonics in the bunched electron-beam density where the bunching is created by saturating radiation at the fundamental frequency. The amount of harmonics generated is determined by the magnet strength, the magnet wavelength, the electron energy, and the electron current, as we show below.

In this work we investigate harmonic generation in a single pass through the wiggler, where we assume that initially the harmonics are not present [i.e., $E_n(0)=0$; $n=3,5,\ldots$]. The harmonics are not supposed to be contained in the resonator, so that a modal projection is not appropriate for them. Instead, to account for diffractive spreading, we need to replace Eq. (1) by the paraxial wave equation

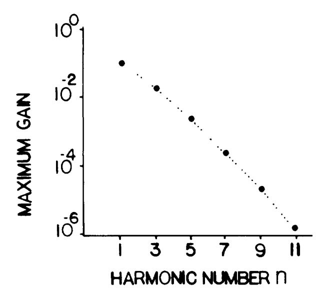

FIG. 1. Maximum gain is shown of the first few harmonics in the small-gain, small-diffraction, and small-signal limits ($B_{\text{peak}}=0.24$ T, $\lambda_l=3.2$ cm, $E=43$ MeV, $I=1$ A).

$$\left[\frac{\partial}{\partial\xi}+\frac{1}{2i\bar{\Delta}nk_s}\nabla_T^2\right]E_n(r,\xi)=\frac{e}{2\epsilon_0 mc^2\gamma_0}u(r)A_n(\xi)\int d\mu_0\mathscr{F}(\mu_0)\frac{1}{2\pi}\int_0^{2\pi}d\theta_0\exp[-in\hat{\theta}(\xi,\mu_0,\theta_0)]\ , \tag{8}$$

where ∇_T^2 is the transverse Laplacian and $u(r)$ is the normalized transverse distribution of the electrons, which we take to be a Gaussian. $E_n(r,\xi)$ are the optical field amplitudes for oscillation at frequency nck_s ($n=3,5,\ldots$). A solution to the wave equation (8) is given by

$$E_n(r,\xi)=\frac{e}{2\pi\epsilon_0 mc^2\gamma_0}\int_0^{\xi}d\xi'\frac{A_n(\xi')}{a_0+2i(\xi-\xi')/\bar{\Delta}nk_s}\exp\{-r^2/[a_0+2i(\xi-\xi')/\bar{\Delta}nk_s]\}$$

$$\times\int d\mu_0\mathscr{F}(\mu_0)\frac{1}{2\pi}\int_0^{2\pi}d\theta_0\exp[-in\hat{\theta}(\xi',\mu_0,\theta_0)]\ , \tag{9}$$

where a_0 is the electron beam area.

Most proposed FEL devices (some under construction) will use linearly polarized magnets.[9] In the present work we concentrate only on the uniform linearly polarized magnet; such a magnet is described by the function

$$\hat{A}_q(z)=2^{1/2}A_l\cos(k_l z)\ , \tag{10}$$

where k_l is the magnet wave vector and A_l is the constant rms amplitude. Although the wiggler field does not have higher harmonic content when viewed as a function of z, such content is nevertheless effectively created by the mass shift. However, for the linearly polarized magnet the even harmonics vanish. For the odd harmonics the coefficients A_n in the relevant Fourier expansion of the wiggler field are given by[3,6]

$$A_n=\frac{A_l}{\sqrt{2}\,\bar{\Delta}_l}(-1)^{(n-1)/2}[J_{(n-1)/2}(n\delta)-J_{(n+1)/2}(n\delta)],\quad n=1,3,5,\ldots \tag{11}$$

where $\delta=\frac{1}{2}(1-1/\bar{\Delta}_l)$ and the average mass shift $\bar{\Delta}_l$ is given by

$$\bar{\Delta}_l=1+\frac{e^2A_l^2}{m^2c^2}\ . \tag{12}$$

One sees from Eq. (9) that light at higher optical frequencies, even if not initially present [$E_n(0)=0$, $n=3,5,\ldots$], can be radiated by higher harmonics in the electron bunching present in the strong-signal regime. In the present work, we assume that the incident electron energy distribution is a Gaussian and that the electron bunching is influenced by the fundamental field only [i.e., the harmonic radiation intensities are sufficiently weak that we can keep only the $n=\pm 1$ terms in Eq. (3)]. Also, we assume that the harmonics initially are not present and leave the cavity at the end, while the fundamental mode is reflected back. Since the wiggler is strictly periodic, the "slow" generalized position coordinate ξ can be replaced by $\xi=\bar{\Delta}_l z$, so Eq. (3) can be rewritten as

$$d\hat{\mu}(z,\mu_0,\theta_0)/dz=-\frac{e^2\bar{\Delta}_l k_s A_1}{m^2c^3\gamma_0^4}\{E_1(z)\exp[i\hat{\theta}(z,\mu_0,\theta_0)]+\text{c.c.}\}\ , \tag{13}$$

while Eqs. (1) and (2) may be simply rewritten by replacing $d\xi$ by $\bar{\Delta}_l dz$. Note that, since it is consistent to take $E_{-n}=E_n^*$, it is sufficient to keep track of the amplitudes for positive n. The power in the fundamental mode is thus $2\,|\sigma_1 E_1|^2$. From Eq. (9), the power of the harmonics can be simply calculated to be

$$P_n(z)=\frac{e^2\bar{\Delta}_l^2A_n^2}{4\pi\epsilon_0 m^2c^3\gamma_0^2}\int_0^z\int_0^z dz'dz''\frac{F_n(z')F_n^*(z'')}{a_0-i(z'-z'')/nk_s}\ , \tag{14}$$

where

$$F_n(z)=\int d\mu_0\mathscr{F}(\mu_0)\frac{1}{2\pi}\int_0^{2\pi}d\theta_0\exp[-in\hat{\theta}(z,\mu_0,\theta_0)]\ .$$

In the numerical results presented below we consider how the harmonic power varies as we vary (one at a time) several of the physical parameters of the linearly polarized FEL system, starting from a configuration with the following parameters: wiggler wavelength, $\lambda_l = 3$ cm; wiggler length, $L = 5.2$ m; electron beam area, $a_0 = 0.1$ mm^2; electron energy, $E = 43$ MeV; fundamental optical wavelength, $\lambda_s = 5.75$ μm; electron current, $I = 2$ A; peak magnetic field, $B = 0.66$ T; energy detuning, $\mu_0 L' = 5$; initial HWHM energy spread, $L'\Delta\mu = 0.1$; input optical power at the fundamental wavelength, $P_{\text{in}} = 65.9$ kW; optical beam waist, $w_1 = 2.23$ mm. The variables λ_s and w_1 are not kept fixed, but are regarded as dependent variables determined by the values of E, λ_l, and B.

An important parameter is the energy detuning. In Figs. 2(a) and 2(b) we show the fundamental gain and the harmonic power (in the strong-signal regime with diffraction) as a function of energy detuning. We see that, the higher the harmonic, the smaller the energy detuning at which maximum harmonic generation occurs. If the device operates as a laser oscillator at the fundamental frequency with the energy detuning giving maximum gain for the fundamental mode, this does not correspond to the optimal detuning for the generation of harmonics. However, it might be possible to shift the detuning to smaller values by use of an intracavity etalon. Of course, in a situation where the FEL is used simply as an amplifier, the detuning is completely at our disposal. In a future publication we will present calculations for the pulsed FEL for situations where the harmonics themselves can contribute to the lasing.

Another important parameter governing the amount of harmonic generation is the spread in the initial electron energy distribution (assumed here to be a Gaussian). From Fig. 3(a) we observe that the higher harmonics are increasingly sensitive to the energy spread, which means that a higher beam quality is needed for higher harmonic generation.

Now we discuss how harmonic generation is affected by the magnetic field, the magnet wavelength, and the electron energy. Figure 3(b) shows the dependence of the harmonic power on the magnetic field (peak value). We see that at no value of B can a higher harmonic produce more intensity than a lower one. For example, to generate 200 W in the third harmonic requires $B = 0.312$ T and $\lambda_3 = 0.978$ μm, but the same power in the ninth harmonic requires $B = 1.66$T and $\lambda_9 = 1.22$ μm. Nevertheless, harmonic generation is still very

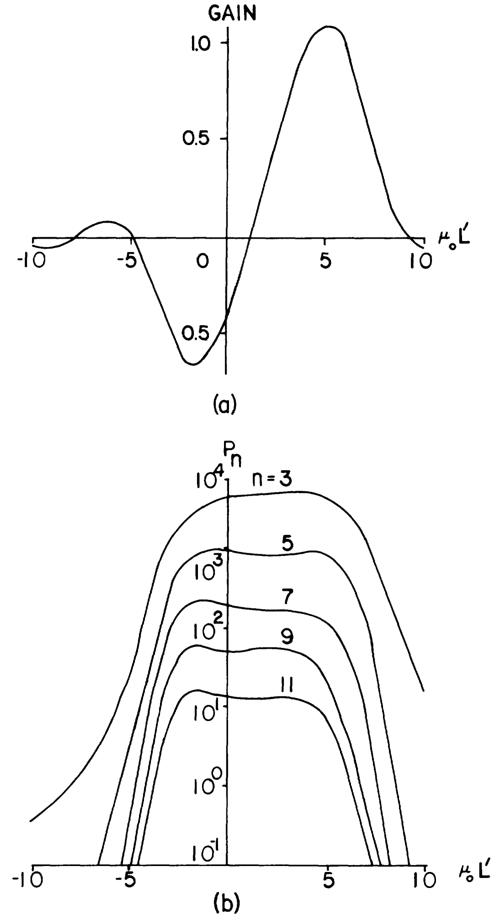

FIG. 2. In (a) the gain of the fundamental mode in the strong-signal regime with diffraction is given as a function of energy detuning $\mu_0 L'$. In (b) the harmonic power in watts as a function of energy detuning is shown for higher harmonics. The energy detuning giving maximum power is less than the detuning giving maximum fundamental gain. The HWHM of the energy distribution is $\Delta\mu = 0.12$ ($B = 0.66$ T, $\lambda_l = 3$ cm, $E = 43$ MeV).

helpful in making the device able to generate appreciable power at shorter wavelengths. For example, let us consider the power generated by a FEL with $B = 0.66$T, $\lambda_l = 3$ cm, $E = 43$ MeV, and an output mirror with reflectivity 97% at the fundamental frequency. We assume oscillation at the point of maximum gain of the fundamental, $\mu_0 L' = 5$. The input fundamental power in this case is 1.84 MW which is larger than optimal for harmonic generation. At the entrance of the wiggler the harmonics are not present. The mirror is assumed to be transparent to the harmonics. Then the powers leaving the cavity are $P_1 = 113$ kW at $\lambda_s = 5.75$ μm; $P_3 = 2025$ W; $P_5 = 253$ W;

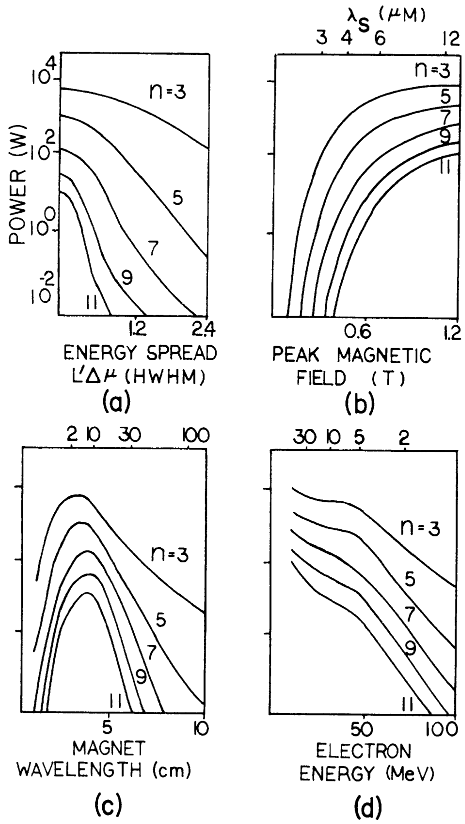

FIG. 3. Harmonic power P_n in watts is shown: in (a) as a function of initial electron energy spread, in (b) as a function of peak magnetic field, in (c) as a function of the magnet wavelength, and in (d) as a function of the electron energy. The upper axis in (b)–(d) is the fundamental optical wavelength $\lambda_s(\mu m)$.

$P_7 = 52$ W; $P_9 = 22$ W; $P_{11} = 6$ W, where $\lambda_n = \lambda_s / n$.

In Fig. 3(c) the harmonic powers are shown as a function of the magnet wavelength. One can see that for the higher harmonics a slightly larger magnet wavelength is optimal. Although the range of magnet wavelength which gives appreciable harmonic generation is fairly limited, assuming other FEL parameters are held fixed, the wavelength of the radiated field [Eq. (15)] is minimized by keeping the magnet wavelength at the lower end of this range.

The decrease in harmonic power at short magnet wavelengths comes about because of the decrease in mass shift, which leads to a reduction in the harmonic coefficients A_n. The power decrease at long magnet wavelengths is caused by the larger mode area at longer λ_s. Since the input power is held constant, on-axis intensity decreases and higher harmonics in the electron bunching become weaker.

If we vary the electron energy, Fig. 3(d) shows that the power of the harmonics decreases as the electron energy is increased. We notice that it decreases faster for $E > 50$ MeV ($B = 0.66$ T, $\lambda_l = 3$ cm). However, one must keep in mind that the wavelength of the radiated field is also decreasing as $\lambda_s \propto E^{-2}$. For $E = 100$ MeV the wavelength of the radiated field in the ninth harmonic is about 1000 Å.

To a large extent one can compensate for the decrease in harmonic power at large E by increasing the power at the fundamental frequency, but eventually the initial energy spread becomes a limiting factor, preventing strong harmonics in the electron bunching and reducing the gain at the fundamental frequency.

Finally, we want to look at the dependence of the harmonic powers on the input fundamental power. The harmonic powers are very small in the small-signal regime, because the pondermotive force is too small to get the electrons bunched in phase. Harmonic generation becomes appreciable in the strong-signal regime, but decreases again somewhat if the fundamental power is too large. This is shown in Fig. 4(a) and is what one would suspect, because the larger the signal the faster the saturation. If the fundamental power is excessive, the electron bunching required for harmonic generation gets smeared out well before the end of the wiggler. By looking at the harmonic power as a function of position through the magnet [Fig. 4(b)], we see that one can do better in the very strong-signal regime (VSSR) by reducing the length of the magnet. In the moderately strong-signal regime (MSSR) the maximum power is generated at the

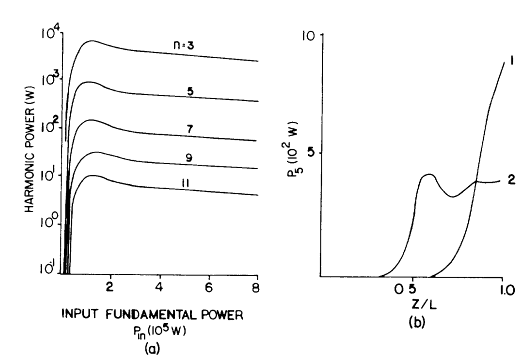

FIG. 4. In (a) harmonic power is given as a function of the input fundmental power P_{in}. In (b) the power of the fifth harmonic P_5 is shown as a function of position in (1) a moderately strong-signal case with $P_{in} = 33$ kW, and (2) for very strong-signal operation with $P_{in} = 297$ kW.

end of the magnet, while in the VSSR, harmonic generation starts early and the maximum harmonic power is generated before the end of the magnet. This power is not as large as that generated in the MSSR, where the power peaks at the end of the magnet. To understand this behavior we look at the electron phase-space distribution. In the MSSR the electrons deaccelerate as they pass through the magnet and get bunched in phase, keeping bunched until the end of the magnet, as shown in Fig. 5(a). But in the VSSR [Fig. 5(b)] the electrons deaccelerate and get bunched early in their traversal of the wiggler. As they continue their path, they start to accelerate again and lose their strong bunching. The maximum power in the MSSR is larger than that in the VSSR because the electrons in the first case continue to be bunched in phase for a long distance. That gives the electrons the chance to produce more radiation. In the second case they remain bunched in phase for a shorter distance.

In conclusion, harmonic generation adds another mechanism for extending the spectral range of the FEL. The tunable range of the device can be further extended by this means toward UV. In the present work we looked at the case where the harmonics were too weak to appreciably affect the electron motion and where the harmonics were not present at the entrance of the wiggler. We saw that the harmonics could be generated in appreciable amounts by the nonlinearity of the device in the strong-signal regime and by using a strong magnetic field ($B_{peak} > 0.62$ T). One difficulty is that harmonic generation needs a narrow distribution in the electron energies. In a future publication we will present short pulse calculations of harmonic generation, taking into account the combined influence of the harmonics and the fundamental mode in driving the electron bunching, and we will look at the possibility of getting the harmonics to lase.

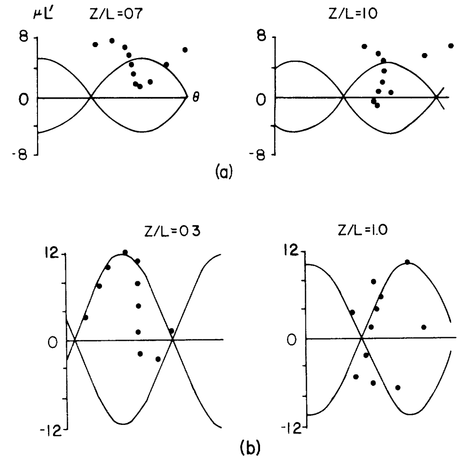

FIG. 5. Phase-space electron distribution is graphed (a) in the moderately strong-signal case $P_{in} = 33$ kW, and (b) in the very strong-signal case $P_{in} = 297$ kW. Notice that in case (b) the electrons get bunched in phase early in the wiggler, and that the bunches have largely washed out by the end of the wiggler.

This research was supported by the Office of Naval Research.

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Wash.; G. Neil at TRW, Redondo Beach, California; H. Winick at SPEAR, Stanford, California; C. Pelligrini at Brookhaven National Laboratory, Upton, New York; V. Baier at Novosobirsk, USSR; L. Elias at University of California at Santa Barbara; and Y. Farge at Orsay. J. M. J. Madey at the ONR Workshop on Free-Electron Lasers, Sun Valley, Idaho, 1981, has reported observations of second-harmonic emission from the Stanford helical wiggler. It is not known if the emission is coherent. Such emission is not predicted by our theory, although off-axis incoherent harmonic radiation has been calculated for helical wiggler by W. B. Colson (Ref. 6).