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Laser cooling of ions stored in harmonic and Penning traps

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Laser (light-pressure) cooling of two-level ions stored in a Penning trap is treated theoretically, in the limit that the frequencies of motion of the ions are much smaller than the natural linewidth of the optical transition. Rate equations for the mean-squared amplitudes of motion are derived from a semiclassical analysis, which is confirmed by quantum-mechanical perturbation theory. Simultaneous cooling of all three modes of motion is shown to require a spatially nonuniform laser-beam profile, unlike the case of the harmonic trap. Comparison is made with recent experiments. Also, the three-dimensional harmonic trap is treated by two simple methods. One is based on the energy rate equations and the other on Langevin-type equations. Identical results are obtained by the two methods for the steady-state energies. These results are compared with the works of others.

I. INTRODUCTION

Laser (light-pressure) cooling of atoms and atomic ions has been demonstrated in recent experiments which were stimulated by the original proposals for laser cooling of free atoms[1] and of ions bound in electromagnetic traps.[2] Cooling is of interest for high-resolution spectroscopy largely because all Doppler effects are fundamentally reduced. Cooling of Na atoms in an atomic beam has been reported by Balykin *et al.*[3]

Two types of electromagnetic traps[4] have been used for the laser-cooling experiments on ions. Cooling of Ba^+ ions stored in an rf quadrupole trap was demonstrated by a group at Heidelberg[5,6] and of Mg^+ ions stored in a Penning trap by a group at the National Bureau of Standards (NBS).[7-9] The rf quadrupole trap uses an alternating inhomogeneous electric field, whose average effect is to create a three-dimensional harmonic potential well. The Penning trap uses a static uniform magnetic field. The ions undergo harmonic motion along the magnetic field and a superposition of circular motions in the plane perpendicular to the field.

Laser cooling of neutral atoms has been treated theoretically by several authors.[1,10-17] Various proposals have been made for trapping neutral atoms in potential wells created by nearly resonant optical fields,[10-12] by static electric fields,[18] and by static magnetic fields,[19] all of which include laser cooling. Previous theoretical treatments of laser cooling of trapped ions have been restricted to harmonic potential wells.[6,16,20-23]

The purposes of this paper are twofold. First, we augment the work of Ref. 16 by explicitly including the effects of light and atomic polarization and of the angular distribution of scattered photons. Unpolarized atoms and isotropic scattering of reemitted photons were usually assumed in Ref. 16. Cooling limits are derived from energy rate equations and also from a simple force fluctuation model. These results are briefly compared with the work of others.

Second, we discuss laser cooling of ions stored in a Penning trap. The natural linewidth of the optical transition used for cooling is assumed to be much greater than any of the frequencies of motion of the trapped ions. This is the case which is most easily realized experimentally. Rate equations are obtained for the mean-squared values of the amplitudes of motion of the ion. Saturation effects are not included, so the results are valid only for low light intensity. Configurations of light beams which can cool all three modes of oscillation are described. For the Penning trap, unlike the harmonic trap, this requires a spatially nonuniform laser beam profile. Some of the general methods were discussed in our previous paper,[16] in which they were applied only to laser cooling of free and harmonically bound atoms. Some of the results of the calculations presented here have been quoted in a previous publication,[9] in which the laser cooling of a single Mg^+ ion in a Penning trap was reported.

Laser cooling of an atom bound in a harmonic well is treated in Sec. II, in order to illustrate the basic ideas in some simple cases. A model of laser cooling of an ion in a Penning trap in which the motion of the ion is treated classically is given in Sec. III. These results are briefly compared with experiment. A quantum-mechanical treatment is given in Sec. IV. The results are discussed in Sec. V.

II. LASER COOLING IN A HARMONIC TRAP

A. Formulation of the problem

The problem to be considered here is the laser cooling of a two-level atom bound in a three-dimensional harmonic well. The term atom includes atomic ions. The ground electronic state $|g\rangle$ is assumed to be stable; the excited electronic state $|e\rangle$ is assumed to decay only to $|g\rangle$ by a one-photon electric-dipole transition at a rate γ. The energy difference between $|g\rangle$ and $|e\rangle$ is $\hbar\omega_0$.

Two-level atoms do not exist, but in practice they can be approximated. In zero external field, either the ground or excited states have some Zeeman degeneracy, since a $(J=0)$-to-$(J'=0)$ transition is forbidden for any one-photon process. (J and J' are the electronic angular momentum quantum numbers in the ground and excited states, respectively.) If a strong magnetic field is applied so that the Zeeman splitting of the optical transition is greater than either the Doppler or natural broadenings, then the laser can be tuned close to resonance with one Zeeman component. The corresponding magnetic sublevels of the ground and excited states will form a two-level system, provided that the excited-state sublevel cannot decay to any other ground-state sublevel. The simplest case involves $J=0$ and $J'=1$. More generally, if $J'=J+1$, then the $M=J$ ground-state sublevel and the $M'=J+1$ excited-state sublevel or the $M=-J$ ground-state sublevel and the $M'=-J-1$ excited-state sublevel will form two-level systems.[7] There are also possibilities when hyperfine structure is present. If the Zeeman splitting is not large compared to the Doppler broadening, it is still possible to form an effective two-level system, but only for certain polarizations and directions of propagation of the light. Systems which are not effectively two level, including those in which hyperfine and Zeeman structure must be taken into account, can be treated by straightforward extension of the two-level methods. For such cases, it may be necessary to introduce additional optical or rf frequencies to ensure that the atoms continue to interact with the laser beam or beams used for cooling. For the Ba^+ experiments,[5,6] the magnetic field is zero and the $6s\,^2S_{1/2}\rightarrow 6p\,^2P_{1/2}$ transition is used ($J=J'=\frac{1}{2}$), so four levels are involved (for $I=0$ isotopes), aside from the metastable D states, but this case can be treated with only minor modifications to a two-level theory.[23]

The harmonic potential causes the atom, which has mass M, to oscillate in the x, y, and z directions at angular frequencies Ω_x, Ω_y, and Ω_z. These frequencies are assumed to be much less than the natural linewidth γ, which is much less than the optical transition frequency ω_0. This is the "weak binding" case of Ref. 16. This case is easily satisfied in practice, since, for a typical resonance line, $\gamma/2\pi \gtrsim 20$ MHz, and for an rf quadrupole trap, $\Omega_i/2\pi \lesssim 2$ MHz $(i=x,y,z)$.

The atom interacts with one or more monochromatic polarized laser beams. The electric field of such a beam is assumed to be given by a classical plane-wave solution:

$$\vec{E}(\vec{r},t)=\hat{\epsilon}\,\mathrm{Re}E_0\exp(i\vec{k}\cdot\vec{r}-i\omega t)\ , \tag{1}$$

where Re stands for the real part, $\hat{\epsilon}$ is a unit polarization vector perpendicular to $\vec{k}$, and $|\vec{k}|\equiv k=\omega/c$.

Consider a scattering event in which a free atom, moving with velocity $\vec{v}$, absorbs a photon of wave vector $\vec{k}$, emits a photon of wave vector $\vec{k}_s$, and has its velocity changed to $\vec{v}'$. From conservation of energy and momentum in the nonrelativistic $(v/c\ll 1)$ limit,

$$\Delta\vec{v}\equiv\vec{v}'-\vec{v}=\hbar(\vec{k}-\vec{k}_s)/M\ , \tag{2}$$

$$\Delta E\equiv\tfrac{1}{2}M(v')^2-\tfrac{1}{2}Mv^2 =\hbar^2(\vec{k}-\vec{k}_s)^2/2M+\hbar(\vec{k}-\vec{k}_s)\cdot\vec{v}\ , \tag{3}$$

$$\Delta E_i\equiv\tfrac{1}{2}M(v_i')^2-\tfrac{1}{2}Mv_i^2 =\hbar^2(k_i-k_{si})^2/2M+\hbar(k_i-k_{si})v_i\quad (i=x,y,z)\ . \tag{4}$$

The subscripts i denote Cartesian components of vectors, except in ΔE_i, which is the kinetic energy due to motion in the i direction. These relations were derived for a free atom, but they also hold for the harmonically bound atom if the scattering takes place in a time much shorter than the period of oscillation, since the atom behaves as a free particle for such short times. Since resonance scattering takes place in a time roughly equal to γ^{-1}, this condition is equivalent to our weak binding assumption. The conservation laws which led to Eqs. (2)—(4) do not depend on the photon scattering being near a resonance. However, we wish to consider only the near-resonance case, where ω is within roughly the combined Doppler and natural broadening of the optical transition frequency. Since it was assumed that $v/c << 1$ and that $\gamma << \omega_0$, this implies that $k \cong \omega_0/c \equiv k_0$. We will also assume that $R << \hbar\gamma$, where $R \equiv (\hbar k)^2/2M \cong (\hbar k_0)^2/2M$. This is easily satisfied for an allowed optical electric-dipole transition. For example, for the $6s\,^2S_{1/2} \rightarrow 6p\,^2P_{1/2}$ 493-nm resonance line in $^{138}Ba^+$, $R = h \times 5.9$ kHz, $\gamma = 2\pi \times 21$ MHz,[24] and $R/\hbar\gamma = 2.8 \times 10^{-4}$. For the $3s\,^2S_{1/2} \rightarrow 3p\,^2P_{3/2}$ 280-nm resonance line in $^{24}Mg^+$, $R = h \times 106$ kHz, $\gamma = 2\pi \times 43$ MHz,[25] and $R/\hbar\gamma = 2.5 \times 10^{-3}$. Since the fractional change in k, $|k - k_s|/k$, is at most about $2v/c + 2\hbar k/Mc$, we can also assume $k \cong k_s$. Equation (4) can be rewritten as

$$\Delta E_i = R(\hat{k}_i^2 - 2\hat{k}_i\hat{k}_{si} + \hat{k}_{si}^2) + \hbar(k_i - k_{si})v_i \; , \qquad (5)$$

where $\hat{k}_i$ and $\hat{k}_{si}$ are the i components of the unit vectors $\hat{k} = \vec{k}/k$ and $\hat{k}_s = \vec{k}_s/k_s$.

The angular distribution of the scattered photons depends only on the orientation of the electric-dipole moment of the transition $\vec{d}_{ge} \equiv \langle g | \vec{d} | e \rangle$. We note here that the ground and excited states we consider are single M states and that an effective two-level system has been created by, for example, applying a magnetic field large enough to resolve the Zeeman structure. The probability that a photon is emitted into a solid angle $d\Omega$ in the $\hat{k}_s$ direction is

$$P_s(\hat{k}_s)d\Omega = (3/8\pi)\sum_j |\hat{d}_{ge}\cdot\hat{\epsilon}_j(\hat{k}_s)^*|^2 d\Omega \; , \qquad (6)$$

where

$$\hat{d}_{ge} = \vec{d}_{ge}/|\vec{d}_{ge}| \; ,$$

and the summation is over the two unit polarization vectors for propagation in the $\hat{k}_s$ direction. These vectors must satisfy

$$\hat{\epsilon}_k(\hat{k}_s)^*\cdot\hat{\epsilon}_j(\hat{k}_s) = \delta_{kj} \; ,$$
$$\hat{\epsilon}_k(\hat{k}_s)\cdot\hat{k}_s = 0 \; , \qquad (7)$$

where δ_{ij} is the Kronecker delta, but are otherwise arbitrary. The normalization is such that $\int P_s(\hat{k}_s)d\Omega = 1$. Equation (6) applies to a nonmoving atom, but it is approximately true for a moving atom, provided $v/c << 1$, which has been assumed. This treatment could be generalized to include other types of multipole radiation, such as magnetic dipole or electric quadrupole. The decay rates for such transitions are usually so slow ($\lesssim 10^3\ s^{-1}$) that it becomes difficult to satisfy the weak binding condition, so they are not treated here. Equation (6) can be evaluated if $|g\rangle$ and $|e\rangle$ are eigenstates of the component of the total angular momentum along a quantization axis defined, for example, by a magnetic field. For a $\Delta M = 0$ transition (M is the eigenvalue of the total angular momentum in the $\hat{B}$ direction) we have

$$P_s(\hat{k}_s) = (3/8\pi)\sin^2\theta_s \; . \qquad (8a)$$

For a $\Delta M = \pm 1$ transition

$$P_s(\hat{k}_s) = (3/16\pi)(1 + \cos^2\theta_s) \; . \qquad (8b)$$

Here θ_s is defined by $\hat{B}\cdot\hat{k}_s = \cos\theta_s$. The isotropic distribution, which has been used in much of the previous work,[12,15,16] corresponds to letting $P_s = 1/4\pi$. This is not of the form of Eq. (6). Given P_s, we can average Eq. (5) over all angles of the scattered photon

$$\langle \Delta E_i \rangle_s = \int P_s(\hat{k}_s)\Delta E_i d\Omega = R(f_i + f_{si}) + \hbar k_i v_i \; , \qquad (9)$$

where

$$f_i \equiv \hat{k}_i^2 \qquad (10a)$$

and

$$f_{si} \equiv \int P_s(\hat{k}_s)\hat{k}_{si}^2 d\Omega \; . \qquad (10b)$$

The terms linear in $\hat{k}_{si}$ drop out because $P_s(\hat{k}_s) = P_s(-\hat{k}_s)$ for any P_s of the form of Eq. (6) or for the isotropic distribution. The total energy change per scattering event, averaged over all angles of the scattered photon, is

$$\langle \Delta E \rangle_s = \langle \Delta E_x \rangle_s + \langle \Delta E_y \rangle_s + \langle \Delta E_z \rangle_s = 2R + \hbar\vec{k}\cdot\vec{v} \; . \qquad (11)$$

This is the same as Eq. (3) of Ref. 16.

The average rate of energy change is obtained by multiplying the scattering rate by the average energy change per scattering event. The scattering rate γ_s is equal to the product of the number of photons per unit area per unit time $I/\hbar\omega$ and the absorption cross section for photons of that particular polarization and propagation direction ($I = c\,|E_0|^2/8\pi$). For a particular atomic velocity $\vec{v}$, the cross section can be obtained from perturbation theory,[16] provided the intensity is below saturation. This is proportional to the Doppler-shifted natural line shape

$$\sigma(\omega,\vec{v})=\sigma_0(\gamma/2)^2/[(\omega_0+\vec{k}\cdot\vec{v}+R/\hbar-\omega)^2+(\gamma/2)^2]\ , \qquad (12)$$

where

$$\sigma_0=6\pi\lambda\!\!\!^{-}_0{}^2\,|\hat{\epsilon}\cdot\hat{d}_{eg}|^2,\quad \lambda\!\!\!^{-}_0\equiv c/\omega_0\ . \qquad (13)$$

This is Eq. (24) of Ref. 16. Terms of order $(v/c)^2$ and $(v/c)(R/\hbar\omega)$ are neglected (see Secs. IV A and IV B of Ref. 16). If we take $|\hat{\epsilon}\cdot\hat{d}_{eg}|^2=\frac{1}{3}$, which is the average over all polarizations $\hat{\epsilon}$, then $\sigma_0=2\pi\lambda\!\!\!^{-}_0{}^2$. As a more realistic example, for the $3p\,^2P_{3/2}\ (M_J=\pm\frac{3}{2})\leftarrow 3s\,^2S_{1/2}\ (M_J=\pm\frac{1}{2})$ transitions in $^{24}Mg^+$ excited by light polarized perpendicular to the magnetic field, $\sigma_0=3\pi\lambda\!\!\!^{-}_0{}^2$. (Note that the expression given in Ref. 7 is too small by a factor of 2.) The rate equations for the energies in the x, y, and z vibrational modes, which in the absence of collisions are separately conserved, are

$$\frac{dE_i}{dt}+(I/\hbar\omega)\langle\sigma(\omega,\vec{v})[\hbar k_i v_i+R(f_i+f_{si})]\rangle_v \quad (i=x,y,z)\ . \qquad (14)$$

Here the velocity average $\langle g(\vec{v})\rangle_v$ of an arbitrary function $g(\vec{v})$ is defined in terms of the velocity probability function $P(\vec{v})$ as $\langle g(\vec{v})\rangle_v\equiv\int P(\vec{v})\times g(\vec{v})d^3\vec{v}$. If a form for $P(\vec{v})$ as a function of $\{E_i\}$ is assumed, then Eq. (14) can be solved to obtain $\{E_i\}$ as a function of time. This might, for example, be a product of Maxwell-Boltzmann distributions for motion in the x, y, and z directions, characterized by three temperatures, which need not be equal,[16,23] unless there is rapid energy transfer between modes by collisions between bound atoms. A special case of Eq. (14), with additional collisional relaxation terms, was discussed in Sec. V F of Ref. 16. If more than one laser beam is present, this can be taken into account by adding additional terms of the same form, with different $\hat{\epsilon}$, E_0, $\vec{k}$, and ω, to the right-hand side of Eq. (14). Equation (14) holds for a single atom or for the ensemble average of a cloud of noninteracting atoms. If collisions between atoms are important, they can be taken into account by introducing phenomenological relaxation constants which tend to equalize the energies in the different modes.[16] If the evolution of the entire velocity distribution, and not of just the average kinetic energy, is of interest, methods based on the Fokker-Planck or another kinetic equation can be used.[13–15,20–23,26]

B. Cooling limits derived from energy rate equations

1. One laser beam along x axis

Consider a single laser beam propagating along the x axis ($\vec{k}=k\hat{x}$). Equation (14) becomes

$$\frac{dE_x}{dt}=(I/\hbar\omega)\langle\sigma(\omega,\vec{v})[\hbar k v_x+R(1+f_{sx})]\rangle_v\ , \qquad (15a)$$

$$\frac{dE_y}{dt}=(I/\hbar\omega)\langle\sigma(\omega,\vec{v})\rangle_v R f_{sy}\ , \qquad (15b)$$

$$\frac{dE_z}{dt}=(I/\hbar\omega)\langle\sigma(\omega,\vec{v})\rangle_v R f_{sz}\ . \qquad (15c)$$

For negative laser detuning ($\omega<\omega_0+R/\hbar\cong\omega_0$), $\sigma(\omega,\vec{v})$ is larger for negative v_x than for positive v_x so that the right-hand side of Eq. (15a) is negative, until very low temperatures are reached. This is the basic principle of laser cooling. If considerable cooling has already taken place, so that the Doppler broadening is much less than the natural linewidth or the detuning, i.e., $k\langle v_x\rangle_{rms}<<\gamma/2$ or $k\langle v_x\rangle_{rms}<<(\omega_0+R/\hbar-\omega)$, then Eq. (12) for the cross section can be approximated by

$$\sigma(\omega,\vec{v})\cong\sigma_0(\gamma/2)^2[(\gamma/2)^2+(\omega_0-\omega)^2]^{-1}\{1-2(\omega_0-\omega)\vec{k}\cdot\vec{v}/[(\gamma/2)^2+(\omega_0-\omega)^2]\}\ . \qquad (16)$$

R has been dropped, because it is assumed that $R/\hbar << (\omega_0-\omega)$. Note that

$$\langle \sigma(\omega,\vec{v})\rangle_v \cong \sigma_0(\gamma/2)^2[(\gamma/2)^2+(\omega_0-\omega)^2]^{-1} , \tag{17}$$

provided $P(\vec{v})=P(-\vec{v})$, which we assume to be true. Let γ_s be the average scattering rate:

$$\gamma_s \equiv (I/\hbar\omega)\langle \sigma(\omega,\vec{v})\rangle_v . \tag{18}$$

Equations (15a)–(15c) now become

$$\frac{dE_x}{dt}=\gamma_s\{-2(\omega_0-\omega)\hbar k^2\langle v_x^2\rangle_v/[(\gamma/2)^2+(\omega_0-\omega)^2]+R(1+f_{sx})\} , \tag{19a}$$

$$\frac{dE_y}{dt}=\gamma_s R f_{sy} , \tag{19b}$$

$$\frac{dE_z}{dt}=\gamma_s R f_{sz} . \tag{19c}$$

The motion in the y and z directions is heated by recoil and does not reach a steady state.[16] The kinetic energy in the x direction reaches a steady-state value when the right-hand side of Eq. (19a) is zero:

$$E_{Kx}\equiv \tfrac{1}{2}M\langle v_x^2\rangle_v$$

$$=\hbar(1+f_{sx})[(\gamma/2)^2+(\omega_0-\omega)^2]/8(\omega_0-\omega) . \tag{20}$$

The minimum value is obtained by setting $(\omega_0-\omega)=\gamma/2$:

$$E_{Kx}=(1+f_{sx})\hbar\gamma/8 . \tag{21}$$

For isotropic scattering, $f_{sx}=\frac{1}{3}$, so the minimum $E_{Kx}=\hbar\gamma/6$. This agrees with the result of Javanainen and Stenholm,[20] in the unsaturated limit. They treated this situation by using a Fokker-Planck equation. If the atomic electric-dipole moment is oriented along a quantization axis perpendicular to the x axis, corresponding to a $\Delta M=0$ transition, then from Eqs. (8a) and (10),

$$f_{sx}=(3/8\pi)\int_{-1}^{1}\int_0^{2\pi}\sin^2\theta(\sin\theta\cos\phi)^2 d\phi\, d\cos\theta=\tfrac{2}{5} , \tag{22}$$

where θ and ϕ are spherical polar angles, so the minimum $E_{Kx}=7\hbar\gamma/40$. This is in agreement with the results obtained by Cook,[14] who used a Fokker-Planck equation, for the one-dimensional cooling of an unbound atom in a weak standing wave, when the dipole is perpendicular to the light propagation direction. This is essentially the same problem that we have treated here. For an unbound atom, a standing wave is required in order that the average radiation pressure force be zero, so that is not accelerated continuously in one direction. Cooling from a single running wave can occur for the harmonically bound atom, since a steady force only shifts the equilibrium point in the potential well. The harmonic restoring force was required in order to ensure that $P(\vec{v})=P(-\vec{v})$, which led to the cancellation of a term in deriving Eq. (16) from Eq. (14). The method we have used here and in Ref. 16, which is based on calculating the average energy imparted to the atom in the photon scattering process, takes into account heating due to both induced fluctuations (recoil on absorption) and spontaneous fluctuations (recoil on emission), at least in the low-intensity limit. These two sources of fluctuations were introduced separately in Ref. 14. In our treatment, the induced fluctuations are associated with the term proportional to f_i and the spontaneous fluctuations with the term proportional to f_{si} in Eq. (14).

The treatment of the atomic motion has been essentially classical. This should be valid in the limit of large harmonic oscillator quantum numbers. The mean occupation number for motion in the x direction is $\langle n_x\rangle \cong 2E_{Kx}/\hbar\Omega_x \gtrsim \gamma/3\Omega_x$, so in the weak binding approximation the motion remains in the classical regime.

2. *Laser beams along x, y, and z axes*

In Sec. II B 1 it was shown that a single laser beam parallel to one of the principal axes of the harmonic potential cools motion parallel to the beam but heats the perpendicular motion by recoil. One simple configuration which is capable of cooling all modes of oscillation consists of three laser beams, one along each of the x, y, and z axes. Denote the laser frequencies by ω_i ($i=x,y,z$) and assume that all are tuned below resonance ($\omega_i<\omega_0$). Then the atoms are cooled, and, assuming that the Doppler widths are small compared with either the natural linewidth or the detunings, [$k_i(v_i)_{\rm rms}<<\gamma/2$ or $k_i(v_i)_{\rm rms}<<(\omega_0-\omega_i)$, where $k_i\equiv\omega_i/c$],

$$\frac{dE_i}{dt}=-2\gamma_{si}(\omega_0-\omega_i)\hbar k_i^2\langle v_i^2\rangle_v/[(\gamma/2)^2+(\omega_0-\omega_i)^2]+R[\gamma_{si}+(\gamma_s)_{\rm tot}f_{si}]\ , \tag{23}$$

where γ_{si} is the average scattering rate due to the laser beam along the i axis and $(\gamma_s)_{\rm tot}=\gamma_{sx}+\gamma_{sy}+\gamma_{sz}$. In the steady state,

$$E_{Ki}\equiv\tfrac{1}{2}M\langle v_i^2\rangle_v$$

$$=\hbar[1+f_{si}(\gamma_s)_{\rm tot}/\gamma_{si}][(\gamma/2)^2+(\omega_0-\omega_i)^2]/8(\omega_0-\omega_i)\ . \tag{24}$$

This is minimized, [for constant $(\gamma_s)_{\rm tot}/\gamma_{si}$], by setting $(\omega_0-\omega_i)=\gamma/2$:

$$E_{Ki}=[1+f_{si}(\gamma_s)_{\rm tot}/\gamma_{si}]\hbar\gamma/8\ . \tag{25}$$

The total mean kinetic energy can be minimized by adjusting the relative laser powers so that

$$(\gamma_{si})/(\gamma_s)_{\rm tot}=(f_{si})^{1/2}/[(f_{sx})^{1/2}+(f_{sy})^{1/2}+(f_{sz})^{1/2}]\ . \tag{26}$$

For the isotropic case, the minimum kinetic energy in each mode is $\hbar\gamma/4$, which agrees with Ref. 16 (Sec. V E 2).

3. *One laser beam at oblique angle to axes*

Another method of cooling all three modes of oscillation is with a single laser beam which is not parallel to any of the principal axes. Equation (14) can be used to obtain the cooling rate. As has been pointed out previously, it is necessary that all of the motional frequencies (Ω_x, Ω_y, and Ω_z) be different.[6,16,23] Otherwise there is at least one mode of oscillation perpendicular to the laser beam which is not cooled, but rather is heated by recoil, like the y and z modes discussed in Sec. II B 1. In the low Doppler limit [$(\vec{k}\cdot\vec{v})_{\rm rms}<<\gamma/2$ or $(\vec{k}\cdot\vec{v})_{\rm rms}<<(\omega_0-\omega)$],

$$\frac{dE_i}{dt}=\gamma_s\{-2(\omega_0-\omega)\hbar k^2f_i\langle v_i^2\rangle_v/[(\gamma/2)^2+(\omega_0-\omega)^2]+R(f_i+f_{si})\}\ , \tag{27}$$

where we have assumed that $\langle v_iv_j\rangle_v=0$ if $i\neq j$. This should be valid if the frequencies are sufficiently different and the intensity is sufficiently low. In the steady state,

$$E_{Ki}=\hbar(1+f_{si}/f_i)[(\gamma/2)^2+(\omega_0-\omega)^2]/8(\omega_0-\omega)\ , \tag{28}$$

and this is minimized by setting $(\omega_0-\omega)=\gamma/2$. Then

$$E_{Ki}=(1+f_{si}/f_i)\hbar\gamma/8\ . \tag{29}$$

If one has control over the direction of the beam, then the total average kinetic energy can be minimized if

$$f_i=(f_{si})^{1/2}/[(f_{sx})^{1/2}+(f_{sy})^{1/2}+(f_{sz})^{1/2}]\ . \tag{30}$$

For an isotropic angular distribution, this can be accomplished by pointing the laser along the $\hat{x}+\hat{y}+\hat{z}$ direction, in which case, $E_{Ki}=\hbar\gamma/4$. This is close to the configuration used by Neuhauser *et al.*[6] Equation (29) is equivalent to the low-intensity limit of Eq. (31) of Javanainen.[23]

C. Cooling limit derived from force fluctuations

The steady-state cooling limits derived in Sec. II B can also be obtained from a simple classical model, in which the effect of the scattered photons on the atomic motion is divided into an average damping force and a random, fluctuating force. The advantage of this approach is its great simplicity; it uses concepts which are familiar from the theory of noise in electronic circuits.

The one-dimensional cooling case, which was solved by the energy rate equation method in Sec. II B 1, will be treated, in order to demonstrate the basic method in the simplest case. The method can also be applied to the other cases treated in Sec. II B.

The equation of motion (Newton's second law) for the x coordinate of the atom is

$$M\ddot{x}+M\Omega_x^2 x=F_x(t)\ , \quad (31)$$

where $F_x(t)$ is the x component of the force on the atom due to photon scattering. Each scattering event imparts a momentum impulse $\hbar(\vec{k}-\vec{k}_s)$ in a time less than about γ^{-1}, which we have assumed to be very short compared to the period of oscillation. Therefore we can represent $F_x(t)$ by a sum of Dirac delta functions:

$$F_x(t)=\sum_l \delta(t-t_l)\hbar[k-k_{sx}(l)]\ , \quad (32)$$

where $k_{sx}(l)$ is the value of k_{sx} for the lth scattering. The scattering events take place at random times at an average rate $I\sigma(\omega,\vec{v})/\hbar\omega$. The time-averaged force on an atom of velocity $\vec{v}$ is

$$\langle F_x\rangle=(I/\hbar\omega)\sigma(\omega,\vec{v})\hbar k\ , \quad (33)$$

since the $k_{sx}(l)$ terms average to zero. In the low Doppler limit,

$$\langle F_x\rangle=\gamma_s\{1-2(\omega_0-\omega)k\dot{x}/[(\gamma/2)^2+(\omega_0-\omega)^2]\}\hbar k$$
$$\equiv\gamma_s\hbar k-M\Gamma\dot{x}\ . \quad (34)$$

The equation of motion can now be written as

$$M\ddot{X}+M\Gamma\dot{X}+M\Omega_x^2 X=F'(t)\equiv F_x(t)-\langle F_x\rangle\ , \quad (35)$$

which describes a damped harmonic oscillator driven by a random force $F'(t)$. We have eliminated the constant force $\gamma_s\hbar k$ by shifting the origin, so that $X(\text{new})=x(\text{old})-\gamma_s\hbar k/M\Omega_x^2$. This equation has the same form as the one which describes the response of a series RCL (resistance-capacitance-inductance) circuit to a fluctuating emf.

The fluctuations in X can be calculated by standard methods.[27] Let $G(f)$ be the complex gain function which describes the steady-state response X to a sinusoidal F' with frequency f, and which is defined by

$$X(f)\exp(2\pi ift)=G(f)F'(f)\exp(2\pi ift)\ . \quad (36)$$

Here the physical quantities $X(t)$ and $F'(t)$ are given by the real parts of $X(f)\exp(2\pi ift)$ and $F'(f)\exp(2\pi ift)$, respectively. Let $w(f)$ be the one-sided power spectral density of $F'(t)$, defined by

$$w(f)=4\int_0^\infty \langle F'(t)F'(t+\tau)\rangle\cos 2\pi f\tau\, d\tau\ .$$

Then the mean-squared value of X is

$$\langle X^2\rangle=\int_0^\infty |G(f)|^2 w(f)df\ . \quad (37)$$

For the system described by Eq. (35),

$$G(f)=1/M[\Omega_x^2-(2\pi f)^2+i2\pi\Gamma f]\ . \quad (38)$$

The spectral density function can be calculated by following the standard derivation for the power spectrum of shot noise.[28] The only difference is that the impulses vary in size because of the random direction of the emitted photon. A derivation is given in the Appendix. We have

$$w(f)=2\gamma_s\hbar^2\langle(k-k_{sx})^2\rangle_s$$
$$=2\gamma_s\hbar^2(k^2+\langle k_{sx}^2\rangle_s)$$
$$=4M\gamma_s R(1+f_{sx})\ , \quad (39)$$

which is valid for frequencies f less than about $\gamma/2\pi$. The fact that Eq. (39) is plausible can be seen by examining the more familiar expression for the spectral density of current for shot noise, $w_I(f)=2eI=2e^2\dot{N}$, where I is the average current, and $\dot{N}$ is the average number of electrons per second passing through a temperature limited diode. Substituting the rms change in momentum for e and the average scattering rate for $\dot{N}$, we arrive at Eq. (39).

The term $|G(f)|^2$ in the integrand in Eq. (37) is sharply peaked around $f=\Omega_x/2\pi$, where $w(f)$ is nearly constant, so

$$\langle X^2\rangle\cong w(\Omega_x/2\pi)\int_0^\infty |G(f)|^2 df$$
$$=\frac{4\gamma_s R}{M}(1+f_{sx})\int_0^\infty \frac{df}{[\Omega_x^2-(2\pi f)^2]^2+(2\pi\Gamma f)^2}$$
$$=\gamma_s R(1+f_{sx})/M\Gamma\Omega_x^2\ . \quad (40)$$

The average kinetic energy is

$$E_{Kx}=\tfrac{1}{2}M\langle v^2\rangle=\tfrac{1}{2}M\Omega_x^2\langle X^2\rangle=\tfrac{1}{2}\gamma_s R(1+f_{sx})/\Gamma$$
$$=\hbar(1+f_{sx})[(\gamma/2)^2+(\omega_0-\omega)^2]/8(\omega_0-\omega)\ . \quad (41)$$

In the last step Eq. (34) was used to express Γ in terms of other parameters. This is the same result obtained previously from the energy rate equation [see Eq. (20)], and the minimum energy is obtained when $(\omega_0-\omega)=\gamma/2$ [see Eq. (21)]. Equation (35) is the Langevin equation, with the addition of the harmonic restoring force. Therefore the theory developed for the solution of the Langevin equation could be applied to the present problem.[12,26]

III. LASER COOLING IN A PENNING TRAP (SEMICLASSICAL TREATMENT)

A. Classical motion of an ion in a Penning trap

The classical motion of a single ion in a Penning trap has been treated previously.[4,29] Here we summarize the important results.

The idealized Penning trap consists of a uniform magnetic field $\vec{B}=B_0\hat{z}$ along the z axis and a quadrupole electrostatic potential of the form

$$V(x,y,z)=A_0(2z^2-x^2-y^2)\ . \tag{42}$$

Consider an ion of mass M and charge q. The equation of motion for the ion position $\vec{r}$ is

$$M\dot{\vec{v}}=-q\vec{\nabla}V(\vec{r})+(q/c)\vec{v}\times\vec{B}\quad(\vec{v}\equiv\dot{\vec{r}}) \tag{43}$$

and the equations for the Cartesian components (x,y,z) of $\vec{r}$ are

$$\dot{v}_x=(2qA_0/M)x+(qB_0/Mc)v_y\ , \tag{44a}$$

$$\dot{v}_y=(2qA_0/M)y-(qB_0/Mc)v_x\ , \tag{44b}$$

$$\dot{v}_z=-(4qA_0/M)z\ . \tag{44c}$$

The general solutions to these equations are

$$x(t)=r_m\cos(\omega_m t+\theta_m)+r_c\cos(\omega_c' t+\theta_c)\ , \tag{45a}$$

$$y(t)=-r_m\sin(\omega_m t+\theta_m)-r_c\sin(\omega_c' t+\theta_c)\ , \tag{45b}$$

$$z(t)=r_z\cos(\omega_z t+\theta_z)\ . \tag{45c}$$

We can also write the expressions for x and y in the complex form[29]

$$x(t)+iy(t)=r_m\exp[-i(\omega_m t+\theta_m)]+r_c[-i(\omega_c' t+\theta_c)]\ . \tag{45d}$$

We have assumed that q, A_0, and B_0 are all positive. The amplitudes (r_m,r_c,r_z), defined to be greater than or equal to zero, and the phases $(\theta_m,\theta_c,\theta_z)$ are determined by the initial conditions. The frequencies are the magnetron frequency ω_m, the modified cyclotron frequency ω_c', and the axial frequency ω_z, where

$$\omega_z=(4qA_0/M)^{1/2}\ , \tag{46a}$$

$$\omega_m=\omega_c/2-(\omega_c^2/4-\omega_z^2/2)^{1/2}\ , \tag{46b}$$

$$\omega_c'=\omega_c/2+(\omega_c^2/4-\omega_z^2/2)^{1/2}\ . \tag{46c}$$

Here $\omega_c=qB_0/Mc$ is the ordinary cyclotron frequency. These solutions assume that $\omega_c^2>2\omega_z^2$, since otherwise the motion is unstable. A typical experimental configuration might have $M=25u$, $q=e$, $B_0=1$ T (10 kG), and $A_0=10$ V/cm^2. For these values of the parameters, $\omega_z\cong2\pi\times197.7$ kHz, $\omega_m\cong2\pi\times33.7$ kHz, and $\omega_c'\cong2\pi\times580.6$ kHz. The ion executes simple harmonic motion at frequency ω_z in the z direction and a superposition of circular motions at frequencies ω_c' and ω_m in the xy plane. Figure 1 shows a typical xy orbit for the case $\omega_m<<\omega_c'$ and $r_m>r_c$. In this case the magnetron motion can be interpreted as an $\vec{E}\times\vec{B}$ drift of the center of the cyclotron orbit around the trap axis.

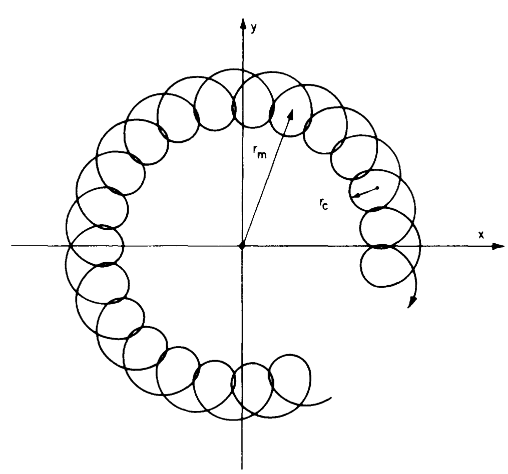

FIG. 1. Orbit of an ion in a Penning trap, projected onto the xy plane. The magnetic field direction is out of the drawing and the charge of the ion is assumed to be positive.

The kinetic energy of the ion is

$$E_K \equiv \tfrac{1}{2}Mv^2 = \tfrac{1}{2}Mr_m^2\omega_m^2 + \tfrac{1}{2}Mr_c^2\omega_c'^2 + Mr_m r_c \omega_m \omega_c' \cos(\omega_m t - \omega_c' t + \theta_m - \theta_c) + \tfrac{1}{2}Mr_z^2\omega_z^2 \sin^2(\omega_z t + \theta_z) . \tag{47}$$

The potential energy is

$$E_P \equiv qV(\vec{r}) = \tfrac{1}{2}M\omega_z^2[r_z^2\cos^2(\omega_z t + \theta_z) - \tfrac{1}{2}(r_m^2 + r_c^2) - r_m r_c \cos(\omega_m t - \omega_c' t + \theta_m - \theta_c)] . \tag{48}$$

The total energy is a constant of the motion,

$$E = E_K + E_P = \tfrac{1}{2}Mr_z^2\omega_z^2 + M\Omega(\omega_c' r_c^2 - \omega_m r_m^2) , \tag{49}$$

where

$$\Omega = \tfrac{1}{2}(\omega_c' - \omega_m) . \tag{50}$$

We are interested primarily in reducing E_K, rather than E, since this will reduce Doppler shifts. The time-averaged kinetic energies in the axial, magnetron, and cyclotron modes are

$$\langle E_{Kz} \rangle = \tfrac{1}{4}Mr_z^2\omega_z^2 , \tag{51a}$$

$$\langle E_{Km} \rangle = \tfrac{1}{2}Mr_m^2\omega_m^2 , \tag{51b}$$

$$\langle E_{Kc} \rangle = \tfrac{1}{2}Mr_c^2\omega_c'^2 . \tag{51c}$$

Hence, we are interested in reducing r_z, r_m, and r_c. Note, however, that an increase in r_m leads to a decrease in the total magnetron energy, because of the potential energy contribution. Therefore, the orbit is unstable toward an increase of r_m if the ion is perturbed, for example, by collisions with neutral molecules. In the following, the term "cooling" will refer to the reduction of the kinetic energy,[9] regardless of what happens to the total energy.

We note that the canonical momentum $\vec{p}$ is defined by $\vec{p} = M\vec{v} + (q/c)\vec{A}$, where $\vec{A}$ is the vector potential. We choose the symmetric gauge, where $\vec{A}(\vec{r}) = \tfrac{1}{2}\vec{B} \times \vec{r}$. In this gauge, the axial component of the canonical angular momentum is a constant of the motion given by

$$L_z \equiv xp_y - yp_x = M\Omega(r_m^2 - r_c^2) . \tag{52}$$

B. Effect of photon interactions

We treat the effect of a laser beam on the motion of an ion with essentially the same assumptions that were made for the case of a harmonically bound atom. The ion is assumed to have only two internal energy levels, which are connected by an electric-dipole transition. The decay rate of the upper state is assumed to be much greater than any of the frequencies of motion (ω_c', ω_m, and ω_z). Hence a photon scattering event can be considered to take place instantaneously, as far as the motion of the ion is concerned. In this semclassical treatment, the ion moves according to the classical equations of motion between scattering events, which perturb its velocity. The scattering events occur at a rate determined by quantum-mechanical perturbation theory. We consider only the limit of low laser intensity. The various quantities ω_0, γ, R, f_i, f_{si}, $\vec{d}_{ge}$, P_s, and σ_0 are defined as in Sec. II.

Consider an event in which an ion moving according to Eqs. (45a)–(45c) absorbs a photon of wave vector $\vec{k}$ and emits a photon of wave vector $\vec{k}_s$ at time t_0. For $t < t_0$, the motion is described by the amplitudes r_i and the phases θ_i ($i = m,c,z$); for $t > t_0$ they are modified to r_i' and θ_i'. Let $\Delta\vec{v} \equiv \hbar(\vec{k} - \vec{k}_s)/M$. Then

$$\Delta r_z^2 \equiv (r_z')^2 - r_z^2 = (\Delta v_z/\omega_z)^2 - (2r_z\Delta v_z/\omega_z)\sin(\omega_z t_0 + \theta_z) , \tag{53a}$$

$$\Delta r_m^2 \equiv (r_m')^2 - r_m^2 = [(\Delta v_x)^2 + (\Delta v_y)^2]/4\Omega^2 + (r_m/\Omega)[\sin(\omega_m t_0 + \theta_m)\Delta v_x + \cos(\omega_m t_0 + \theta_m)\Delta v_y] , \tag{53b}$$

$$\Delta r_c^2 \equiv (r_c')^2 - r_c^2 = [(\Delta v_x)^2 + (\Delta v_y)^2]/4\Omega^2 - (r_c/\Omega)[\sin(\omega_c' t_0 + \theta_c)\Delta v_x + \cos(\omega_c' t_0 + \theta_c)\Delta v_y] . \tag{53c}$$

These equations are derived from combinations of Eqs. (45a)–(45c) and their time derivatives. Equation (53a) is, of course, equivalent to Eq. (4), since the axial mode is harmonic. From Eqs. (53b) and (53c) it can be shown that the magnetron radius is reduced if $\Delta\vec{v}$ is tangential to and in the same direction as the magnetron motion, while the cyclotron radius is reduced if $\Delta\vec{v}$ is tangential to and in the opposite direction as the cyclotron motion. The radii are increased for the opposite cases. This behavior has been noted previously.[7-9] The requirements for cooling the cyclotron mode are similar to those for a harmonic oscillator, while those for the magnetron mode are different.

C. Rate equations for mean-squared amplitudes

1. Uniform laser beams

Here we consider the effect of a plane-wave laser beam of the form given by Eq. (1). We assume the intensity to be constant over the orbit of the ion. Consider a laser beam parallel to the z axis. If the laser frequency ω is less than ω_0, the axial motion is cooled. Throughout the rest of Sec. III we neglect $R/\hbar$ compared with $(\omega_0-\omega)$ or γ. The magnetron and cyclotron modes are heated by recoil. As far as the axial mode is concerned, this is the same as the problem treated in Sec. II B 1. In the steady state

$$\langle E_{Kz}\rangle=\hbar(1+f_{sz})[(\gamma/2)^2+(\omega_0-\omega)^2]/8(\omega_0-\omega) ,$$

and for the optimum detuning,

$$(\omega_0-\omega)=\gamma/2,\quad \langle E_{Kz}\rangle=(1+f_{sz})\hbar\gamma/8 .$$

The second terms on the right-hand sides of Eqs. (53b) and (53c) average to zero; therefore, the heating rates of the magnetron and cyclotron modes are given by

$$\frac{d\langle r_m^2\rangle}{dt}=\frac{d\langle r_c^2\rangle}{dt}=\gamma_s R(f_{sx}+f_{sy})/2M\Omega^2 . \tag{54}$$

The brackets around r_m^2 and r_c^2 denote ensemble averages. Now consider a laser beam propagating along the x direction ($\vec{k}=k\hat{x}$). Assume that the system has somehow been cooled, so that $k(v_y)_{\rm rms}<<\gamma/2$ or $k(v_y)_{\rm rms}<<|\omega_0-\omega|$. Then, using Eqs. (16), (45a), and (45b), the scattering cross section is

$$\sigma(\omega,\vec{v})\cong\sigma_0(\gamma/2)^2[(\gamma/2)^2+(\omega_0-\omega)^2]^{-1}\{1+2(\omega_0-\omega)k[r_m\omega_m\sin(\omega_m t+\theta_m)+r_c\omega_c'\sin(\omega_c' t+\theta_c)]/[(\gamma/2)^2+(\omega_0-\omega)^2]\} . \tag{55}$$

For an individual ion, the amplitudes and phases $(r_m,r_c,\theta_m,\theta_c)$ are perturbed by each scattering event. To find the average rate of change of $\langle r_z^2\rangle,\langle r_m^2\rangle,\langle r_c^2\rangle$, we multiply $(I/\hbar\omega)$ by the cross section [Eq. (55)] and by the change per scattering [Eqs. (53a)–(53c)] and average over r_m, r_c, θ_m, θ_c, and $\hat{k}_s$:

$$\frac{d\langle r_z^2\rangle}{dt}+2\gamma_s Rf_{sz}/M\omega_z^2 , \tag{56a}$$

$$\frac{d\langle r_m^2\rangle}{dt}=\gamma_s\{(\omega_0-\omega)\hbar k^2\omega_m\langle r_m^2\rangle/M\Omega[(\gamma/2)^2+(\omega_0-\omega)^2]+R(1+f_{sx}+f_{sy})/2M\Omega^2\} , \tag{56b}$$

$$\frac{d\langle r_c^2\rangle}{dt}=\gamma_s\{-(\omega_0-\omega)\hbar k^2\omega_c'\langle r_c^2\rangle/M\Omega[(\gamma/2)^2+(\omega_0-\omega)^2]+R(1+f_{sx}+f_{sy})/2M\Omega^2\} . \tag{56c}$$

For negative laser detuning, $(\omega<\omega_0)$, Eq. (56c) predicts that $\langle r_c^2\rangle$ is reduced until it reaches a steady-state value, while Eq. (56b) predicts that $\langle r_m^2\rangle$ increases without limit. This is because the Doppler shift causes the scattering rate to increase when either the cyclotron or magnetron contribution to the velocity is opposed to $\vec{k}$. For positive laser detuning, $(\omega>\omega_0)$, the opposite occurs. The Doppler shift due to the heated mode will eventually cause the approximation for the cross section [Eq. (55)] to break down, which will modify the heating and cooling rates, but not change these qualitiative features. The axial motion is heated regardless of detuning. It could be cooled by introducing a laser beam with some component along z, with negative detuning. However, there is no combination of uniform laser beams which is capable of cooling all three modes.

2. Nonuniform beam along x and uniform beam along z

Cooling of both the magnetron and cyclotron modes can be accomplished with a laser beam whose direction of propagation is in the xy plane and whose intensity increases on the side of the z axis on which the magnetron velocity is along the direction of propagation. The photon scattering then tends to cool the magnetron motion. With negative laser detuning, the cyclotron motion will be cooled, due to Doppler effect velocity selection. This method has been demonstrated experimentally.[8,9] The axial motion can be cooled with a separate beam or by tilting the nonuniform beam so that it has a component along the z axis.

Here we treat the case of a nonuniform beam of frequency ω_1 and wave vector $k_1\hat{x}$ and a uniform beam of frequency ω_2 and wave vector $k_2\hat{z}$ which cools the axial motion. The laser intensity of the nonuniform beam is assumed to be a function of y with value I_1 at $y=0$ and a positive slope I_1/y_0 at $y=0$ (see Fig. 2). We approximate the intensity profile by $I(y)=I_1(1+y/y_0)$ for $|y| << y_0$. The rms magnetron and cyclotron radii are assumed to be much less than y_0, so that this linear approximation is valid. Let I_2 be the intensity of the laser beam along z. Let the scattering cross sections for the beams along the x and z axes, which depend upon the light polarizations through Eq. (13), be $\sigma_1(\omega_1,\vec{v})$ and $\sigma_2(\omega_2,\vec{v})$, respectively, with resonant $(\vec{v}=0,\omega=\omega_0)$ cross sections σ_{01} and σ_{02}. The rate equations for the mean-squared amplitudes are

$$\frac{d\langle r_z^2\rangle}{dt}=(I_2/\hbar\omega_2)\langle\sigma_2(\omega_2,\vec{v})\Delta r_z^2\rangle+(I_1/\hbar\omega_1)\langle(1+y/y_0)\sigma_1(\omega_1,\vec{v})\Delta r_z^2\rangle\ , \tag{57a}$$

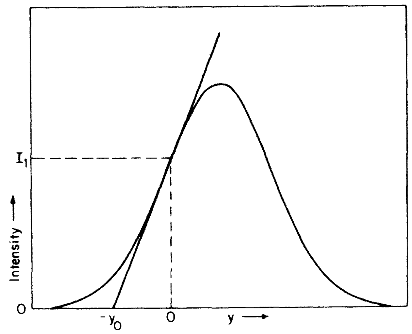

FIG. 2. Intensity profile of nonuniform laser beam. I_1 is the intensity at $y=0$ and I_1/y_0 is its slope. The tangent to the curve is used as an approximation to the actual profile in the calculation.

$$\frac{d\langle r_i^2\rangle}{dt}=(I_1/\hbar\omega_1)\langle(1+y/y_0)\sigma_1(\omega_1,\vec{v})\Delta r_i^2\rangle+(I_2/\hbar\omega_2)\langle\sigma_2(\omega_2,\vec{v})\Delta r_i^2\rangle\ , \tag{57b}$$

where $i=m,c$. The Δr_z^2 and Δr_i^2 are evaluated from Eqs. (53a)–(53c) using the $\vec{k}$ appropriate to each laser beam. We now make the low Doppler approximation to the cross section and perform the ensemble average. The average scattering rates γ_{sl} $(l=1,2)$ are given by

$$\gamma_{sl}=(I_l\sigma_{0l}/\hbar\omega_l)(\gamma/2)^2/[(\gamma/2)^2+(\omega_0-\omega_l)^2]\ . \tag{58}$$

The rate equations become

$$\frac{d\langle r_z^2\rangle}{dt}=-2\gamma_{s2}(\omega_0-\omega_2)\hbar k_2^2\langle r_z^2\rangle/M[(\gamma/2)^2+(\omega_0-\omega_2)^2]+(2\gamma_{s2}R/M\omega_z^2)[1+(1+\gamma_{s1}/\gamma_{s2})f_{sz}]\ , \tag{59a}$$

$$\frac{d\langle r_m^2\rangle}{dt}=-\gamma_{s1}\hbar k_1\langle r_m^2\rangle/2M\Omega y_0+\gamma_{s1}(\omega_0-\omega_1)\hbar k_1^2\omega_m\langle r_m^2\rangle/M\Omega[(\gamma/2)^2+(\omega_0-\omega_1)^2]+(\gamma_{s1}R/2M\Omega^2)[1+(1+\gamma_{s2}/\gamma_{s1})(f_{sx}+f_{sy})]\ , \tag{59b}$$

$$\frac{d\langle r_c^2\rangle}{dt}=\gamma_{s1}\hbar k_1\langle r_c^2\rangle/2M\Omega y_0-\gamma_{s1}(\omega_0-\omega_1)\hbar k_1^2\omega_c'\langle r_c^2\rangle/M\Omega[(\gamma/2)^2+(\omega_0-\omega_1)^2]+(\gamma_{s1}R/2M\Omega^2)[1+(1+\gamma_{s2}/\gamma_{s1})(f_{sx}+f_{sy})]\ . \tag{59c}$$

The axial motion can be cooled if $\omega_2<\omega_0$ [see Eq. (59a)]. The first term on the right-hand side of Eq. (59b) shows cooling of the magnetron motion due the intensity gradient (for $y_0>0$), the second shows heating due to Doppler selection (for $\omega_1<\omega_0$), and the third shows heating due to recoil. For $y_0>0$ and $\omega_1<\omega_0$, the first term on the right-hand side of Eq. (59c) shows heating of the cyclotron motion due the intensity gradient, the second shows cooling due to Doppler selection, and the third shows heating due to recoil. Some terms have been neglected in Eqs. (59b) and (59c) because they are at most of relative order $(k_1 v_x/\gamma)(y/y_0)$ compared to the recoil heating term. The term in the first set of parentheses is small, by the low Doppler assumption, while the term in the second set is small because $(r_m)_{\rm rms}$ and $(r_c)_{\rm rms}$ are assumed to be much less than y_0. The condition for simultaneous cooling of the magnetron and cyclotron modes, for $y_0>0$ and $\omega_1<\omega_0$, is

$$\omega_m < [(\gamma/2)^2+(\omega_0-\omega_1)^2]/2k_1 y_0(\omega_0-\omega_1) < \omega_c' \ . \tag{60}$$

Assuming this condition is satisfied, as can be obtained experimentally, the steady state is given by

$$\langle r_z^2\rangle = \hbar[1+(1+\gamma_{s1}/\gamma_{s2})f_{sz}][(\gamma/2)^2+(\omega_0-\omega_2)^2]/2M\omega_z^2(\omega_0-\omega_2) \ , \tag{61a}$$

$$\langle r_m^2\rangle = \frac{\hbar k_1 y_0[1+(1+\gamma_{s2}/\gamma_{s1})(f_{sx}+f_{sy})]}{2M\Omega\{1-2k_1y_0(\omega_0-\omega_1)\omega_m/[(\gamma/2)^2+(\omega_0-\omega_1)^2]\}} \ , \tag{61b}$$

$$\langle r_c^2\rangle = \frac{\hbar k_1 y_0[1+(1+\gamma_{s2}/\gamma_{s1})(f_{sx}+f_{sy})]}{2M\Omega\{k_1y_0(\omega_0-\omega_1)\omega_c'/[(\gamma/2)^2+(\omega_0-\omega_1)^2]-1\}} \ . \tag{61c}$$

The average kinetic energy can be obtained by inserting these values into Eqs. (51a)–(51c). The parameters γ_{s2}/γ_{s1}, $(\omega_0-\omega_1)$, $(\omega_0-\omega_2)$, and y_0 can be adjusted in order to minimize the kinetic energy. In the general case, this is complicated.

We consider a simple case which can be experimentally realized. For this case, the Doppler selection cooling rate of the cyclotron motion is much greater than the intensity gradient heating rate, and the Doppler selection heating rate of the magnetron motion is less than and not too close to the intensity gradient cooling rate. These condition are most easily satisfied when $\omega_c'/\omega_m >> 1$, which is obtained by reducing the electric potential. Also, y_0 must fall within a certain range, so that $k_1 y_0 \omega_c'/\gamma >> 1$ and $(1-2k_1y_0\omega_m/\gamma)$ is not too small, assuming that $(\omega_0-\omega_1)\cong\gamma/2$. Then the magnetron kinetic energy is much less than the cyclotron kinetic energy. We have

$$\langle E_{Kz}\rangle = \hbar[1+(1+\gamma_{s1}/\gamma_{s2})f_{sz}] \times[(\gamma/2)^2+(\omega_0-\omega_2)^2]/8(\omega_0-\omega_2) \ . \tag{62a}$$

$$\langle E_{Kc}\rangle \cong \hbar[1+(1+\gamma_{s2}/\gamma_{s1})(f_{sx}+f_{sy})] \times[(\gamma/2)^2+(\omega_0-\omega_1)^2]/4(\omega_0-\omega_1) \ , \tag{62b}$$

and we neglect $\langle E_{Km}\rangle$. For constant γ_{s1} and γ_{s2}, the kinetic energies are minimized if we set $(\omega_0-\omega_2)=(\omega_0-\omega_1)=\gamma/2$:

$$\langle E_{Kz}\rangle = [1+(1+\gamma_{s1}/\gamma_{s2})f_{sz}]\hbar\gamma/8 \ , \tag{63a}$$

$$\langle E_{Kc}\rangle \cong [1+(1+\gamma_{s2}/\gamma_{s1})(f_{sx}+f_{sy})]\hbar\gamma/4 \ . \tag{63b}$$

The total kinetic energy can be minimized by adjusting the relative laser beam intensities so that

$$\gamma_{s1}/\gamma_{s2} = [2(f_{sx}+f_{sy})/f_{sz}]^{1/2} \ . \tag{64}$$

For the isotropic angular distribution, this condition is $\gamma_{s1}/\gamma_{s2}=2$, in which case, $\langle E_{Kz}\rangle \cong \frac{1}{2}\langle E_{Kc}\rangle \cong \hbar\gamma/4$. This gives the same average kinetic energy in the x, y, and z directions as for the harmonic trap discussed in Secs. II B 2 and II B 3. The minimum energies are slightly different for an anisotropic distribution, as is the case for the harmonic trap.

3. *Nonuniform beam in xz plane*

Cooling of all three modes can be accomplished by rotating the direction of the nonuniform beam

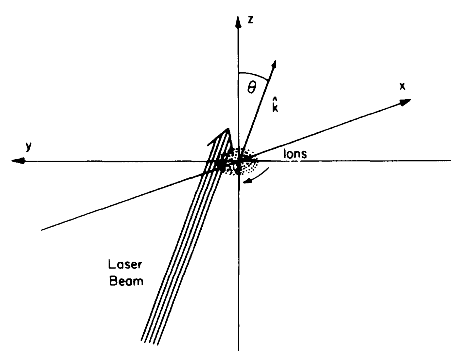

FIG. 3. Nonuniform laser beam configuration which is capable of cooling all three modes of motion of an ion in a Penning trap. The trap coordinate axes are indicated. The ion cloud is at the origin, and the direction of the magnetron rotation is indicated by an arrow. The laser light propagates in the $\hat{k}$ direction, which lies in the xz plane. The laser beam is focused so that it is more intense for $y>0$ than for $y<0$.

described in Sec. III C 2 so that it has a component along z and eliminating the uniform beam. Let $\vec{k}=k\sin\theta\hat{x}+k\cos\theta\hat{z}$, $(0<\theta<\pi/2)$, be the wave vector for the nonuniform beam, which has frequency ω. The geometry is shown in Fig. 3. Let γ_s be the average scattering rate. In the low Doppler approximation,

$$\frac{d\langle r_z^2\rangle}{dt}=-2\gamma_s(\omega_0-\omega)\hbar k^2\cos^2\theta\langle r_z^2\rangle/M[(\gamma/2)^2+(\omega_0-\omega)^2]+(2\gamma_s R/M\omega_z^2)(\cos^2\theta+f_{sz})\,, \tag{65a}$$

$$\frac{d\langle r_m^2\rangle}{dt}=-\gamma_s\hbar k\sin\theta\langle r_m^2\rangle/2M\Omega y_0+\gamma_s(\omega_0-\omega)\hbar k^2\sin^2\theta\omega_m\langle r_m^2\rangle/M\Omega[(\gamma/2)^2+(\omega_0-\omega)^2]$$
$$+(\gamma_s R/2M\Omega^2)(\sin^2\theta+f_{sx}+f_{sy})\,, \tag{65b}$$

$$\frac{d\langle r_c^2\rangle}{dt}=\gamma_s\hbar k\sin\theta\langle r_c^2\rangle/2M\Omega y_0-\gamma_s(\omega_0-\omega)\hbar k^2\sin^2\theta\omega_c'\langle r_c^2\rangle/M\Omega[(\gamma/2)^2+(\omega_0-\omega)^2]$$
$$+(\gamma_s R/2M\Omega^2)(\sin^2\theta+f_{sx}+f_{sy})\,. \tag{65c}$$

Some small terms have been neglected. The axial motion can be cooled if $\omega<\omega_0$. The condition for simultaneous cooling of the magnetron and cyclotron modes, assuming $y_0>0$ and $\omega<\omega_0$, is given by Eq. (60), with $k\sin\theta$ substituted for k_1. If this condition is met, which is feasible experimentally, the steady-state mean-squared amplitudes are given by

$$\langle r_z^2\rangle=\hbar(1+f_{sz}/\cos^2\theta)[(\gamma/2)^2+(\omega_0-\omega)^2]/2M\omega_z^2(\omega_0-\omega)\,, \tag{66a}$$

$$\langle r_m^2\rangle=\frac{\hbar k y_0(\sin^2\theta+f_{sx}+f_{sy})}{2M\Omega\sin\theta\{1-2k\sin\theta y_0(\omega_0-\omega)\omega_m/[(\gamma/2)^2+(\omega_0-\omega)^2]\}}\,, \tag{66b}$$

$$\langle r_c^2\rangle=\frac{\hbar k y_0(\sin^2\theta+f_{sx}+f_{sy})}{2M\Omega\sin\theta\{2k\sin\theta y_0(\omega_0-\omega)\omega_c'/[(\gamma/2)^2+(\omega_0-\omega)^2]-1\}}\,. \tag{66c}$$

We consider the same limit as in Sec. III C 3, i.e., the Doppler selection cooling rate of the cyclotron motion is much greater than the intensity gradient heating rate, and the Doppler selection heating rate of the magnetron motion is less than and not too close to the intensity gradient cooling rate. Then

$$\langle E_{Kz}\rangle=\hbar(1+f_{sz}/\cos^2\theta)[(\gamma/2)^2+(\omega_0-\omega)^2]/8(\omega_0-\omega)\,, \tag{67a}$$

$$\langle E_{Kc}\rangle\cong\hbar[1+(f_{sx}+f_{sy})/\sin^2\theta][(\gamma/2)^2+(\omega_0-\omega)^2]/4(\omega_0-\omega)\,, \tag{67b}$$

and $\langle E_{Km}\rangle$ is negligible. These quantities are minimized by setting $(\omega_0-\omega)=\gamma/2$:

$$\langle E_{Kz}\rangle=(1+f_{sz}/\cos^2\theta)\hbar\gamma/8\,, \tag{68a}$$

$$\langle E_{Kc}\rangle\cong[1+(f_{sx}+f_{sy})/\sin^2\theta]\hbar\gamma/4\,. \tag{68b}$$

The total kinetic energy is minimized with respect to θ for

$$\tan^4\theta=2(f_{sx}+f_{sy})/f_{sz}\,. \tag{69}$$

For the isotropic angular distribution, this condition is, $\theta=\tan^{-1}(\sqrt{2})\cong 54.74°$, in which case, $\langle E_{Kz}\rangle\cong\frac{1}{2}\langle E_{Kc}\rangle\cong\hbar\gamma/4$. This is the same limit obtained in Sec. III C 3.

4. *Comparison with experimental data*

The most unambiguous comparison between theory and experiment can be made for the case of a single ion. The details of energy exchange due to collisions are not precisely understood for the case of a cloud of ions; moreover the frequencies of oscillation ω_z, ω_c', and ω_m are all shifted by the presence of space charge. Both of these problems are absent for a single ion, and the simple theory discussed above should apply.

Laser cooling of a single $^{24}Mg^+$ ion in a Penning trap has been performed previously.[9] The cooling achieved was such that the Doppler broadening was significantly less than the natural linewidth of the transition, so the results of the last section should apply. For this experiment,[9] a single $^{24}Mg^+$ ion was stored in the trap and cooled. The trap parameters were $B_0 \cong 1.01$ T and $A_0 = V_0/(r_0^2+2z_0^2) \cong 2.95$ V/cm^2, where the interelectrode voltage was $V_0 \cong 2$ V and the electrode dimensions, in the notation of Ref. 4, were such that $r_0^2+2z_0^2 \cong 0.678$ cm^2. This gave approximate observed frequencies of $\omega_c' \cong 2\pi\times 638.5$ kHz, $\omega_z \cong 2\pi\times 106.9$ kHz, and $\omega_m \cong 2\pi\times 8.95$ kHz. The laser beam intensity profile in the vicinity of the ion was measured to be $I(y) \cong I'\exp[-2(y-d)^2/w_0^2]$, where $d \cong 15$ μm and $w_0 \cong 25$ μm.

From this expression, we estimate

$$\frac{1}{y_0} = \frac{1}{I(y=0)} \frac{dI}{dy}\bigg|_{y=0} = \frac{1}{10.4\,\mu\text{m}} .$$

The transition involved in the laser cooling was the $^{24}Mg^+$ $(3p\,^2P_{3/2}, M_J=-\frac{3}{2}) \leftarrow (3s\,^2S_{1/2}, M_J=-\frac{1}{2})$ transition at $\lambda = 279.6$ nm; for this case [see Eqs. (8b) and (10b)] $f_{sz}=\frac{2}{5}$ and $f_{sx}+f_{sy}=\frac{3}{5}$. From Eq. (69), the angle of laser propagation that minimizes the total kinetic energy is $\theta = \tan^{-1}(3^{1/4}) \cong 52.77°$. Experimentally, θ could be varied between 82° and 90°, and it was set at 82°. The laser detuning was such that $(\omega_0-\omega) \cong 2\pi\times 25$ MHz.

From Eqs. (66a)–(66c) we predict $\langle r_z^2\rangle = (4.17\times 10^{-4}\text{ cm})^2$, $\langle r_c^2\rangle = (2.07\times 10^{-5}\text{ cm})^2$, and $\langle r_m^2\rangle = (5.25\times 10^{-5}\text{ cm})^2$. An important parameter in very high resolution spectroscopy is the second-order Doppler shift $\Delta\omega_0/\omega_0 \cong -\frac{1}{2}\langle v^2\rangle/c^2$. Because this shift is proportional to the kinetic energy of the ions, it is convenient to define a "temperature" in terms of the kinetic energy. The concept of a temperature may not be valid for a single ion, but it is reasonable to define temperatures for the axial and cyclotron degrees of freedom in terms of the time-averaged kinetic energies as $\langle E_{Kz}\rangle \equiv \frac{1}{2}k_B T_z$ and $\langle E_{Kc}\rangle \equiv k_B T_c$, where k_B is Boltzmann's constant. For the magnetron degree of freedom, there is an additional problem since the total energy is negative (when the electric potential is defined to be zero at the origin) and decreases to minus infinity as $\langle r_m^2\rangle$ goes to plus infinity. However, it is still useful to define, by analogy with the cyclotron motion, $\langle E_{Km}\rangle \equiv k_B T_m$. Using these definitions, we find for the experiment of Ref. 9, $T_z \cong 11$ mK, $T_c \cong 1.0$ mK, and $T_m \cong 1.3\times 10^{-6}$ K. From the residual Doppler broadening of the optical lines measured in Ref. 9 we estimated that $T_c = (50 \pm 30)$ mK and $T_m \lesssim 1$ mK. At the present time we do not understand why the measured temperatures are higher than predicted. One difficulty not mentioned previously is the possible collisional heating due to impurity ions, such as NH_3^+, H_3O^+, N_2H^+, and HCO^+, that might be simultaneously trapped.

IV. LASER COOLING IN A PENNING TRAP (QUANTUM-MECHANICAL TREATMENT)

A. Quantum states of an ion in a Penning trap

The Hamiltonian operator for a single ion in the same idealized Penning trap which was treated classically in Sec. II A, is

$$H = [\vec{p} - (q/c)\vec{A}(\vec{r})]^2/2M + qV(\vec{r}) \equiv H_{xy} + H_z , \tag{70}$$

where

$$H_{xy} = (p_x^2+p_y^2)/2M + \tfrac{1}{2}M\Omega^2(x^2+y^2) - \tfrac{1}{2}\omega_c L_z , \tag{71a}$$

and

$$H_z = p_z^2/2M + \tfrac{1}{2}M\omega_z^2 z^2 . \tag{71b}$$

Here $\vec{r}$ and $\vec{p}$ are operators and L_z is an operator defined by $L_z \equiv xp_y - yp_x$. This Hamiltonian has been dealt with by other authors.[30,31] Here we summarize the useful results. H_z is the Hamiltonian of a one-dimensional simple harmonic oscillator. Therefore it can be rewritten as

$$H_z = (N_z + \tfrac{1}{2})\hbar\omega_z , \tag{72}$$

where

$$N_z = a_z^\dagger a_z , \tag{73a}$$

$$z = z_0(a_z^\dagger + a_z) , \tag{73b}$$

$$p_z = iz_0 M\omega_z(a_z^\dagger - a_z) , \tag{73c}$$

$$z_0 = (\hbar/2M\omega_z)^{1/2} . \tag{73d}$$

H_{xy} is the Hamiltonian of a two-dimensional isotropic oscillator, plus a term proportional to L_z. It can be rewritten as

$$H_{xy} = (N_c + \tfrac{1}{2})\hbar\omega_c' - (N_m + \tfrac{1}{2})\hbar\omega_m , \tag{74}$$

where

$$N_i = a_i^\dagger a_i \quad (i = c, m) \tag{75a}$$

$$a_c = (a_x + ia_y)/2^{1/2} , \tag{75b}$$

$$a_m = (a_x - i a_y)/2^{1/2} , \quad (75c)$$

$$x = r_0(a_x^\dagger + a_x)/2^{1/2}, \quad y = r_0(a_y^\dagger + a_y)/2^{1/2} , \quad (75d)$$

$$p_j = r_0 M\Omega(a_j^\dagger - a_j)/2^{1/2} \quad (j = x,y) \quad (75e)$$

$$r_0 = (\hbar/M\Omega)^{1/2} . \quad (75f)$$

The three sets of raising and lowering operators satisfy the commutation relations

$$[a_i, a_j] = 0 , \quad (76a)$$

$$[a_i^\dagger, a_j^\dagger] = 0 , \quad (76b)$$

$$[a_i, a_j^\dagger] = \delta_{ij} I \quad (i = z,c,m) \quad (76c)$$

where I is the identity operator. The axial component of the angular momentum operator can be written in the form

$$L_z = (N_m - N_c)\hbar . \quad (77)$$

The complete set of eigenstates of H is given by the set of simultaneous eigenstates of N_z, N_m, and N_c. We label these states $|n_z, n_m, n_c\rangle$, where n_z, n_m, and n_c are the eigenvalues of N_z, N_m, and N_c and can be any set of nonnegative integers. They can be generated from the $|0,0,0\rangle$ state by applying the raising operators. The normalized states are

$$|n_z, n_m, n_c\rangle = (n_z! n_m! n_c!)^{-1/2} (a_z^\dagger)^{n_z} (a_m^\dagger)^{n_m} (a_c^\dagger)^{n_c} |0,0,0\rangle . \quad (78)$$

the coordinate-space representation of $|0,0,0\rangle$ is

$$\langle x,y,z | 0,0,0\rangle = \pi^{-3/4} (2^{1/2} z_0 r_0^2)^{-1/2} \exp[-z^2/2z_0^2 - (x^2+y^2)/2r_0^2] . \quad (79)$$

Explicit expressions for the wave functions for arbitrary quantum numbers have been given by Sokolov and Pavlenko.[31] By comparing the classical expressions for the energy [Eq. (49)] and the axial angular momentum component [Eq. (52)] with the corresponding quantum-mechanical operators, [Eqs. (72), (74), and (77)], one obtains the following correspondences between classical amplitudes of motion and the quantum numbers:

$$r_c^2 \sim (n_c + \tfrac{1}{2}) r_0^2 , \quad (80a)$$

$$r_m^2 \sim (n_m + \tfrac{1}{2}) r_0^2 , \quad (80b)$$

$$r_z^2 \sim 4(n_z + \tfrac{1}{2}) z_0^2 . \quad (80c)$$

B. Transition rates

We treat the interaction of the ion with the electromagnetic field by second-order time-dependent perturbation theory, in the form of the "golden rule." The motion of the ion as a whole, as well as its internal structure, is described by wave functions. In Sec. III, only the internal structure was treated quantum mechanically. Consider the transition rate from $|n_z^l, n_m^l, n_c^l\rangle \equiv |\{n^l\}\rangle$ to $|\{n^f\}\rangle$ induced by a near-resonant uniform laser beam described by Eq. (1). The ion is in its ground electronic state $|g\rangle$ before and after the transition. This is a kind of spontaneous resonance Raman scattering, through the intermediate electronic state $|e\rangle$. The cross section for this process is given by Eq. (19) of Ref. 16. We make the electric-dipole approximation and sum over all polarizations and directions of the scattered photon.

This cross section is

$$\sigma(\{n^l\} \to \{n^f\}) = \sigma_0(\gamma/2)^2 \int \left| \sum_{\{n^j\}} \frac{\langle \{n^f\} | \exp(-i\vec{k}_s \cdot \vec{r}) | \{n^j\}\rangle \langle \{n^j\} | \exp(i\vec{k} \cdot \vec{r}) | \{n^l\}\rangle}{\omega_0 - \omega - i\gamma/2 + \omega(\{n^j\}) - \omega(\{n^l\})} \right|^2 P_s(\hat{k}_s) d\Omega , \quad (81)$$

where $\omega(\{n^i\}) \equiv (n_z^i + \tfrac{1}{2})\omega_z + (n_c^i + \tfrac{1}{2})\omega_c' - (n_m^i + \tfrac{1}{2})\omega_m$, and the other terms have been defined previously. We have replaced $\vec{x}$ of Ref. 16 with $\vec{r}$ here. The transition rate is obtained by multiplying this cross section by $I/\hbar\omega$.

The transition rate due to a nonuniform laser beam can also be calculated by perturbation theory. The only difference is that the electric field vector due to the laser perturbation $-\vec{d}\cdot\vec{E}(\vec{r},t)$, which induces transitions to the intermediate state, is no longer a uniform plane wave of the form of Eq. (1). We consider only the low-intensity limit, in which case the transition from the intermediate state to the final state is induced by the zero-point field. Consider a nonuniform beam like the one treated in Sec. III C 3, which has an intensity gradient in the y direction and has a wave vector $\vec{k}=k\sin\theta\hat{x}+k\cos\theta\hat{z}$, $(0<\theta<\pi/2)$. Let the electric field be

$$\vec{E}(\vec{r},t)=\hat{\epsilon}(1+y/2y_0)\mathrm{Re}E_0\exp(i\vec{k}\cdot\vec{r}-i\omega t)\ , \tag{82}$$

which gives the intensity profile

$$I(y)=I_0[1+y/y_0+(y/2y_0)^2]\cong I_0(1+y/y_0) \tag{83}$$

for $|y|<<y_0$. We can use this $\vec{E}$ in the perturbation matrix element to reproduce the case treated classically in Sec. III C 3, provided that the wave functions are localized in the region $|y|<<y_0$. The transition rate is

$$\gamma(\{n^l\}\to\{n^f\})=(I_0/\hbar\omega)\sigma^*(\{n^l\}\to\{n^f\})\ , \tag{84}$$

where the effective cross section σ^* is given by

$$\sigma^*(\{n^l\}\to\{n^f\})=\sigma_0(\gamma/2)^2\int\left|\sum_{\{n^j\}}\frac{\langle\{n^f\}|\exp(-i\vec{k}_s\cdot\vec{r})|\{n^j\}\rangle\langle\{n^j\}|(1+y/2y_0)\exp(i\vec{k}\cdot\vec{r})|\{n^l\}\rangle}{\omega_0-\omega-i\gamma/2+\omega(\{n^j\})-\omega(\{n^l\})}\right|^2 P_s(\hat{k}_s)d\Omega\ . \tag{85}$$

C. Rate equations for average quantum numbers

We calculate the rate equations for the ensemble averages of the quantum numbers $\langle n_z\rangle$, $\langle n_m\rangle$, and $\langle n_c\rangle$. We do not consider coherences between $|n_z,n_m,n_c\rangle$ states. Let $P(\{n^l\})$ be the probability that the ion has the set of quantum numbers $\{n^l\}$. We can derive the rate equations for any of the cases considered in Sec. III. For the sake of brevity we consider only the case of the nonuniform laser beam propagating in the xz plane [Sec. III C 3], using the effective cross section defined by Eqs. (84) and (85). The rate equations are given by[16]

$$\frac{d\langle n_i\rangle}{dt}=\sum_{\{n^l\},\{n^f\}}P(\{n^l\})\gamma(\{n^l\}\to\{n^f\})(n_i^f-n_i^l)\ , \tag{86}$$

where

$$i=z,m,c\ .$$

We make the low Doppler approximation. In this limit we can make the approximation

$$[\omega_0-\omega-i\gamma/2+\omega(\{n^j\})-\omega(\{n^l\})]^{-1}\cong(\omega_0-\omega-i\gamma/2)^{-1}\{1+[\omega(\{n^l\})-\omega(\{n^j\})]/(\omega_0-\omega-i\gamma/2)\} \tag{87}$$

in Eq. (85). Equation (87) is valid for

$$|[\omega(\{n^l\})-\omega(\{n^j\})]/(\omega_0-\omega-i\gamma/2)|<<1\ .$$

This condition is violated for some terms in Eq. (85), which correspond to large differences in quantum numbers, since the summation is over all possible $\{n^j\}$, but these terms are of negligible size, because the matrix elements are very small. With this approximation, Eq. (86) can be evaluated by straightforward algebra. We outline the important steps here. The method was previously used for the harmonic trap in Ref. 16. We note that

$$\langle \{n^j\} | A | \{n^l\} \rangle [\omega(\{n^l\}) - \omega(\{n^j\})] = \langle \{n^j\} | [A, H/\hbar] | \{n^l\} \rangle \tag{88}$$

for any operator A. The summations in Eq. (86) over $\{n^f\}$ and $\{n^j\}$ can be carried out using the closure property, and we obtain

$$\frac{d\langle n_i \rangle}{dt} = \gamma_s \sum_{\{n^l\}} P(\{n^l\}) \int \langle \{n^l\} | A^\dagger [N_i, A] | \{n^l\} \rangle P_s(\hat{k}_s) d\Omega \ . \tag{89}$$

Here

$$A \equiv \exp(-i\vec{k}_s \cdot \vec{r}) \{ (1 + y/2y_0) \exp(i\vec{k} \cdot \vec{r}) + [(1 + y/2y_0) \exp(i\vec{k} \cdot \vec{r}),\ H/\hbar(\omega_0 - \omega - i\gamma/2)] \} \ , \tag{90}$$

and

$$\gamma_s = (I_0 \sigma_0 / \hbar\omega)(\gamma/2)^2 / [(\omega_0 - \omega)^2 + (\gamma/2)^2] \tag{91}$$

is the average scattering rate. We define the ensemble averages $\langle r_i^2 \rangle$, using Eqs. (80a)–(80c) by

$$\langle r_z^2 \rangle \equiv 4z_0^2 \langle n_z + \tfrac{1}{2} \rangle \equiv 4z_0^2 \sum_{\{n^l\}} P(\{n^l\})(n_z^l + \tfrac{1}{2}) \ , \tag{92a}$$

$$\langle r_i^2 \rangle \equiv r_0^2 \langle n_i + \tfrac{1}{2} \rangle \equiv r_0^2 \sum_{\{n^l\}} P(\{n^l\})(n_i^l + \tfrac{1}{2}) \quad (i = m, c) \ . \tag{92b}$$

The rate equations for $\langle r_i^2 \rangle$ $(i = z, m, c)$, obtained by evaluating Eq. (89) and dropping small terms, are identical with Eqs. (65a)–(65c). This is not surprising, because the average quantum numbers are large. The other cases treated in Sec. II C can be treated by the same methods. The rate equations obtained are identical to those obtained from the classical analysis. Here the following quantities are regarded as small: r_m/y_0, r_c/y_0, ω_m/γ, ω_c'/γ, ω_z/γ, $R/\hbar\gamma$, $k\omega_m r_m/\gamma$, $k\omega_c' r_c/\gamma$, and $1/ky_0$.

The quantities kz_0 and kr_0 are assumed to be not too large, so that when multiplied by a small quantity, the result is still small.

V. DISCUSSION

We have carried out a semiclassical calculation of the laser cooling of an ion in a Penning trap, for the case where the frequencies of motion are much less than the natural linewidth of the optical transition. The results are confirmed by a quantum-mechanical calculation based on perturbation theory. Also, some simple calculations of laser cooling in a harmonic trap are presented. Where a direct comparison can be made, the results agree with those obtained by more formal methods.[20,23]

Experiments have confirmed the qualitative features of the present analysis.[7–9] Cooling to average kinetic energies so low that the Doppler broadening of the optical transition is smaller than the natural linewidth has been observed.[9] However, the ultimate limits predicted by the theory have not been achieved in these experiments.

Useful improvements to the theory would include taking into account Coulomb forces between ions and assuming a more realistic laser beam profile. Saturation of the optical transition has not been included. However, this is not a severe restriction, especially for the final stages of cooling, since we would expect the minimum energies to be obtained in the low-intensity limit, as for the harmonic trap.[20] It would be interesting to apply the Fokker-Planck formalism to the Penning trap as has been done for the harmonic trap.[20–23]

In this paper, the case of resolved sidebands $(\omega_c', \omega_z, \omega_m >> \gamma)$ has not been treated but should be quite straightforward and could follow the development of Ref. 16 and this paper. We remark that when the mean occupation numbers $\langle n_i \rangle$ are small enough, then all sidebands can be well resolved. This means that cooling can be achieved using spatially *uniform* plane waves provided the lasers are tuned to the correct sidebands.

Motional sideband cooling of the magnetron motion of an electron in a Penning trap[32] has been demonstrated and is closely related to laser cooling. Here the axial resonance, which is analogous to the optical transition of the two-level ion, is driven by an inhomogeneous rf field at the sideband frequency $\omega_z + \omega_m$. The modes are coupled by the field in such a way that the magnetron motion is cooled.

We also remark that, in general, the reduction of the magnetron orbits can be achieved by other means which add (total) energy to the magnetron degree of freedom.[33] This could be achieved, for example, by coupling the magnetron motion to a negative resistance or by collisions with atomic beams that preferentially interact with the side of the cloud where the magnetron motion is in the same direction as the beam. This effect could be important in certain collision experiments.[34]

All previous detailed theoretical treatments of laser cooling of trapped ions have assumed harmonic potential wells. The experiments which most closely approach this situations are performed with rf quadrupole traps, in which the motion of an ion consists of the high frequency, small amplitude "micromotion" superimposed on the low frequency, large amplitude "secular motion."[4] Approximating such a trap by a harmonic potential neglects the micromotion. The average kinetic energy in the micromotion, averaged over an rf cycle, is just an effective potential energy for the secular motion. A formalism which could take account of the micromotion, either classically or quantum mechanically, might be useful for the interpretation of the experiments.

Finally, we wish to point out that Einstein,[35] in the same paper in which the A and B coefficients for spontaneous and stimulated emission were introduced, treated a problem which is closely related to laser cooling. He showed that a gas of two-level atoms comes into thermal equilibrium with radiation having the Planck spectrum, as a result of light pressure forces. In this case, since the radiation is broad band and isotropic, the average damping force is mostly due to an effect that we have neglected: The apparent intensity of a light source increases if the observer moves toward the source, as a result of the transformation law for the electric and magnetic fields. The Doppler shift and angular aberration of the light must also be taken into account. The damping force due to the average light pressure competes with the fluctuating force due to photon recoil so that, in the steady state and in the nonrelativistic limit, $E_{Kx}=\frac{1}{2}k_BT$, if the radiation has the Planck spectrum for temperature T. Recently, the calculation has been extended to demonstrate that the gas has the relativistic Boltzmann distribution after it comes into thermal equilibrium with the radiation.[36]

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APPENDIX: SPECTRAL DENSITY OF A RANDOM SERIES OF IMPULSES

We calculate here the one-sided (i.e., defined for positive frequencies) power spectral density of $G(t)$, where $G(t)$ is a stationary, random function of time consisting of a series of Dirac delta functions. These impulses occur at an average rate γ and they may vary in amplitude. This calculation follows that of Rice.[37]

First we calculate the correlation function,

$$R(\tau)\equiv\langle G(t)G(t+\tau)\rangle\ , \tag{A1}$$

where the angular brackets denote an ensemble average, and R is independent of t, by the assumption of stationarity. We consider $G(t)$ during the time interval $0<t<T$, where T is arbitrary. We denote a particular member of the ensemble by $G_{Kk}(t)$, where K is the number of impulses that occur in the period, and k uniquely identifies the member. This function takes the following form:

$$G_{Kk}(t)=\sum_{l=1}^{K} g_{Kkl}\delta(t-t_{Kkl})\ . \tag{A2}$$

The amplitudes g_{Kkl} can vary, but are not correlated in any way with t_{Kkl}, K, or l. The l index does not indicate an order in time. For fixed K and l, t_{Kkl} can be anywhere in the interval from 0 to T with equal probability. We define ensemble averages over the index k as

$$\langle g_{Kkl}\rangle_k\equiv\langle g\rangle\ , \tag{A3}$$

$$\langle g_{Kkl}^2\rangle_k\equiv\langle g^2\rangle\ , \tag{A4}$$

where $\langle g\rangle$ and $\langle g^2\rangle$ are independent of l and K. The product $G_{Kk}(t)G_{Kk}(t+\tau)$ takes the form

$$G_{Kk}(t)G_{Kk}(t+\tau)=\sum_{l=1}^{K}\sum_{l'=1}^{K} g_{Kkl}g_{Kkl'}\delta(t-t_{Kkl})\delta(t+\tau-t_{Kkl'})$$
$$=S_1+S_2\ , \tag{A5}$$

where

$$S_1=\sum_{l=1}^{K}(g_{Kkl})^2\delta(t-t_{Kkl})\delta(t+\tau-t_{Kkl}) \tag{A6}$$

and

$$S_2=\sum_{l\neq l'} g_{Kkl}g_{Kkl'}\delta(t-t_{Kkl})\delta(t+\tau-t_{Kkl'})\ . \tag{A7}$$

Now we carry out the ensemble averages over the index k for fixed K:

$$\begin{aligned}\langle S_1\rangle_k&=\sum_{l=1}^{K}(\langle g^2\rangle/T)\int_0^T\delta(t-t_{Kkl})\delta(t+\tau-t_{Kkl})dt_{Kkl}\\&=\sum_{l=1}^{K}(\langle g^2\rangle/T)\delta(\tau)=K(\langle g^2\rangle/T)\delta(\tau)\end{aligned} \tag{A8}$$

and

$$\begin{aligned}\langle S_2\rangle_k&=\sum_{l\neq l'}(\langle g\rangle/T)^2\int_0^T\delta(t-t_{Kkl})dt_{Kkl}\int_0^T\delta(t+\tau-t_{Kkl'})dt_{Kkl'}\\&=\sum_{l\neq l'}(\langle g\rangle/T)^2=K(K-1)(\langle g\rangle/T)^2\ .\end{aligned} \tag{A9}$$

The ensemble average over K is carried out using the Poisson distribution function

$$p(K)=(\gamma T)^K e^{-\gamma T}/K!\ , \tag{A10}$$

where $p(K)$ is the probability that K impulses occur during the period T. Finally, we have

$$R(\tau)=\sum_{K=0}^{\infty}p(K)(\langle S_1\rangle_k+\langle S_2\rangle_k)=\gamma\langle g^2\rangle\delta(\tau)+\gamma^2\langle g\rangle^2\ . \tag{A11}$$

The spectral density is given by

$$w(f)=4\int_0^\infty R(\tau)\cos 2\pi f\tau\,d\tau=2\gamma\langle g^2\rangle+2\gamma^2\langle g\rangle^2\delta(f)\ , \tag{A12}$$

which is defined for frequencies $f\geq 0$. The first (white noise) term corresponds to the spectral density calculated in Eq. (39). The second (dc) term comes from the average value of G, which correspond to $\langle F_x\rangle$ defined in Eq. (33).

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