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Multiplicative noise in the Vinen equation for turbulent superfluid ^{4}He

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We have studied the effects of multiplicative noise in the counterflow velocity on the Vinen phenomenological description of vortex-line turbulence in superfluid helium using the Fokker-Planck equation. The principal result is that the noise pushes the critical velocity for the onset of turbulence toward higher values. We discuss a noise-induced phase transition which is not likely to be physically observable in turbulence experiments.

In recent years, the effects of external noise on the phenomenological description of nonequilibrium, macroscopic systems has received a great deal of attention. Probably the simplest such systems are the first-order, one-dimensional, nonlinear differential equations with multiplicative noise, studied by Horsthemke and Lefever and their collaborators.[1] Their most remarkable result is the discovery of so-called noise-induced phase transitions (NIPT's), wherein new statistically stationary states become accessible to the system when the noise intensity exceeds a certain, well-defined critical value. The applications are to certain biological,[2] chemical,[3] and electronic[4] systems, where some experimental results, at least qualitatively, seem to indicate the existence of NIPT's. Mikhailov has predicted NIPT's in diffusive nonlinear biological systems.[5]

More recently, the transition to turbulence of classical systems with noisy parameters has been described in terms of one-dimensional, difference equations,[6,7] a higher-order, continuous nonlinear equation with harmonic forcing,[8] and the multidimensional continuous Lorenz system,[9] which represents the Rayleigh-Benard instability. At least one experiment on classical turbulence with deliberately noisy parameters has been reported.[10]

Motivated by these results, and by the likelihood that a superfluid turbulence experiment with noisy parameters could easily be done, we have studied the Vinen equation as modified by Tough,[11] which describes the onset and growth of vortex-line turbulence.[12] We have superimposed Gaussian white noise on the counterflow velocity and used a Fokker-Planck analysis following Horsthemke *et al.*[1] In a previous study by Northby,[13] the Vinen equation was linearized and noise added to the counterflow velocity. The results of these two approaches are not immediately comparable, in that the present approach yields pseudoprobability density functions of the line density, while the linear approach results in the power spectra of the line density fluctuations.

The Vinen equation reads

$$\frac{dL}{dt}=Av\left[1-\frac{\alpha}{dL^{1/2}}\right]L^{3/2}-C\kappa L^2\ , \tag{1}$$

where A, C, and α are dimensionless constants, d is the tube diameter, and where L, v, and κ are the vortex-line density, counterflow velocity, and circulation quantum, respectively. For a detailed discussion of these parameters and the remarkably accurate description of superfluid turbulence provided by the steady-state solutions of Eq. (1), see Refs. 11 and 12. The well-known steady-state solutions $L_0(v)$, obtained from setting $dL/dt=0$ in Eq. (1), are pairs of real values for $v\geq v_c=4C\kappa\alpha/Ad$. We can convert Eq. (1) to dimensionless form with the following definitions: $\nu=v/v_c$, $X=L/L_0(v_c)$, and $\tau=4C\kappa(\alpha/d)^2t$ with the compact result that

$$\frac{dX}{d\tau}=2\nu X(X^{1/2}-\tfrac{1}{2})-X^2 \tag{2a}$$

$$=\nu g(X)+h(X)\ . \tag{2b}$$

The steady-state solutions of Eq. (2a) are

$$X_s^{1/2}=\nu(1\pm\sqrt{1-1/\nu})\ , \tag{3}$$

which are real when $\nu\geq 1$.

We now allow the dimensionless velocity to vary randomly with time: $\nu\rightarrow V+\sqrt{S}\,\xi_\tau$ such that

$\langle v \rangle = V$, and the variance is S. We suppose this noise to be Gaussian and white, so that $\langle \xi_\tau \xi'_\tau \rangle = \delta(\tau - \tau')$. Then Eq. (2) becomes a stochastic differential equation, and $X(\tau)$ can be described by its probability density $\rho(X,t)$, which is the solution of the Fokker-Planck equation[14]

$$\partial_t \rho(X,t) = -\partial_X [f(X,V) + (\tfrac{1}{2}) S g(X) \partial_X g(X)] \rho(X,t) + (\tfrac{1}{2}) S \partial_{XX} g^2(X) \rho(X,t) , \quad (4)$$

where $f(X,V) = Vg(X) + h(X)$, as defined by Eqs. (2). The well-known solution of $\partial_t \rho(X,t) = 0$ is

$$\rho_s(X) = [N/g(X)] \exp(2/S) \int_{X_0}^{X} (f/g^2) dX , \quad (5)$$

which with Eq. (2) results in

$$\rho_s(X) = \frac{N}{2X(X^{1/2} - \frac{1}{2})} \left[\frac{X_0}{X} \right]^{2V/S} \left[\frac{X^{1/2} - \frac{1}{2}}{X_0^{1/2} - \frac{1}{2}} \right]^{4V - 1/S} \exp \left[\frac{X_0^{1/2} - X^{1/2}}{2S(X^{1/2} - \frac{1}{2})(X_0^{1/2} - \frac{1}{2})} \right] , \quad (6)$$

where X_0 has been included to avoid the singularity at $X = 0, \frac{1}{4}$, and N is a normalizing constant which we have set equal to 1. The steady-state solutions of Eq. (1) are real only when the mean spacing between lines $L^{-1/2} \leq d$. This leads to the singularity in Eq. (6) at $X, X_0 = \frac{1}{4}$. The function ρ_s, given by Eq. (6), thus represents some measure of the physically observable line density only for $X, X_0 > \frac{1}{4}$, though it is real also for $X, X_0 < \frac{1}{4}$, as we discuss below. We call ρ_s a pseudodensity because the singularities are nonintegrable.

Figure 1 shows $\rho_s(X)$ for $S = 0.5$ and for three values of V, indicating a critical value $V_c = 1.41, \ldots$. For $V < V_c$, the only mode is the singularity at $X \rightarrow \frac{1}{4}$, while for $V > V_c$ a minimum and a maximum appear at unobservable and observable X, respectively. The locations of these extrema X_m, are obtained from $d\rho_s/dX = 0$, which results in the quadratic in $X_m^{1/2}$

$$\left[\frac{3}{2} + \frac{1}{2S} \right] X_m - \left[\frac{5}{4} + \frac{V}{S} \right] X_m^{1/2} + \frac{1}{4} + \frac{V}{2S} = 0 . \quad (7)$$

The solutions are shown by the solid curves in Fig. 2 for several values of S as a parameter. The broken curve divides the plane into stable solutions above and unstable solutions below, and intersects the solid curves at the values $V = V_c$ and $X_m = X_{mc}$ which mark the onset of turbulence. The curve marked $S = 0$ is the deterministic result using Eq. (3) for which $V_c = 1$ and $X_{mc} = 1$, and is identical with the result obtained in the $S = 0$ limit of the

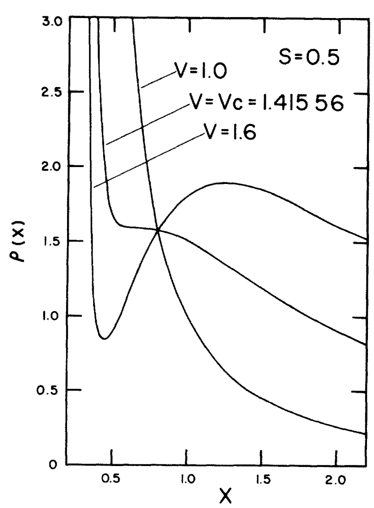

FIG. 1. Stationary probability density functions of the dimensionless vortex-line density calculated from Eq. (6) with $X_0 = 0.8$. Equation (6) is not normalized, so the vertical scale is arbitrary.

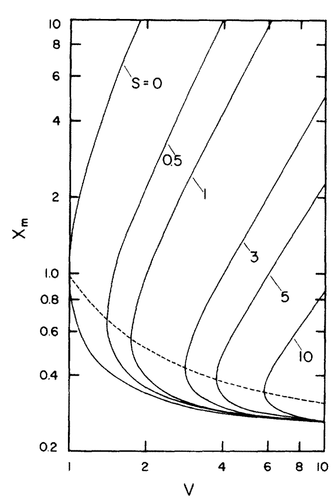

FIG. 2. Extrema of the density functions calculated from Eq. (7), and for $S = 0$ from Eq. (3).

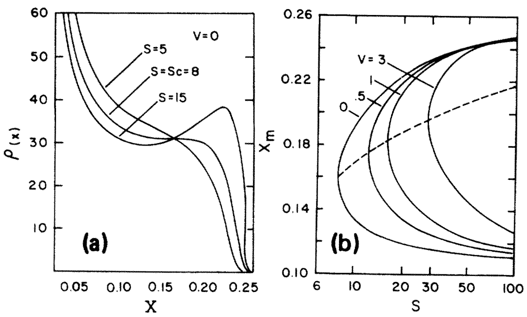

FIG. 3. (a) Probability density functions from Eq. (6) with $X_0=0.16$, showing the NIPT at $S_c=8$. (b) The extrema of the density functions.

solutions of Eq. (7). Figure 2 shows our principal result that *noise superimposed on the counterflow velocity enhances the stability of the flow* by pushing V_c toward higher values and X_{mc} toward lower values compared to the deterministic case.

The effect of the noise is not small. The curve marked $S=1$ in Fig. 2 shows a critical velocity nearly twice the $S=0$ value, and this should be easily observable if Eq. (1) is a good dynamical description of vortex-line turbulence.[15] A similar effect of comparable magnitude has, in fact, already been observed in superfluid turbulence by Oberly and Tough[16] who modulated V with a harmonic function.

Enhancement of the stability of classical flows by harmonic modulation of a flow parameter has been well known since the Couette flow experiments of Donnelly *et al.*[17] The interpretation of this enhancement effect has been that it is the result of a decreasing viscous penetration depth δ with increasing modulation frequency, and that stability is maximized when $d/\delta \simeq 1$. It is interesting that the *comparable enhancements predicted here emerge solely from the inherent nonlinearities of the system.*[18]

We have also found a NIPT in Vinen's equation, but it lies in the unobservable region for $X, X_0 < \frac{1}{4}$. We report it here simply as an additional example and, to our knowledge, the first in a fluid dynamic system, of that class of fascinating objects discovered by the Brussels group.[1-3] Equation (7) again shows pairs of real roots for small V and even for $V=0$, *so long as* $S > S_c$. The pseudodensity functions, calculated from Eq. (6) with $X_0 < \frac{1}{4}$, are shown in Fig. 3(a), where for $V=0$ we find $S_c=8$. The extrema from Eq. (7) are shown by the solid curves in Fig. 3(b), where the broken line again divides the stable (upper) and unstable (lower) branches. Here, increasing velocity pushes the critical noise variance toward higher values. The physical significance, if any, of these solutions is unknown to us at this time.

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[14]We write here the Stratonovic version only and leave the Itô definition for a more complete description of this work. For a discussion of the two definitions, see especially the first item in Ref. 1.

[15]Figure 2 plots the loci of the maxima shown on Fig. 1. In order to observe these, ρ_s itself must be measured. We cannot predict $\langle X \rangle$ since Eq. (6) is nonintegrable.

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[18]While this paper was in review we have determined that these enhancements are generic to a certain class of nonlinear systems with multiplicative noise. Equation (1) is only one such example. Another example is the Schlögl model. See G. V. Welland and F. Moss, Phys. Lett. (in press).