Period doubling: Universality and critical-point order

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The dependence of the universal bifurcation ratio on the order of the local maximum is studied for a family of maps $f(x)=1-ax^z$ ($z=2,4,6,8$). Although these maps all exhibit period doubling, the precise bifurcation ratio depends on the order z, and seems to be an increasing function of z. We have employed a renormalization-group method to calculate $\delta(z)$ to second order. They agree very well with the exact values.

I. INTRODUCTION

The period-doubling route to chaotic behavior has become a subject of intense theoretical and experimental interest. The discovery of universal ratios[1] and subsequent experimental evidences[2–4] further excited workers in the field. The nonlinear dynamics displayed here also seems to be common to a variety of phenomena in a diversity of disciplines.[5]

The concept of universality is indeed very important in this regard. The mathematical model equation Feigenbaum studied is a simple one-dimensional quadratic map

$$f(x)=1-ax^2 . \tag{1}$$

However, the many features present in this map also seem to be embodied in much more complicated higher-dimensional systems with a large number of degrees of freedom.[6] Theoretically this was found to be the case when Franceschini *et al.*[7] studied a five-mode and a seven-mode truncation of the Navier-Stokes equations for a two-dimensional incompressible fluid on a torus and found that the results are compatible with Feigenbaum's theory. Experimentally, the Rayleigh-Bernard system was studied by Libchaber and Maurer[2] with a low Prandtl number, by Gollub *et al.*[3] with an intermediate Prandtl number, and by Giglio *et al.*[4] with a low aspect ratio. They have all found period-doubling bifurcations, and the measured universal ratios are consistent with the numerical values obtained by Feigenbaum. All these results seem very impressive indeed.

The concept of universality is common to both critical and chaotic phenomena. However, despite the importance and usefulness of this concept, if one asks for the defining criteria of universality classes, one cannot but conclude that universality is a less well-developed concept in chaotic than in critical phenomena. In critical phenomena, one can cite, for example, the dimensionality of space, the number of components of the order parameter, and the symmetry of the system, as the criteria for defining universality classes. These criteria have also met ample theoretical and experimental verifications. However, in chaotic phenomena, our knowledge is less complete. Yet we do know, for example, that δ depends on the order of the local maximum and on the conservative or dissipative nature of the map.[8] In this paper we will use a simple renormalization-group method to calculate explicitly the dependence of δ on the order of the local maximum for a family of maps

$$f(x)=1-ax^z \tag{2}$$

for $z=2,4,6,8$. They all exhibit a period-doubling route to chaotic behavior, and yet the rate of convergence depends on the order z. As will be shown later, the higher the order the faster the rate. In Sec. II the method of the renormalization group to calculate the bifurcation ratio will be described. Applications and the results will be presented and discussed. Finally, a summary will be given in Sec. III.

II. METHOD OF THE RENORMALIZATION GROUP

Many analogies have been drawn between chaotic and critical phenomena: e.g., the bifurcation ratio and the critical exponent, the Lyapunov exponent and the order parameter, etc. As a matter of fact, when Wilson first proposed the renormalization-group theory of critical phenomena, he already alluded to the relevance of the renormalization group to the study of turbulence. Feigenbaum in his first works further expanded on this point. However, it was not until Derrida *et al.*,[9] following the earlier discovery by Metropolis, Stein, and Stein (MSS)[10] of the universal ordering pattern of the MSS sequences, found the important property of self-similarity that a theoretical basis for the utility of the renormalization-group approach was laid. They developed a simple renormalization-group method and calculated the Feigenbaum universal ratio for the quadratic map to second order. Very good agreement with the exact numerical value was achieved. Recently the calculation has been extended to third order,[11] and extremely good agreement was attained. We will therefore adopt this method for our purposes in this paper.

The basic idea of Derrida *et al.* is to associate, for each value a, a value a' such that the $2n$th iterated map $f_{a'}^{(2n)}$ locally resembles the nth iterated map $f_a^{(n)}$. An approximate way to do it is to linearize the function in the neighborhood of a fixed point and equate the eigenvalue of the n cycle $\lambda_n(a)$ to that of the $2n$ cycle $\lambda_{2n}(a')$

$$\lambda_n(a)=\lambda_{2n}(a') \ . \tag{3}$$

The fixed point of this recursion relation then gives the limit point of the bifurcation sequence:

$$\lambda_n(a^*)=\lambda_{2n}(a^*) \ . \tag{4}$$

The derivative of a with respect to a' at the fixed point gives the bifurcation ratio

$$\delta=\left.\frac{\partial a}{\partial a'}\right|_{a^*} \ . \tag{5}$$

Successive higher cycles provide increasingly accurate approximations to the exact values. In the case of the quadratic map, analytic expressions can be derived for the lowest order. However, even for the second-order approximation, Derrida *et al.*[12] had to resort to ingenious tricks to derive an implicit analytic equation and solve it numerically. Obviously this algebraic approach very quickly becomes prohibitively difficult, if at all possible. One alternative is to use algebraic programs like MACSYMA or REDUCE to write a very complicated equation and solve it numerically. However, since we are ultimately interested in numbers anyway, it seems much simpler to solve the constitutive equations directly. Let us briefly describe the method.

Let $\{x_i\}$ be an n cycle, i.e.,

$$x_{i+1}=f(x_i) \ \ (\mathrm{mod}\ n) \ . \tag{6}$$

Each element x_i is then a fixed point of the nth iterate of f:

$$x_i=f^{(n)}(x_i) \ . \tag{7}$$

The chain rule then implies that the eigenvalue at each point of the cycle is the same:

$$\lambda_n(a)=\prod_{i=1}^{n} f'(x_i) \ . \tag{8}$$

Now, let $\{x_i'\}$ be a $2n$ cycle, then similarly

$$x_{i+1}'=f(x_i') \ \ (\mathrm{mod}\ 2n) \ , \tag{9}$$

$$\lambda_{2n}(a')=\prod_{i=1}^{2n} f'(x_i') \ . \tag{10}$$

If we denote by f_{a^*} the function f with parameter value a^*, then the set of equations to be solved are the following:

$$x_{i+1}=f_{a^*}(x_i) \ \ (\mathrm{mod}\ n) \tag{11}$$

$$x_{i+1}'=f_{a^*}(x_i') \ \ (\mathrm{mod}\ 2n) \tag{12}$$

$$\prod_{i=1}^{n} f_{a^*}'(x_i)=\prod_{i=1}^{2n} f_{a^*}'(x_i') \ . \tag{13}$$

This set of $(3n+1)$ equations solves for the $(3n+1)$ unknowns: $\{x_i\}$, $\{x_i'\}$, and a^*. After a^* has been found, δ can be found easily by solving the corresponding set of derived linear equations with known constant coefficients.

Using this approach, we have calculated to second order the universal bifurcation ratios for the family of maps as described in Eq. (2). Our results, together with the exact values[13] and the analytic approximations[13,14] of Vilela Mendes are tabulated in Table I. The averaged deviation of our results from the exact values is approximately 4%, whereas that of Vilela Mendes is approximately 16%. Moreover, it is evident that Vilela Mendes' analytic approximations are already quite unreliable for $z=6,8$ (17% and 36% in error, respectively), let alone higher z's. Our results, on the other hand, admit of systematic improvements. For example, a third-order calculation for the quadratic map gives the Feigenbaum ratio $\delta=4.675\,3244$, which is different from the exact value by only 0.1%.

It is also evident from Table I that δ seems to be an increasing function of z. Although present ex-

TABLE I. The universal bifurcation ratio for a family of period-doubling maps $f(x)=1-ax^z$ ($z=2,4,6,8$). The subscript denotes the order of approximation.

$f(x)$	$\delta_{(1)}$	$\delta_{(2)}$	Exact	Ref. 13	$a^*_{(1)}$	$a^*_{(2)}$
$1-ax^2$	5.1224	4.6142	4.669	4.83	1.3904	1.4014
$1-ax^4$	9.3160	6.9992	7.284	6.87	1.5692	1.5958
$1-ax^6$	13.3721	8.8071	9.296	10.91	1.6525	1.6842
$1-ax^8$	17.3987	10.3294	10.948	14.93	1.7043	1.7366

periments are consistent with the quadratic map, it would be premature to preclude the relevance of higher z's since it may happen that, for symmetry or other reasons, the quadratic term vanishes. Moreover, these large δ values make them more difficult to discern experimentally.

III. SUMMARY

In summary, we have employed a simple renormalization-group method to calculate the dependence of the universal bifurcation ratio on the order of the critical point. To second order, the calculated values agree very well with the exact values. We expect the agreement to be even better when higher-order calculations are performed. For the four low even orders we have studied, $\delta(z)$ is an increasing function of z. These larger bifurcation ratios may be relevent to the description of experiments in the future. The challenging important problem is indeed to find the full set of defining criteria for universality classes. Any progress made in this direction will undoubtedly shed light on the entire subject of nonlinear dynamics.

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